

# A note on Clifford-Klein forms

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**Abstract.** We consider the problem of finding Clifford-Klein forms in a class of homogeneous spaces determined by inclusions of real Lie algebras of a special type which we call strongly regular. This class of inclusions is described in terms of their Satake diagrams. For example, the complexifications of such inclusions contain the class of subalgebras generated by automorphisms of finite order. We show that the condition of strong regularity implies the restriction on the real rank of subalgebras. This in part explains why the known examples of Clifford-Klein forms are rare. We make detailed calculations of some known examples from the point of view of the Satake diagrams.

## 1. Introduction

We say that a pair of reductive Lie groups  $(G, H)$  is a Clifford-Klein form, if there is a discrete subgroup  $\Gamma \subset G$  acting properly discontinuously on  $G/H$  such that the double coset  $\Gamma \backslash G/H$  is compact. Riemannian locally symmetric spaces yield a class of examples of such pairs. In this case,  $H$  is compact. We are interested in the case when  $H$  is non-compact. It is important in geometry to solve the problem of finding Clifford-Klein forms. We refer to the survey [5], as well as research papers [4, 6, 7, 8, 10] and references therein for motivation and known results. The importance of Clifford-Klein forms comes from the fact that such homogeneous spaces  $G/H$  may carry various important invariant geometric structures which descend onto compactifications  $\Gamma \backslash G/H$ . For example this kind of compactification was used in [11, 3] to construct new examples of symplectic structures. The problem of determining whether a pair  $(G, H)$  is a Clifford-Klein form is difficult. There is the "Calabi-Markus phenomenon" that homogeneous space  $G/H$  with  $G$  and  $H$  of equal real ranks cannot have infinite discrete subgroups  $\Gamma \subset G$  acting freely and uniformly on  $G/H$ . The existence of some invariant geometric structures on  $G/H$  eliminates the possibility of  $G/H$  being a Clifford-Klein form. For example, para-hermitian symmetric spaces cannot be Clifford-Klein forms. Lorentz space forms admit compact forms only in the flat case and in case of negative sectional curvature, if the dimension is odd, see [5]. The following method of proof that  $(G, H)$  is a Clifford-Klein form was found in [4]. We restrict ourselves to homogeneous spaces  $G/H$  of reductive type, which means that both  $G$  and  $H$  are reductive and their Cartan decompositions can be made compatible, see details in [4]. To prove that  $G/H$  is a Clifford-Klein form, one can consider the Lie algebras  $\mathfrak{g} \supset \mathfrak{h}$  and look for a second subalgebra  $\mathfrak{l}$  (which also corresponds to a closed Lie subgroup  $L \subset G$ ) with the following properties. Both  $\mathfrak{h}$  and  $\mathfrak{l}$  are compatible with the Cartan decomposition of  $\mathfrak{g}$ :

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad \mathfrak{h} = \mathfrak{h} \cap \mathfrak{k} \oplus \mathfrak{h} \cap \mathfrak{p}, \quad \mathfrak{l} = \mathfrak{l} \cap \mathfrak{k} \oplus \mathfrak{l} \cap \mathfrak{p}, \quad \dim \mathfrak{p} = \dim \mathfrak{h} \cap \mathfrak{p} + \dim \mathfrak{l} \cap \mathfrak{p}$$

and maximal Cartan subspaces  $\mathfrak{a} \subset \mathfrak{p}$ ,  $\mathfrak{a}' \subset \mathfrak{h} \cap \mathfrak{p}$ ,  $\mathfrak{a}'' \subset \mathfrak{l} \cap \mathfrak{p}$  can be chosen in a way that

$$\mathfrak{a}' \cap \cup_{w \in W} w \cdot \mathfrak{a}'' = \{0\}$$



for the little Weyl group

$$W = W(\mathfrak{g}, \mathfrak{a}) = N_G(\mathfrak{a})/Z_G(\mathfrak{a})$$

Thus, given a triple  $(G, H, L)$  one can check if it yields a Clifford-Klein form by the described method. However there is no way in sight of looking for triples  $(G, H, L)$  satisfying these conditions. The purpose of this note is to propose a procedure of detecting Clifford-Klein forms among a certain class of homogeneous spaces. The idea is to restrict ourselves to a special family of embeddings  $H \subset G$  and  $L \subset G$  (and, on the Lie algebra level, to inclusions  $\mathfrak{h} \subset \mathfrak{g}, \mathfrak{l} \subset \mathfrak{g}$ ), which we call strongly regular. It seems that singling out much more "tame" class of subgroups is useful for understanding the problem of constructing Clifford-Klein forms. Our inclusions are obtained by a certain procedure applied to "generalized" Dynkin diagrams of the complexification  $\mathfrak{g}^c$ , these are called the diagrams of type  $S(A)$  in [2]. In particular, subalgebras generated by automorphisms of finite order  $\varphi : \mathfrak{g}^c \rightarrow \mathfrak{g}^c$  belong to this class. Introducing this class of subalgebras allows us to describe the problem of finding Clifford-Klein forms in terms of the Satake diagrams. We make all calculations in detail for some known examples of Clifford-Klein forms. These calculations, as well as the main theorem reveal the algebraic nature of the difficulties arising in constructing Clifford-Klein forms, and explain to some extent why they are so rare. The main result of this paper shows that the strong regularity condition together with the condition of Kobayashi put restrictions on the real rank of subgroups, which should be  $\leq 2$ . It is worth mentioning that we don't know any examples of Clifford-Klein forms obtained by the described method [4] which are not strongly regular.

## 2. Notation and preliminaries

We use basics of the theory of (semisimple) real and complex Lie algebras without further explanations. Our terminology and notation are close to [9]. Let  $\mathfrak{g}$  be a real semisimple Lie algebra. Consider a Cartan decomposition  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ . Fix a maximal abelian subalgebra  $\mathfrak{a} \subset \mathfrak{p}$  and consider the real root system  $\Sigma \subset \mathfrak{a}^*$  of  $\mathfrak{g}$ . The choices made yield the decompositions

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\lambda \in \Sigma} \mathfrak{g}_\lambda, \quad \mathfrak{g}_0 = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}), \quad \mathfrak{g}_\lambda = (\mathfrak{g}_0 \cap \mathfrak{k}) \oplus \mathfrak{a}$$

Let  $\Delta$  be the root system of  $\mathfrak{g}^c$ . In what follows we make several observations and choices which relate  $\Sigma$  and  $\Delta$ . Choose a Cartan subalgebra  $\mathfrak{t} \subset \mathfrak{g}$  as a maximal abelian subalgebra in  $\mathfrak{g}$  containing  $\mathfrak{a}$ . This yields the decompositions

$$\mathfrak{t} = \mathfrak{t}^+ \oplus \mathfrak{a}, \quad \mathfrak{t}^c = (\mathfrak{t}^+)^c \oplus \mathfrak{a}^c$$

It is known that  $\mathfrak{t}^c$  is an algebraic subalgebra consisting of semisimple elements, and, therefore

$$\mathfrak{t}^c(\mathbb{R}) = (i\mathfrak{t}^+) \oplus \mathfrak{a}$$

The latter enables one to introduce the restriction map  $r : \mathfrak{t}^c(\mathbb{R}) \rightarrow \mathfrak{a}^*$ , and to show that  $r(\Delta \cup \{0\}) = \Sigma \cup \{0\}$ . Let

$$\Delta_0 = \{\alpha \in \Delta \mid r(\alpha) = 0\}, \quad \Delta_1 = \Delta \setminus \Delta_0$$

The decomposition  $\Delta = \Delta_0 \cup \Delta_1$  yields the decompositions

$$(\mathfrak{g}_0 \cap \mathfrak{k})^c = \mathfrak{t}^c \oplus \sum_{\alpha \in \Delta_0} \mathfrak{g}_\alpha^c, \quad \mathfrak{g}_\lambda^c = \sum_{r(\alpha)=\lambda} \mathfrak{g}_\alpha^c, \quad \mathfrak{g}^c = \mathfrak{t}^c \oplus \sum_{\alpha \in \Delta_0} \mathfrak{g}_\alpha^c \oplus \sum_{\lambda \in \Sigma} \sum_{r(\alpha)=\lambda} \mathfrak{g}_\alpha^c, \quad \alpha \in \Delta_1, \lambda \in \Sigma$$

It is visible from the latter formula that  $\Delta_0$  is a root system for the semisimple part of  $(\mathfrak{g}_0 \cap \mathfrak{k})^c$ . One assigns to any real semisimple Lie algebra the *Satake diagram* as follows. It is a Dynkin diagram of  $\mathfrak{g}^c$  with vertexes of two types, black and white. Black vertexes  $\alpha_i$  are those for which  $r(\alpha_i) = 0$ . White vertexes  $\alpha_j$  and  $\alpha_k$  are joined by an arrow if  $r(\alpha_j) = r(\alpha_k)$ . It is known that two semisimple real Lie algebras are isomorphic if and only if so are their Satake diagrams. As usual, we consider the dual root systems  $\Sigma$  and  $\Sigma^\vee$  and relate the corresponding vectors by the formula  $\mu(\alpha^\vee) = 2(\mu, \alpha)/(\alpha, \alpha) = \langle \mu | \alpha \rangle$ . The Cartan matrix  $A = A_\Sigma$  corresponding to  $\Sigma$  is a matrix with elements  $a_{ij} = \langle \alpha_i | \alpha_j \rangle$ , where  $\alpha_i$  are simple roots of  $\Sigma$ . Note that  $\alpha_i$  is a black root in the Satake diagram, if and only if  $\alpha_i^\vee$  centralizes  $\mathfrak{a}^c = (\langle \Sigma^\vee \rangle)^c$ .

### 3. On Clifford-Klein forms generated by strongly regular triples

In order to introduce the class of subalgebras we are interested in, we need to describe the notion of the diagram of type  $S(A)$  in the sense of [2]. Note that our description is very brief and we recommend the reader to consult [2], Chapter X. Let  $E$  be a real vector space of dimension  $n$ . A subset  $\{\alpha_0, \dots, \alpha_n\} \subset E$  is called a subset of type  $S(A)$  if

- (i)  $a_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \in -\mathbb{Z}^+$  for  $i \neq j$
- (ii)  $S(A) = \{\alpha_0, \dots, \alpha_n\}$  is an indecomposable system of vectors (that is,  $S(A)$  cannot be decomposed into a sum of orthogonal non empty subsets),
- (iii)  $S(A)$  is a linearly dependent system of vectors generating  $E$ , in particular

$$\det(a_{ij}) = \det A = 0$$

One associates to each matrix  $A$  a diagram  $S(A)$  as follows (note that in the sequel we use the same notation for the set  $S(A)$ , and for the corresponding diagram). We take  $n + 1$  vertexes, join the  $i$ -th and  $j$ -th vertexes by  $a_{ij} \cdot a_{ji}$  lines, and if  $|a_{ij}| < |a_{ji}|$ , these lines have an arrow pointing towards the  $i$ -th vertex. The corresponding diagram will be called the diagram of type  $S(A)$ . All such diagrams are classified and one can find this classification in [2] on page 503. Note that we don't reproduce this table, but just mention that it consists of extended Dynkin diagrams (these are of types  $A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}, F_4^{(1)}, G_2^{(1)}$ ), and two other types:  $A_{2n}^{(2)}, D_{n+1}^{(2)}, A_{2n-1}^{(2)}, E_6^{(2)}$ , and  $D_4^{(3)}$ .

Let us mention the relation between the diagrams of type  $S(A)$  and the Dynkin diagrams of simple complex Lie algebras. The diagrams of types  $S(A)$  correspond to those automorphisms  $\nu : \mathfrak{g}^c \rightarrow \mathfrak{g}^c$  of simple complex Lie algebras  $\mathfrak{g}^c$ , which are induced by the automorphisms (of order  $k = 1, 2, 3$ ) of the Dynkin diagrams. Then, the subalgebra  $(\mathfrak{g}^c)^\nu$  of fixed points has the Dynkin diagram determined by the subdiagram  $S(A) \setminus \{\rho_0\}$  for a suitable choice of a root  $\rho_0$  (we refer to Section 5 of Chapter X in [2], especially Examples 1 and 2 and arguments on page 508). Moreover, simple roots of  $S(A) \setminus \{\rho_0\}$  are restrictions of simple roots from the Dynkin diagram of  $\mathfrak{g}^c$  onto a Cartan subalgebra of  $(\mathfrak{g}^c)^\nu$ . Let us adopt the following notation:  $\Pi$  is the set of simple roots for the root system  $\Delta$ ,  $\bar{\Pi}$  will be reserved for the set  $\{\alpha_0, \dots, \alpha_n\}$  of roots generating the extended Dynkin diagram of rank  $n$ , while  $\hat{\Pi}$  will denote *any* of the possible generating sets of diagrams  $S(A) \setminus \{\rho_0\}$ . Note that  $\hat{\Pi}$  can be considered as a subset of  $\bar{\Pi}$ , but in general not as a subdiagram.

**Definition 3.1.** We say that an inclusion of real semisimple Lie algebras is *strongly regular*, if

- the inclusion of their complexifications  $\mathfrak{h}^c \subset \mathfrak{g}^c$  is described as follows: the Dynkin diagram of  $\mathfrak{h}^c$  is obtained by removing certain vertexes from one of the diagrams of  $\hat{\Pi} = S(A) \setminus \{\rho_0\}$ ,
- if  $\Sigma = r(\Delta)$  and  $r_1(\Delta') = \Sigma'$  are real root systems of  $\mathfrak{g}$ , and  $\mathfrak{h}$  respectively, then  $\Pi'_\Sigma \subset r(\hat{\Pi})$ .

Note that semisimple part of any subalgebra  $\mathfrak{h}^c = (\mathfrak{g}^c)^\varphi$  of fixed points of an automorphism  $\varphi$  of finite order satisfies the first condition of the definition. This follows from Theorem 5.15 in [2]. Indeed, the Dynkin diagram of such subalgebra is obtained by removing certain vertexes from  $\hat{\Pi}$  (according to a certain rule which is not used here).

**Remark 3.2.** In the proof of the main result we use only the second condition of the definition. However, in order to calculate examples we need to use the Satake diagrams, and, therefore, some conditions which enable us to calculate the Dynkin diagrams of  $\mathfrak{h}^c$  and  $\mathfrak{l}^c$ . This is the reason why we prefer to define our class using both restrictions (on  $\Pi', \Pi''$  and  $\Pi'_\Sigma, \Pi''_\Sigma$ ).

There is the notion of regularity for subalgebras of complex Lie algebras (see [9], Chapter 6). Note that maximal regular subalgebras are obtained by removal one or two vertexes from the extended Dynkin diagram (see Theorem 1.2 in [9]). It is known that maximal semisimple subalgebras of maximal rank are of the form  $(\mathfrak{g}^c)^\nu$  for an inner involutive automorphism  $\nu : \mathfrak{g}^c \rightarrow \mathfrak{g}^c$ , and therefore, are obtained from diagrams  $S(A)$  as well. We don't know if all regular subalgebras can be obtained from diagrams of types  $S(A)$ . The triple  $(\mathfrak{g}, \mathfrak{h}, \mathfrak{l})$  is called strongly regular, if  $\mathfrak{h}$  and  $\mathfrak{l}$  are strongly regular subalgebras in  $\mathfrak{g}$ . We say that a Clifford-Klein form is generated by a strongly regular triple  $(G, H, L)$ , if the corresponding triple  $(\mathfrak{g}, \mathfrak{h}, \mathfrak{l})$  is strongly regular,  $L$  acts properly on  $G/H$  and the double coset space  $L \backslash G/H$  is compact.

Recall that such triple always produces a Clifford-Klein form if there is a cocompact torsionfree lattice  $\Gamma \subset L$  [4]. Now we are ready to formulate the main result of this work.

**Theorem 3.3.** *Assume that  $G$  is a simple Lie group which is a non-compact real form of a complex classical Lie group. Any strongly regular triple of semisimple Lie groups  $(G, H, L)$  generating the Clifford-Klein form has the property that at least one of the subgroups has real rank  $\leq 2$ .*

*Proof.* Let  $\Pi$  be the system of simple roots of  $\mathfrak{g}^c$ ,  $\Pi'$  and  $\Pi''$  be the systems of simple roots for  $\mathfrak{h}^c$  and  $\mathfrak{l}^c$ , respectively. Then, for the restriction maps  $r_1$  and  $r_2$  (built for  $\mathfrak{h}$  and  $\mathfrak{l}$ , respectively),  $r_1(\Pi') = \Pi'_\Sigma$ ,  $r_2(\Pi'') = \Pi''_\Sigma$ . By the assumption of strong regularity

$$\Pi'_\Sigma \cup \Pi''_\Sigma \subset r(\tilde{\Pi})$$

it follows that

$$\mathfrak{a}' = \langle \Pi'_\Sigma \rangle, \quad \mathfrak{a}'' = \langle \Pi''_\Sigma \rangle, \quad \mathfrak{a} = \langle \Pi_\Sigma \rangle = \langle r(\tilde{\Pi}) \rangle$$

The latter implies that the condition  $\cup_{w \in W} w \cdot \mathfrak{a}' \cap \mathfrak{a}'' = \{0\}$  is satisfied if and only if  $w(\Pi'_\Sigma) \cup \Pi''_\Sigma$  is linearly independent in  $\mathfrak{a}$  for any  $w \in W$ . We are going to show that the latter implies the condition of the Theorem. This will be done by considering each of the classical Lie groups separately. Let  $\mathfrak{g}$  be a non-compact real form of a complex simple Lie algebra. The type of  $\Pi_\Sigma$  is known for all simple Lie algebras, for example, from [9]. Note also that

$$r(\tilde{\Pi}) = \begin{cases} \Pi_\Sigma, & \text{if } r(\alpha_0) = 0 \\ \Pi_\Sigma \cup \{r(\alpha_0)\}, & \text{otherwise} \end{cases}$$

This implies the inclusions

$$\Pi'_\Sigma \subset \Pi_\Sigma \cup \{r(\alpha_0)\}, \quad \Pi''_\Sigma \subset \Pi_\Sigma \cup \{r(\alpha_0)\}$$

because of the strong regularity assumption. Looking at Table 4 in [9] one can notice that only the following types of  $\Sigma$  are possible:

- (i)  $\mathfrak{g}^c$  of type  $A_n$ :  $\Sigma$  may have types  $A_n$ ,  $BC_p$  and  $C_p$ , with  $p \leq \lfloor \frac{n}{2} \rfloor - 1$
- (ii)  $\mathfrak{g}^c$  of type  $B_n$ :  $\Sigma$  has type  $B_p$ , with  $p < n$
- (iii)  $\mathfrak{g}^c$  of type  $C_n$ :  $\Sigma$  has type  $C_n$ , or  $BC_p$ ,  $p < n$
- (iv)  $\mathfrak{g}^c$  of type  $D_n$ :  $\Sigma$  has type  $C_p$  or  $BC_p$ ,  $p < n$

Let us begin our case by case analysis.

- **type  $B_p$**  In this case,

$$\Pi_\Sigma = \{\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3, \dots, \alpha_{p-1} = \varepsilon_{p-1} - \varepsilon_p, \alpha_p = \varepsilon_p\}$$

Since  $W \cong W_\Sigma$  acts on  $\Pi_\Sigma$  permuting all  $\varepsilon_i$  and possibly changing signs  $\varepsilon_i \rightarrow (\pm)\varepsilon_i$ , it has two orbits  $\mathcal{O}_1$  consisting of roots  $\alpha_1, \dots, \alpha_{p-1}$ , and  $\mathcal{O}_2$  consisting of  $\alpha_p = \varepsilon_p$ . Note that  $r(\alpha_0)$  may belong to  $\Pi'_\Sigma$ , or to  $\Pi''_\Sigma$  or may not.

- (i) if  $r(\alpha_0) \notin \Pi'_\Sigma \cup \Pi''_\Sigma$ , then, from the definition of  $r$  one can see that

$$\Pi'_\Sigma \cup \Pi''_\Sigma \subset \Pi_\Sigma$$

and the only possibility of their linear independence is that one of these sets is  $\mathcal{O}_2$ , and the other is a subset of  $\mathcal{O}_1$ . But the latter means that the rank of the subalgebra corresponding to  $\Pi'_\Sigma$  must be one.

- (ii) if  $r(\alpha_0) \in \Pi'_\Sigma$ , then the following possibilities may occur:

$$r(\alpha_0) \in \mathcal{O}_1, r(\alpha_0) \in \mathcal{O}_2, r(\alpha_0) \notin \mathcal{O}_1 \cup \mathcal{O}_2$$

In the first case, the linear independence is achieved, if one of the sets  $\Pi'_\Sigma$  or  $\Pi''_\Sigma$  has a form  $\{\alpha_n\}$ , and the other consists of vectors in  $\mathcal{O}_2$ . This case again, yields the rank one. If  $r(\alpha_0) \in \mathcal{O}_2$ , necessarily  $\Pi'_\Sigma = \{r(\alpha_0), \alpha_n\}$ , and the rank of the corresponding real Lie subalgebra is 2. If  $r(\alpha_0)$  does not belong to  $\mathcal{O}_1 \cup \mathcal{O}_2$ , this means that still the rank is 1 or 2, because in any case one of the subspaces  $\mathfrak{a}'$  or  $\mathfrak{a}''$  does not contain any vector from  $\mathcal{O}_1$ .

- **type  $C_p, BC_p$** . The argument is similar. We have

$$\Pi_\Sigma = \langle \alpha_1 = \varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{p-1} - \varepsilon_p, \alpha_p = 2\varepsilon_p \rangle$$

It follows that still  $\Pi_\Sigma$  has two orbits  $\mathcal{O}_1 = \{\alpha_1, \dots, \alpha_{p-1}\}$  and  $\mathcal{O}_2 = \{\alpha_p\}$ .

- **type  $A_n$**  differs, because in this case  $W$  acts transitively on  $\Pi_\Sigma$ , and the only possible new orbit of  $W$  may appear if  $r(\alpha_0)$  does not belong to this orbit. Thus, one of the subalgebras must have real rank 1. □

Note that our theorem contains a bit more than the restriction on the real rank. It contains a procedure of looking for possible regular Clifford-Klein forms. The latter goes as follows.

- Given a (say, simple) classical Lie algebra  $\mathfrak{g}$ , consider its complexification  $\mathfrak{g}^c$  and all the diagrams of type  $S(A)$  with the set of vertexes  $\hat{\Pi} = S(A) \setminus \{\rho_0\}$ , and write down possible strongly regular subalgebras,
- looking at the corresponding Satake diagrams of these subalgebras, choose those which satisfy  $\Pi'_\Sigma \subset \Pi_\Sigma \cup \{r(\alpha_0)\}$  and such that one of them has real rank  $\leq 2$ ,
- check the condition  $\dim \mathfrak{p} = \dim \mathfrak{p}' + \dim \mathfrak{p}''$ .

**4. Examples**

Let us analyze examples constructed by T. Kobayashi in [4, 5] from the point of view of strongly regular triples.

**Example 4.1.** The triple

$$(G = SU(2n, 2), H = SU(2n, 1), L = Sp(n, 1))$$

is a strongly regular Clifford-Klein form.

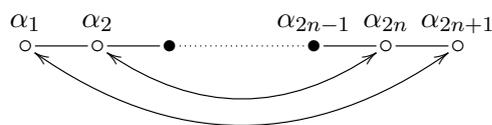
To prove this, we pass to the Lie algebra level. Thus

$$\begin{aligned} \mathfrak{g} &= su(2n, 2), & \mathfrak{h} &= su(2n, 1), & \mathfrak{l} &= sp(n, 1) \\ \mathfrak{g}^c &= sl(2n + 2, \mathbb{C}), & \mathfrak{h}^c &= sl(2n + 1, \mathbb{C}), & \mathfrak{l} &= sp(n + 1, \mathbb{C}) \end{aligned}$$

which shows that all considered complex Lie algebras are simple, and have types, respectively,  $A_{2n+1}$ ,  $A_{2n}$  and  $C_n$ . Consider the embedding  $\mathfrak{h}^c \subset \mathfrak{g}^c$  determined by the corresponding inclusion

$$\Pi' = \{\alpha_2, \dots, \alpha_{2n+1}\} \subset \Pi = \{\alpha_1, \dots, \alpha_{2n+1}\}$$

Looking at the corresponding Satake diagrams for the real forms  $su(2n, 2)$  and  $su(2n, 1)$  we determine  $\Sigma$  and  $\Sigma'$ . Let us begin with the Satake diagram on for  $su(2n, 2)$ :



**Figure 4.1.** Satake diagram for  $su(2n, 2)$ .

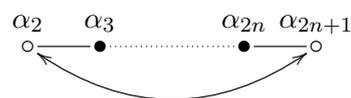
We see that  $\mathfrak{a} = \langle \Sigma^\vee \rangle$  and

$$\mathfrak{g} = \mathfrak{z}_{\mathfrak{g}}(\langle \Sigma^\vee \rangle) \oplus \sum_{\lambda \in \Sigma} \mathfrak{g}_\lambda$$

where  $\Sigma = \langle \bar{\alpha}_1, \bar{\alpha}_2 \rangle$ . Note that the black roots  $\alpha_3, \dots, \alpha_{2n-1}$  determine the semisimple part of  $\mathfrak{z}_{\mathfrak{g}}(\langle \Sigma^\vee \rangle)$ . Consider

$$\Sigma' = \langle \bar{\alpha}_2 \rangle \subset \Sigma = \langle \bar{\alpha}_1, \bar{\alpha}_2 \rangle, \quad \mathfrak{h} = \mathfrak{z}_{\mathfrak{g}}(\langle (\Sigma')^\vee \rangle) \oplus \sum_{\lambda \in \Sigma'} \mathfrak{g}_\lambda$$

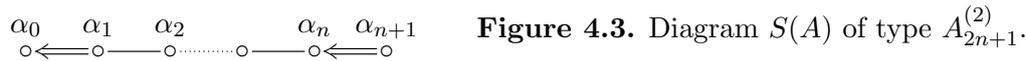
Clearly,  $\mathfrak{h} \subset \mathfrak{g}$ . In order to determine the Satake diagram of the semisimple part  $\mathfrak{h}_s$  of  $\mathfrak{h}$ , we need to determine the Dynkin diagram for  $\mathfrak{h}_s^c$  and vertices which represent roots  $\alpha_i$ , whose duals  $\alpha_i^\vee$  centralize  $\langle (\Sigma')^\vee \rangle$ . Since  $\alpha_3^\vee, \dots, \alpha_{2n-1}^\vee$  centralize  $\langle (\Sigma')^\vee \rangle$ , they belong to the Dynkin diagram of  $\mathfrak{h}_s^c$ , and represent black vertexes of the corresponding Satake diagram. It is known that the Cartan matrix  $(a_{ij})$  of  $\mathfrak{g}^c$  has  $a_{ii} = 2$ ,  $a_{i,i+1} = -1$  and  $a_{ij} = 0, j \neq i$ . For example, one can consult [1]. It follows that  $\alpha_{2n}(\alpha_2^\vee) = 0$ , for  $n > 1$ . Thus,  $\alpha_{2n}$  must belong to the Dynkin diagram of  $\mathfrak{h}_s^c$ , and must be black. Hence, the Satake diagram for  $\mathfrak{h}_s^c$  must contain  $\alpha_2, \dots, \alpha_{2n}$ . Looking at the table of all possible Satake diagrams of simple real Lie algebras, one concludes that there is only one possibility for the Satake diagram which contains the part  $\{\alpha_2, \dots, \alpha_{2n+1}\}$  with black roots  $\alpha_3, \dots, \alpha_{2n}$ . This is the diagram



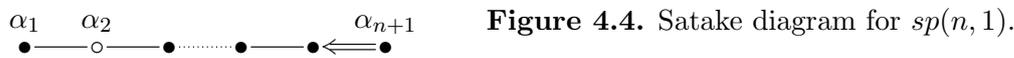
**Figure 4.2.** Satake diagram for  $su(2n, 1)$ .

The latter is the Satake diagram of  $su(2n, 1)$ , which means that  $\mathfrak{h}_s = \mathfrak{h} = su(2n, 1)$ . We see that  $\mathfrak{h}$  is embedded into  $\mathfrak{g}$  in a strongly regular way, as required, because  $\Pi'_\Sigma \subset r(\Pi) \subset r(\tilde{\Pi})$  (in this case, the diagram of type  $S(A)$  is the extended Dynkin diagram with the set of roots  $\tilde{\Pi}$ ).

Consider the case of  $\mathfrak{l} = sp(n, 1) \subset sl(2n + 1, \mathbb{C})$ . On the complex Lie algebra level, the Lie algebra of type  $A_{2n+1}$  admits an involutive automorphism  $\nu$  such that  $\mathfrak{l}^c = (\mathfrak{g}^c)^\nu$ . The root system of  $(\mathfrak{g}^c)^\nu$  is obtained from the diagram  $S(A)$  of type  $A_{2n+1}^{(2)}$  (in the terminology of [2]). This diagram has the form



The Dynkin diagram of  $\mathfrak{l}^c$  is obtained from the diagram of type  $A_{2n+1}^{(2)}$  by removing vertex  $\alpha_0$ . Looking again at the Satake diagrams of  $su(2n, 2)$  and  $sp(n, 1)$  one can argue exactly as in the previous case and calculate that  $r_2(\alpha_1) = 0, r_2(\alpha_2) = \alpha_2$ . Here is the required Satake diagram:



Thus,  $\Pi''_\Sigma \subset r(\hat{\Pi}) \subset r(\tilde{\Pi})$ , where  $\hat{\Pi}$  is determined by the diagram of type  $A_{2n+1}^{(2)}$ . This shows that  $\mathfrak{l}$  is a strongly regular subalgebra in  $\mathfrak{g}$ .

It remains to prove that the corresponding triple yields a Clifford-Klein form. Looking at the Satake diagrams and consulting Table 4 on page 231 in [9] we get the types of the root systems

$$\Sigma = \langle \bar{\alpha}_1, \bar{\alpha}_2 \rangle, \quad \Sigma' = \langle \bar{\alpha}_1 \rangle, \quad \Sigma'' = \langle \bar{\alpha}_2 \rangle$$

where  $\Sigma$  has type  $BC_2$ , while  $\Sigma'$  and  $\Sigma''$  both have rank 1. It follows that

$$\Sigma = \langle \varepsilon_1 - \varepsilon_2, \varepsilon_2 \rangle, \quad \Sigma' = \langle \varepsilon_1 - \varepsilon_2 \rangle, \quad \Sigma'' = \langle \varepsilon_2 \rangle$$

Since the Weyl group  $W$  of  $\Sigma$  acts on  $\Sigma$  by permuting indices and changing signs, for any  $w \in W$ ,  $w\bar{\alpha}_1, \alpha_2$  always constitute a base for  $\mathfrak{a}$ , and this is equivalent to the statement we wanted to prove. The condition

$$\dim \mathfrak{p} = \dim \mathfrak{p}' \oplus \dim \mathfrak{p}''$$

is checked directly, in this case one can also use [9], Table 4 on page 229.

**Example 4.2.** In [5] the author analyzed the problem of existence of Clifford-Klein forms in the class of (non-Riemannian) symmetric spaces and pointed out some of them which are Clifford-Klein. Our method yields similar results (we restrict ourselves to the case when  $G$  is a real form of non-compact type of a complex simple Lie group). Here is an example.

**Theorem 4.3.** *The following triples are strongly regular, satisfy the restrictions of Theorem 3.3 and generate Clifford-Klein forms:*

- $(SU(2, 2n), Sp(1, n), U(1, 2n))$
- $(SO(2, 2n), U(1, n), SO(1, 2n))$
- $(SO(4, 4n), SO(3, 4n), Sp(1, n))$
- $(SO(4, 4n), SO(3, 4n), Sp(1, n) \times Sp(1))$
- $(SO(3, 4), G_{2(2)}, SO(1, 4))$
- $(SO(8, 8), SO(7, 8), SO(1, 8))$
- $(SO(4, 4), SO(3, 4), SO(1, 4))$
- $(SO(4, 4), SO(3, 4), SO(1, 4) \times SO(3))$

*Proof.* There are several ways of proof. One of the possibilities is to use the method described in Example 4.1 (the first triple has been already worked out in detail).  $\square$

## References

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