Group theory aspects of chaotic strings

S Groote[1,2] **and R Saar**[2]

[1]Institut für Physik, Johannes-Gutenberg-Universität, Staudinger Weg 7, 55099 Mainz, Germany
[2]Füüsika Instituut, Tartu Ülikool, Ravila 4c, 50411 Tartu, Estonia

Abstract. Chaotic strings are a special type of non-hyperbolic coupled map lattices, exhibiting a rich structure of complex dynamical phenomena with a surprising correspondence to physical contents. Chaotic strings are generated by the Chebyshev maps $T_2(\phi)$ and $T_3(\phi)$. In this paper we connect the Chebyshev maps via the Galois theory to the cyclic groups Z_2 and Z_3 and give some ideas how this fundamental connection might lead to the emergence of the familiar Lie group structure of particle physics and, finally, even to the emergence of space-time. The Z_3-graded cubic and ternary algebras presented here have been introduced by R. Kerner in 1991 and then developed and elaborated in collaboration with many researches. We present here the most important results associated with these papers.

1. Introduction

As far as we understand by now, the relation of chaotic strings with elementary particle physics is given by the energy dependence of the couplings (usually called running) caused by the renormalization group of the theories of the Standard Model. Chaotic strings have a single coupling which, if being identified with the couplings of the Standard Model, unveils a surprising fact: The minima of the self energy of the chaotic string are related to the masses of the elementary particles [1, 2]. The details of this relation, however, are still quite unknown. Even though first progress has been made in understanding the perturbation theory for chaotic strings [3, 4] including also nonperturbative effects [5], the origin of the pattern for the self energy of the chaotic string appearing also for strings of length 3 [6] is still unclear.

In this paper we undertake the attempt to shed light on the relation by employing group theory. As in case of the theories of the Standard Model of elementary particles, group theory helps to organize particle states. However, in looking at the group theory aspects of chaotic strings it turns out that group theory appears prior to quantum mechanics and, in addition, space and time naturally emerge in the course of this. The approach we are following here is motivated more by inspiration than by manifest calculation and proof.

2. Symmetry group of chaotic strings

Chaotic strings are mathematical objects defined by the iterative map. For simplicity, we look only at one special case of chaotic strings given by

$$\phi_i^{n+1} = (1-a)T_N(\phi_i^n) + \frac{a}{2}\left(\phi_{i+1}^n + \phi_{i-1}^n\right)$$

As can be seen, besides a Laplace coupling between nearest neighbours chaotic strings are based on the Chebyshev polynomials $T_N(x)$, and it turns out that only the degrees $N = 2$ and $N = 3$ are relevant. Interesting for the relation searched for are characteristic points (maxima, minima

and zeros) not only in the self energy as mentioned in the introduction but also in the map itself. According to the equation $T_N(\phi) = \cos(N \arccos \phi)$, characteristic points are the real values of $\phi^N = \pm 1$ and $\phi^N = i$, as it is displayed in Figure 1 as symbolic "wheels". To be more precise, the Galois group of the Chebyshev polynomial $T_N(x)$ is S_N, the symmetric group with N elements. The minimal (nontrivial) normal subgroup of S_N, finally, is the cyclic group $Z_N = Z/(NZ)$ which are something like turns of the weel by an angle of $120°$.
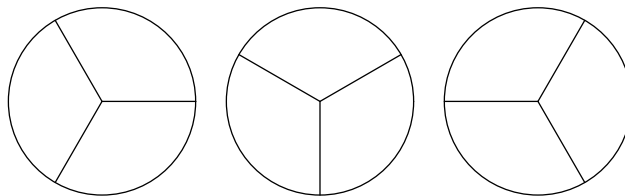

Figure 1. "Wheels" for maxima (left), zeros (middle) and minima (right)

During his short live, Évariste Galois (1811–1832) has enriched the mathematics of his time with interesting considerations about polynomials. One of these was the observation that a polynomial equation is solvable by roots if a chain of normal subgroups exists. This for instance exclude a polynomial equation of degree $N \geq 5$ from being solvable. Modern Galois theory is formulated in terms of (number) field extensions. In case of the Chebyshev polynomials, these field extensions would be $Q(\sqrt{2})$ and $Q(\sqrt{3})$.

3. Graded Z_N algebras

For this paper we stick with the Galois normal subgroups Z_2 and Z_3. For these groups one can define a graded algebra. Cubic and ternary Z_3-graded algebras have been introduced by Richard Kerner in 1991 [7] and developed and elaborated by R. Kerner and his co-authors V. Abramov, B. Le Roy, L. Vainerman and A. Borowiec in a series of publications dating from 1992 till 2012 [8, 9, 10, 11, 12]. If for instance θ is generator of the cyclic group Z_N, the graduation means that $\theta^N = 0$. A physical example of a graded Z_2 algebra is the creation operator θ^A of a quantum state $|A\rangle$, for one has

$$\theta^A|0\rangle = |A\rangle \quad \text{and} \quad (\theta^A)^2|0\rangle = \theta^A|A\rangle = 0$$

where the second equation expressing Pauli's exclusion principle for fermions. If there is more than one generator, we can speak of a graded algebra. In our example, with $M = 2$ generators there will be $M = 2$ different fermionic (spin) states, and the creation operators obey a Grassmann algebra[1]

$$\theta^A\theta^B + \theta^B\theta^A = 0$$

In general, for M states and M creation operators, in mathematical terms one can speak of a graded Z_2 algebra with M generators.

Surprisingly, there is a generalization of the graded Z_2 algebra well known from quantum mechanics to a graded Z_3 algebra which makes sense physically. To start from mathematics, the graduation of generators θ^A of Z_3 is established by distinguishing between elements θ^A of grade 1, $\theta^A\theta^B$ of grade 2, and $\theta^A\theta^B\theta^C$ of grade 3, the last one identified with grade 0. For this graded Z_3 algebra $\mathcal{A}$ a generalized Grassmann algebra holds,

$$\theta^A\theta^B\theta^C = j\theta^B\theta^C\theta^A = j^2\theta^C\theta^A\theta^B$$

---

[1] including also annihilation operators, one ends up with a Heisenberg algebra.

where $j = e^{2\pi i/3}$ so that $1 + j + j^2 = 0$ [8, 9, 10]. In addition, one can define a graded conjugate Z_3 algebra $\bar{\mathcal{A}}$ with generators $\bar{\theta}^{\bar{A}}$ of grade -1 (which, according to the graduation, is equivalent to grade 2) with the property

$$\theta^A \bar{\theta}^{\bar{B}} = -j \bar{\theta}^{\bar{B}} \theta^A$$

(note the minus sign on the right hand side). The graded (conjugate) Z_3 algebras can be used as model for (anti)quarks with three colours. The generators θ^A of grade 1 in this model can be seen as quarks, the generators $\bar{\theta}^{\bar{A}}$ of grade -1 as antiquarks, forming colour neutral mesonic and baryonic states with grade 0. In the following we will analyse these mesonic and baryonic forms for $M = 2$ and $M = 3$ generators [Kerner R]
[].

3.1. Graded Z_3 algebra with $M = 2$ generators

Using simple combinatorics we can determine how many independent elements are contained in the zero-graded subalgebras $\mathcal{A}_0$, $\bar{\mathcal{A}}_0$ and $(\mathcal{A} \times \bar{\mathcal{A}})_0$. There are

- $(M^3 - M)/3$ independent elements in $\mathcal{A}_0$ and $\bar{\mathcal{A}}_0$ (baryonic states),
- M^2 independent elements in $(\mathcal{A} \times \bar{\mathcal{A}})_0$ (mesonic states).

In case of $M = 2$ (two flavours for instance) one has two 3-forms (baryonic states p, n) and four 2-forms (mesonic states $\pi^{\pm}$, π^0, ρ). As seen for the baryonic states, the generalized Grassmann algebra implies that there are no three quarks in the same state. This restriction (no uuu and ddd states, only uud (proton) and udd (neutron)) can be named accordingly as generalized Pauli principle [11, 12]. Mathematically, these two possibilities are given by two independent choices for the 3-form

$$\psi^\alpha = \psi^\alpha_{ABC} \theta^A \theta^B \theta^C$$

Namely $\psi^1_{121} = 1$ (from which follows $\psi^1_{211} = j$ and $\psi^1_{112} = j^2$) and $\psi^2_{212} = 1$ ($\psi^2_{122} = j$, $\psi^2_{221} = j^2$). Claiming covariance under the transformation $\theta^{A'} = U^{A'}_A \theta^A$ of the generators, the corresponding 3-forms are related by $\psi' = L\psi$ where

$$L^{\alpha'}_\alpha \psi^\alpha_{ABC} = U^{A'}_A U^{B'}_B U^{C'}_C \psi'^{\alpha'}_{A'B'C'}$$

The matrix

$$L = \begin{pmatrix} L^{1'}_1 & L^{1'}_2 \\ L^{2'}_1 & L^{2'}_2 \end{pmatrix} = \begin{pmatrix} U^{1'}_1 & -U^{1'}_2 \\ -U^{2'}_1 & U^{2'}_2 \end{pmatrix} \det U$$

obeys $\det L = (\det U)^3 = 1$. It turns out to be isomorphic to the covering $SL(2,C)$ of the Lorentz group. L is unitary but U is unitary only up to j and j^2. This means that ψ transform as a free spinor (baryon) but the generator transform as bounded spinors (quarks). Looking at the mesonic states, there are four mixed 2-forms

$$\eta^\mu = \eta^\mu_{A\bar{B}} \theta^A \bar{\theta}^{\bar{B}}$$

Due to the reality condition $\eta^\mu_{A\bar{B}} \theta^A \bar{\theta}^{\bar{B}} = \bar{\eta}^\mu_{\bar{B}A} \bar{\theta}^{\bar{B}} \theta^A$ one obtains $\eta^\mu_{A\bar{B}} = -j^2 \bar{\eta}^\mu_{\bar{B}A}$. One possible representation (but not the only one) is via the Pauli spin matrices,

$$\eta^\mu_{A\bar{B}} = \frac{i\sigma^\mu_{A\bar{B}}}{\sqrt{2}}, \quad \bar{\eta}^\mu_{\bar{B}A} = -j \frac{i\bar{\sigma}^\mu_{\bar{B}A}}{\sqrt{2}}$$

where σ^0 is the 2×2 unit matrix. Using the "spinorial metric"

$$\epsilon^{12} = 1 = -\epsilon^{21}, \quad \epsilon^{\bar{1}\bar{2}} = 1 = -\epsilon^{\bar{2}\bar{1}}$$

the given representation makes it easy to see that

$$\eta^{\mu\nu} = \eta^{\mu}_{A\bar{B}} \eta^{\nu A\bar{B}} = \text{diag}\,(+1; -1, -1, -1)$$

Surprising enough, we end up with the metric of flat space-time. One might speculate at this point that the forms η^{μ} themselves are something like tetrad fields, for instance of the Poincaré gauge theory of gravitation (see e.g. Ref. [13]). Correspondingly, under the transformation of θ^A and $\bar{\theta}^{\bar{B}}$ one has to claim covariance of the form, $\eta' = \Lambda\eta$ with

$$\Lambda^{\mu'}_{\mu} \eta^{\mu}_{A\bar{B}} = U^{A'}_A \bar{U}^{\bar{B}'}_{\bar{B}} \eta'^{\mu'}_{A'\bar{B}'}$$

This claim defines the 4×4 Lorentz group with elements Λ for the Lorentz covariant 4-vector η.

3.2. Graded Z_3 algebra for $M = 3$ generators
Given three generators Q^a, one has eight 3-forms (baryonic states) and nine mixed 2-forms (mesonic states). Analysing the transformation, it turns out that the 3-forms transform under the adjoint representation of $SU(3)$, the mixed 2-forms transform under the $3 \otimes \bar{3}$ representation of $SU(3)$, and the generators themselves transform under the fundamental representation of $SU(3)$, for a more detailed consideration we refer to [Kerner R arXiv:0901.3961].

4. Conclusions
We have seen that the graded Z_3 algebra with $M = 2$ generators transform under the Lorentz group while the algebra with $M = 3$ generators transform under the Lie group $SU(3)$. Looking at the whole stream of arguments, one can state that the group properties of the outer and inner degrees of freedom emerge from a graded algebra originating from the Chebyshev polynomials $T_3(\phi)$ and associated with (confined) quark states.

4.1. First the hen, then the egg – or vice versa?
Concerning the Lorentz transformations, we obtain a new point of view for the relation between the two representations [11, 12]. The traditional view (we call it "first the hen, then the egg") is that the Lorentz transformations $x \mapsto \Lambda x$ (explicitly $x^{\mu} \mapsto x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu}$) and $j \rightarrow \Lambda j$ with vector currents

$$j^{\mu} = \langle \psi | \gamma^{\mu} | \psi \rangle \quad \mapsto \quad j'^{\mu} = \langle \psi' | \gamma^{\mu} | \psi' \rangle = \Lambda^{\mu}_{\nu} j^{\nu}$$

lead to the Lorentz transformation $\psi(x) \mapsto \psi'(x') = \psi'(\Lambda x) = L(\Lambda)\psi(x)$. The new point of view ("first the egg, then the hen") states that the transformation of spinors

$$\psi(x) \rightarrow \psi'(x') = \psi'(\Lambda(L)x) = L\psi(x)$$

is the primary transformation and the space-time transformation is the secondary one. This makes sense from the mathematical point of view because the covering group $SL(2, C)$ of the Lorentz group as the larger group releases information to the particle–antiparticle symmetry in covering the Lorentz group Λ of space-time.

4.2. Outlook

The ideas presented here are linked with a lot of more directions to think which are not included in this paper:

- Group theory seems to be the link between chaotic strings and quantum mechanics. This link should be developed as an instrument to specify properties related to the minima of the self energy of the chaotic strings.

- The considerations are done only for the Chebyshev polynomials which are the "motors" for the chaotic strings. While iterated Chebyshev polynomials are related to chaotic strings at coupling $a = 0$, these iterated polynomials are deformed for non-vanishing coupling $a \neq 0$, resulting in a modification of the Galois group. It is intuitively clear that from this we might obtain tetrad fields for a non-flat space-time and interactions between particles.

- As a link again between group theory and space-time emergence, quantum mechanics turns out to be "space-timeless". This point of view would support a quantum gravity in which the geometry of space-time (via general relativity or possible generalizations) is founded on quantum mechanics – and not vice versa (see, for instance, Ref. [14]).

In any case, new ideas are always welcome to our "think tank".

References

[1] Beck C 2002 *Physica* D **171** 62
[2] Beck C 2002 *Spatio-Temporal Chaos and Vacuum Fluctuations of Quantized Fields* (World Sci.)
[3] Groote S and Beck C 2006 *Phys. Rev.* E **74** 046216
[4] Groote S and Beck C 2007 *Dyn. Sys.* **22** 219
[5] Groote S, Veermäe H and Beck C 2013 *Chaos Sol. Fract.* **53** 18
[6] Groote S and Veermäe H 2009 *Chaos Sol. Fract.* **41** 2354
[7] Kerner R 1991 *C. R. Acad. Sci. Paris* **312** série II 191
[8] Kerner R 1992 *J. Math. Phys.* **33** 403
[9] Abramov V, Kerner R and Le Roy B 1997 *J. Math. Phys.* **38** 1650
[10] Kerner R 1997 *Classical Quantum Gravity* **14** A203
[11] Kerner R 2012 *J. Phys.: Conf. Ser.* **343** 012056
[12] Kerner R 2012 *AIP Conf. Proc.* **1483** 144
[13] Hehl F W, von der Heyde P, Kerlick G D and Nester J M 1976 *Rev. Mod. Phys.* **48** 393
[14] Wüthrich C 2012 *Die allgemeine Relativitätstheorie als Ausgangspunkt einer Theorie der Quantengravitation* (Philosophie der Physik) Hrsg M Esfeld (Berlin: Suhrkamp) p 306