$3j$-symbol for the modular double $SL_q(2, \mathbb{R})$ revisited

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Abstract. Modular double of quantum group $\mathrm{SL}_q(2, \mathbb{R})$ with $|q| = 1$ has a series of selfadjoint irreducible representations π_s parameterized by $s \in \mathbb{R}_+$. Ponsot and Teschner in [*Comm. Math. Phys.* **224** (2001) 613] considered a decomposition of the tensor product $\pi_{s_1} \otimes \pi_{s_2}$ into irreducibles. In our paper we give more detailed derivation and some new proofs.

1. Introduction

Conformal Field Theory is one of the main sources of quantum groups. The very first example of deformed algebra G_q of functions on $\mathrm{SL}(2)$ was given by the monodromy matrix for the quantized Lax operator of the Liouville model [1]. The variable τ, entering the deformation parameter $q = e^{i\pi\tau}$ played the role of the coupling constant. The duality $\tau \to -1/\tau$ observed in [2, 3] was formalized in [4] in the notion of modular double.

The irreducible representations of the modular double of $\mathrm{SL}_q(2, \mathbb{R})$, introduced in [4], were investigated in [5, 6]. In particular in a remarkable paper [5] the problem of the decomposition of the tensor product is solved.

There is an intriguing connection of the representations of the modular double of $\mathrm{SL}_q(2, \mathbb{R})$ and primary fields of the Liouville model. Both the representation π_s of the modular double and vertex operators $V_\alpha(x) = \exp(\alpha\phi(x))$ are labeled by the same number $\alpha = 1/2 + is$, $s > 0$. Apparently there should be a correspondence between the operator expansion of $V_{\alpha_1}(x_1)V_{\alpha_2}(x_2)$ and the decomposition of $\pi_{s_1} \otimes \pi_{s_2}$ into irreducibles. Some indications on such connection can be found in [7]. However the work in this direction is still to be done. Having this in mind we decided to rederive the results of the paper of Ponsot and Teschner [5] and supply more details of derivations and proofs. Our paper is a complete exposition of the talks of the second author (L.D.F) in the early summer of 2012. The first author (S.E.D) joined the company in the late summer and his contribution let to important improvement of the full exposition.

2. Modular double of $\mathrm{SL}_q(2, \mathbb{R})$

The algebra has six generators, combined in two mutually commuting triplets E, F, K and $\tilde{E}, \tilde{F}, \tilde{K}$. The usual relations [8, 9] for E, F, K read

$$KE = q^2 EK, \quad KF = q^{-2}FK, \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$$

with $q = e^{i\pi\tau}$ are supplemented by similar relations for $\tilde{E}, \tilde{F}, \tilde{K}$ with $\tilde{q} = e^{i\pi/\tau}$. Generators E, F, K and $\tilde{E}, \tilde{F}, \tilde{K}$ commute. The coproduct is given by

$$\Delta(E) = E \otimes K + I \otimes E, \quad \Delta(F) = F \otimes I + K^{-1} \otimes F, \quad \Delta(K) = K \otimes K \qquad (2.1)$$

and similarly for $\tilde{E}, \tilde{F}$ and $\tilde{K}$.

We shall use the irreducible representations which are equivalent to the used in [5, 6]. Let u and v realize the Weyl relations $uv = q^2 vu$ and act in $L_2(\mathbb{R})$ by the explicit formulae

$$u f(x) = \exp\left(-\frac{i\pi x}{\omega}\right) f(x), \quad v f(x) = f(x + 2\omega')$$

Here ω and ω' are half-periods which substitute periods 1 and τ

$$\tau = \omega'/\omega, \quad \omega\omega' = -1/4, \quad \omega'' = \omega + \omega'$$

We shall consider the case $\tau > 0$, so that ω and ω' are pure imaginary with positive imaginary part. Operators u and v are unbounded and can be defined on the dense domain $\mathcal{D}$ consisting of the entire functions $f(x)$, rapidly vanishing at infinity along the lines $\text{Im}\, x = const$. For instance we can take $f(x)$ in the form

$$f(x) = e^{-\alpha x^2} e^{\beta x} P(x)$$

for positive α, arbitrary complex β and polynomial $P(x)$. The operators u and v are nonnegative and essentially selfadjoint.

The representation π_s is given by formulae

$$E = \frac{i}{q - q^{-1}} e(s), \quad F = \frac{i}{q - q^{-1}} f(s)$$

where

$$e(s) = u^{-1}\left(qv + Z\right) = \left(q^{-1}v + Z\right) u^{-1}, \quad f(s) = u\left(1 + qZ^{-1}v^{-1}\right) = \left(1 + q^{-1}Z^{-1}v^{-1}\right) u$$

and

$$Z = \exp\left(-i\pi s/\omega\right), \quad \forall s \in \mathbb{R}, \quad K = v.$$

The representation for the second triple is given by similar formulae in terms of $\tilde{u}, \tilde{v}$ and $\tilde{Z}$

$$\tilde{u} = u^{1/\tau}, \quad \tilde{v} = v^{1/\tau}, \quad \tilde{Z} = Z^{1/\tau}$$

so that

$$\tilde{u} f(x) = \exp\left(-i\pi x/\omega'\right) f(x), \quad \tilde{v} f(x) = f(x + 2\omega)$$

and $\tilde{Z} = \exp\left(-i\pi s/\omega'\right)$. Thus $\tilde{u}, \tilde{v}$ and $\tilde{Z}$ are obtained from u, v and Z by interchange $\omega \rightleftarrows \omega'$.

Operators $\tilde{u}, \tilde{v}$ have the same domain and are nonnegative and essentially selfadjoint.

Let us note that there is the second regime for τ which could be called real form for $\text{SL}_q(2)$. It is the case $|\tau| = 1$ or $\omega' = -\bar{\omega}$. In this case the involution interchanges pairs u, v and $\tilde{u}, \tilde{v}$ by $u^* = \tilde{u}, v^* = \tilde{v}$. This regime has many interesting features. In particular it corresponds to the value of central charge of Liouville model in the interval between 1 and 25. However in this paper we shall consider only regime $\tau > 0$.

3. Modular quantum dilogarithm

We shall use this term for the function

$$\gamma(x) = \exp\left\{ -\frac{1}{4} \int_{-\infty}^{+\infty} \frac{\mathrm{d}t}{t} \frac{e^{itx}}{\sin(\omega t)\sin(\omega' t)} \right\}$$

where the contour goes above the singularity at $t = 0$. This function has long history, different names, normalizations and applications. The normalization used in this paper is used in [10, 11].

The term modular quantum dilogarithm is used for the function

$$\Phi(u) = \gamma(x), \quad u = \exp\left(-i\pi x/\omega\right)$$

The adjective "modular" is due to the symmetry of $\Phi(u)$ after exchange $\omega \rightleftarrows \omega'$ and term "dilogarithm" is due to the asymptotic for $\tau \to 0$

$$\Phi(u) \to \exp\frac{1}{2\pi i \tau}\mathrm{Li}_2(-u)$$

where

$$\mathrm{Li}_2(u) = \sum_{n=1}^{\infty} u^n/n^2$$

containing the Euler dilogarithm. Finally "quantum" is a conventional term for q-deformation.

The main property is the functional equation

$$\frac{\Phi(qu)}{\Phi(q^{-1}u)} = \frac{1}{1+u} \tag{3.1}$$

and similar one for the shift with $\tilde{q}$. In terms of function $\gamma(x)$ these equations look as follows

$$\gamma(x+\omega')/\gamma(x-\omega') = 1 + e^{-\frac{i\pi}{\omega}x}, \quad \gamma(x+\omega)/\gamma(x-\omega) = 1 + e^{-\frac{i\pi}{\omega'}x}$$

We shall need also the reflection formula

$$\gamma(x)\gamma(-x) = e^{i\beta}e^{i\pi x^2}, \quad \beta = \frac{\pi}{12}\left(\frac{\omega'}{\omega} + \frac{\omega}{\omega'}\right) \tag{3.2}$$

and formula for the complex conjugation

$$\overline{\gamma(x)} = 1/\gamma(\overline{x}) \tag{3.3}$$

Asymptotic behaviour is $\gamma(x) \to 1$ for $\mathrm{Re}(x) \to +\infty$ and reflection formula (3.2) can be used to get asymptotic for $\mathrm{Re}(x) \to -\infty$.

The function $\gamma(x)$ has a pole at the point $x = -\omega''$ and zero at the point $x = +\omega''$. The first terms of the series expansions are [10, 11, 14]

$$\gamma(-\omega''+z) = -\frac{1}{2\pi i c}\frac{1}{z} + \ldots, \quad \gamma(\omega''+z) = \frac{2\pi i}{c}z + \ldots \tag{3.4}$$

Comparison of (3.2) and (3.4) gives

$$c^2 = e^{-i\beta}e^{-i\pi\omega''^2} = ie^{2i\beta}, \quad c = e^{i\beta + \frac{i\pi}{4}}$$

There are main integral identities [10, 11]

$$\int_{\mathbb{R}} \mathrm{d}t\, e^{-2\pi itz} \frac{1}{\gamma(\omega''-i0-t)} = c\gamma(z-\omega'') \tag{3.5a}$$

$$\int_{\mathbb{R}} \mathrm{d}t\, e^{-2\pi itz} \frac{\gamma(x-t)}{\gamma(\omega''-i0-t)} = c\frac{\gamma(x)\gamma(z-\omega'')}{\gamma(x+z)} \tag{3.5b}$$

$$\int_{\mathbb{R}} \mathrm{d}t\, e^{-2\pi itz} \frac{\gamma(x-t)\gamma(y-t)}{\gamma(\omega''-i0-t)\gamma(x+y+z+\omega''-i0-t)} = c\frac{\gamma(x)\gamma(y)\gamma(z-\omega'')}{\gamma(x+z)\gamma(y+z)} \tag{3.5c}$$

3.1. Intertwining operator

Let us use the functional equation to show equivalence of the representations π_s and π_{-s}. We should find an operator $A(s)$ such that

$$e(s)A(s) = A(s)e(-s), \quad f(s)A(s) = A(s)f(-s), \quad vA(s) = A(s)v \tag{3.6}$$

The last equation indicates that $A(s)$ is an operator of convolution with kernel of the form

$$A(x, y, s) = A(x - y, s) = \int_{-\infty}^{+\infty} dt\, e^{2\pi i t(x-y)} \hat{A}(t)$$

It is advisable to make Fourier transform. Let F be operator

$$[Ff](x) = \int_{-\infty}^{+\infty} dy\, e^{-2\pi i x y} f(y) = \hat{f}(x)$$

We have $uF = Fv$ and $vF = Fu^{-1}$ and after conjugation by F we get from (3.1)

$$v^{-1}\left(qu^{-1} + Z\right)\hat{A}(u) = \hat{A}(u)v^{-1}\left(qu^{-1} + Z^{-1}\right)$$
$$v\left(1 + qZ^{-1}u\right)\hat{A}(u) = \hat{A}(u)v\left(1 + qZu\right)$$

We move v^{-1} and v to the right and cancel. After that we obtain the equations

$$\left(q^{-1}u^{-1} + Z\right)\hat{A}(q^2 u) = \hat{A}(u)\left(q^{-1}u^{-1} + Z^{-1}\right)$$
$$\left(1 + q^{-1}Z^{-1}u\right)\hat{A}(q^{-2}u) = \hat{A}(u)\left(1 + q^{-1}Zu\right)$$

which are equivalent and differs by the change $u \to q^2 u$ so that really we have only one equation

$$\left(1 + qZu\right)\hat{A}(q^2 u) = \hat{A}(u)\left(1 + qZ^{-1}u\right)$$

It is evident that solution of this equation is

$$\hat{A}(u) = \frac{\Phi(Zu)}{\Phi(Z^{-1}u)}$$

and finally

$$A(s) = F^{-1}\frac{\Phi(Zu)}{\Phi(Z^{-1}u)}F \tag{3.7}$$

The unitarity of the operator $A(s)$ is evident due to (3.3).

4. The main equation for the 3j-symbol

In this section we shall solve the system of equations for the function $S(x_1, x_2, x_3 | s_1, s_2, s_3)$

$$e_{12}(s_1, s_2)S = e'_3(s_3)S \tag{4.1a}$$
$$f_{12}(s_1, s_2)S = f'_3(s_3)S \tag{4.1b}$$
$$K_{12}S = K'_3 S \tag{4.1c}$$

where the operators e_{12}, f_{12} and K_{12} act on variables x_1 and x_2 by

$$\Delta(E) = E_{12} = E_1 K_2 + E_2, \quad \Delta(F) = F_{12} = F_1 + K_1^{-1}F_2, \quad \Delta(K) = K_{12} = K_1 K_2$$

and e_3', f_3', K_3' acts on variable x_3. These operators can be obtained from the transposition $u' = u$, $v' = v^{-1}$ and are given by $K_3' = v_3^{-1}$,

$$e_3' = \left(Z_3 + qv_3^{-1}\right) u_3^{-1} = u_3^{-1}\left(Z_3 + q^{-1}v_3^{-1}\right), \quad f_3' = \left(1 + qZ_3^{-1}v_3\right)u_3 = u_3\left(1 + q^{-1}Z_3^{-1}v_3\right)$$

It is already clear that function S realizes the decomposition of the representation $\pi_{s_1} \otimes \pi_{s_2}$ into irreducibles. More will be said in the end of the section.

Following the previous section we shall often use variables u_i instead of x_i. Equation (4.1c),

$$v_1 v_2 v_3 S = S \tag{4.2}$$

allows to exclude v_2 from the equation (4.1a) and v_2^{-1} from the equation (4.1b) to get the system of equations for $v_1^{-1}S/S$ and $v_3 S/S$ as follows:

$$\left(u_2 + qZ_1^{-1}u_1\right)v_1^{-1}S + \left(qZ_2^{-1}u_2 - q^{-1}Z_3^{-1}u_3\right)v_3 S = \left(u_3 - u_1\right)S$$
$$\left(Z_1 u_1^{-1} + qu_2^{-1}\right)v_1^{-1}S + \left(Z_2 u_2^{-1} - q^2 Z_3 u_3^{-1}\right)v_3 S = qu_1^{-1}u_3^{-1}\left(u_1 - u_3\right)S$$

or after diagonalization

$$\frac{v_1^{-1}S}{S} = qZ_1\frac{u_1 - u_3}{Z_1 u_3 + qZ_2 Z_3 u_1}\frac{u_2 - Z_2 Z_3 u_1}{qu_1 + Z_1 u_2}, \quad \frac{v_3 S}{S} = qZ_2 Z_3\frac{u_1 - u_3}{Z_1 u_3 + qZ_2 Z_3 u_1}\frac{qu_2 + Z_1 u_3}{Z_2 u_3 - qZ_3 u_2}$$

More explicitly these equations read

$$S(q^2 u_1, u_2, u_3) = \frac{1}{Z_2 Z_3}\frac{1 - \frac{u_3}{u_1}}{1 + \frac{Z_1}{qZ_2 Z_3}\frac{u_3}{u_1}}\frac{1 - Z_2 Z_3\frac{u_1}{u_2}}{1 + \frac{q}{Z_1}\frac{u_1}{u_2}}S(u_1, u_2, u_3)$$

$$S(u_1, u_2, q^{-2}u_3) = \frac{Z_1}{Z_2}\frac{1 - \frac{u_3}{u_1}}{1 + \frac{Z_1}{qZ_2 Z_3}\frac{u_3}{u_1}}\frac{1 + \frac{q}{Z_1}\frac{u_2}{u_3}}{1 - \frac{qZ_3}{Z_2}\frac{u_2}{u_3}}S(u_1, u_2, u_3)$$

and can be supplemented by

$$S\left(u_1, q^{-2}u_2, u_3\right) = \frac{1}{Z_1 Z_3}\frac{1 - Z_2 Z_3\frac{u_1}{u_2}}{1 + \frac{q}{Z_1}\frac{u_1}{u_2}}\frac{1 - \frac{Z_3}{qZ_2}\frac{u_2}{u_3}}{1 + \frac{1}{qZ_1}\frac{u_2}{u_3}}S(u_1, u_2, u_3)$$

which is a corollary of (4.2). Let us look for solution in the form

$$S = \frac{\Phi\left(\alpha_1\frac{u_2}{u_3}\right)\Phi\left(\alpha_2\frac{u_3}{u_1}\right)\Phi\left(\alpha_3\frac{u_1}{u_2}\right)}{\Phi\left(\beta_1\frac{u_2}{u_3}\right)\Phi\left(\beta_2\frac{u_3}{u_1}\right)\Phi\left(\beta_3\frac{u_1}{u_2}\right)}S_1 \tag{4.3}$$

and due to (3.1) the choice

$$\alpha_1 = -qZ_3 Z_2^{-1}, \quad \alpha_2 = -q, \quad \alpha_3 = Z_1^{-1}, \quad \beta_1 = Z_1^{-1}, \quad \beta_2 = Z_1 Z_2^{-1}Z_3^{-1}, \quad \beta_3 = -Z_2 Z_3 q^{-1}$$

reduces the equations to

$$S_1(q^2 u_1, u_2, u_3) = Z_2^{-1}Z_3^{-1}S_1(u_1, u_2, u_3)$$
$$S_1(u_1, u_2, q^{-2}u_3) = Z_1 Z_2^{-1}S_1(u_1, u_2, u_3)$$
$$S_1\left(u_1, q^{-2}u_2, u_3\right) = Z_1^{-1}Z_3^{-1}S_1(u_1, u_2, u_3)$$

The dual equations, corresponding to the interchange $\omega \rightleftarrows \omega'$, have the same solution (4.3) and after that the Ansatz

$$S_1(x_1, x_2, x_3) = \exp -2\pi i \left(s_1 x_{23} + s_2 x_{31} + s_3 x_{21}\right) S_0(x_1, x_2, x_3)$$

where $x_{ik} = x_i - x_k$, reduce the freedom to double periodic function S_0, which has to be constant.

Thus the solution of the equations (4.1) is given by

$$S(x_1, x_2, x_3) = S_0 \exp -2\pi i \left(s_1 x_{23} + s_2 x_{31} + s_3 x_{21}\right) \times$$

$$\times \frac{\gamma(x_{12} - s_1)}{\gamma(x_{12} + s_2 + s_3 + \omega'')} \frac{\gamma(x_{23} + s_3 - s_2 - \omega'')}{\gamma(x_{23} - s_1)} \frac{\gamma(x_{31} - \omega'')}{\gamma(x_{31} + s_1 - s_2 - s_3)} \tag{4.4}$$

The appearance of ω'' is due to sign factors in (4.3),

$$-qu = \exp \left(-\frac{i\pi(x - \omega'')}{\omega}\right)$$

and the singularities here have to be understood as $\omega'' \to \omega'' - i0$, which will be explained in the course of the proof of completeness. The expression for S, equivalent to (4.4), was given in [5] without derivation.

Now we can interpret the result in more details. The solution exists for any triplet of real s_1, s_2, s_3 and is unique up to normalization constant. This means that the representation with "spin" s_3 enter the tensor product $\pi_{s_1} \otimes \pi_{s_2}$ once for any s_3. This can be formalized by the relation

$$\pi_{s_1} \otimes \pi_{s_2} = \int_0^{+\infty} \mathrm{d}s_3 \rho(s_3) \pi_{s_3} \tag{4.5}$$

We can consider $S(x_1 x_2 x_3)$ as the kernel of the integral operator S defined by

$$\pi_{s_3} \xrightarrow{\mathrm{S}} \pi_{s_1} \otimes \pi_{s_2}, \quad f(x_3) \mapsto [\mathrm{S}f](x_1, x_2) = \int_{\mathbb{R}} \mathrm{d}x_3 S(x_1 x_2 x_3) f(x_3)$$

and equations (4.1) in operator form are

$$e_{12}\mathrm{S} = \mathrm{S}e_3, \quad f_{12}\mathrm{S} = \mathrm{S}f_3, \quad K_{12}\mathrm{S} = \mathrm{S}K_3 \tag{4.6}$$

The complex conjugate function $\overline{S(x_1 x_2 x_3)}$ has interpretation as the kernel of the projection operator

$$\pi_{s_1} \otimes \pi_{s_2} \xrightarrow{\mathrm{P}} \pi_{s_3}, \quad f(x_1, x_2) \xrightarrow{\mathrm{P}} [\mathrm{P}f](x) = \int_{\mathbb{R}^2} \mathrm{d}x_1 \mathrm{d}x_2 \overline{S(x_1 x_2 x)} f(x_1, x_2)$$

The measure $\rho(s)$ in (4.5) should be found from normalization condition for the kernel $S(x_1, x_2, x_3)$, which will be obtained in the last section.

5. Undressing of the Casimir

To get the normalization for S it is useful to interpret it as an eigenfunction of the Casimir operator

$$C_{12} = f_{12}e_{12} - qK_{12} - q^{-1}K_{12}^{-1}$$

It is clear from (4.6), that as a function of x_1 and x_2, S satisfies the equation

$$C_{12}S = \left(Z_3 + Z_3^{-1}\right)S$$

where s_3 and x_3 play the role of parameters labeling eigenvalue and multiplicity.

Explicitly C_{12} can be written as

$$C_{12} = Z_2\frac{u_1}{u_2} + Z_2^{-1}\frac{u_2}{u_1} + \left(Z_1 + \frac{1}{q}\frac{u_2}{u_1}\right)\left(1 + \frac{q}{Z_1}\frac{u_1}{u_2}\right)v_2 + \left(Z_2 + \frac{Z_1}{qZ_2}\frac{u_2}{u_1}\right)\left(1 + \frac{q}{Z_1}\frac{u_1}{u_2}\right)v_1^{-1}$$
$$+ \frac{Z_1}{q^2}\frac{u_2}{u_1}\left(1 + \frac{q}{Z_1}\frac{u_1}{u_2}\right)\left(1 + \frac{q^3}{Z_1}\frac{u_1}{u_2}\right)v_1^{-1}v_2$$

or

$$C_{12} = Z_2\frac{u_1}{u_2} + Z_2^{-1}\frac{u_2}{u_1} + \left(Z_1 + \frac{1}{q}\frac{u_2}{u_1}\right)V_2 + \left(Z_2 + \frac{Z_1}{qZ_2}\frac{u_2}{u_1}\right)V_1^{-1} + \frac{Z_1}{q^2}\frac{u_2}{u_1}V_1^{-1}V_2$$

offer the substitution

$$V_1 = v_1\frac{1}{1 + \frac{q}{Z_1}\frac{u_1}{u_2}}, \quad V_2 = \left(1 + \frac{q}{Z_1}\frac{u_1}{u_2}\right)v_2$$

We shall introduce a series of adjoint transformations of C_{12} to reduce it to more simple form. These transformations we shall call "undressing".

The first step is to use operator R_1 to cancel factors in front of v_1 and v_2

$$R_1^{-1}V_1R_1 = v_1, \quad R_1^{-1}V_2R_1 = v_2$$

The solution is a multiplication operator by function

$$R_1 = \Phi\left(\frac{1}{Z_1}\frac{u_1}{u_2}\right)$$

The operator K_{12} is invariant under transformation by R_1, i.e $R_1^{-1}K_{12}R_1 = K_{12}$ and C_{12} transforms into

$$C'_{12} = R_1^{-1}C_{12}R_1 = Z_2\frac{u_1}{u_2} + Z_1v_2 + Z_2v_1^{-1} + Z_2^{-1}u_1^{-1}\left(1 + q^{-1}Z_1v_1^{-1}\right)U_2$$

where

$$U_2 = u_2\left(1 + q^{-1}Z_2v_2\right)$$

Now we find R_2 transforming U_2 to u_2, i.e $R_2^{-1}U_2R_2 = u_2$. It is clear that R_2 is similar to R_1 after interchange u_2 and v_2, which is given by the Fourier transformation which respect variable x_2, so that $R_2 = F_2^{-1}\tilde{R}_2F_2$ where $\tilde{R}_2$ is a multiplication by $\Phi(Z_2u_2)$. The operator K_{12} is invariant under transformation R_2 and operator C'_{12} acquires the form

$$C''_{12} = R_2^{-1}C'_{12}R_2 = Z_2\frac{u_1}{u_2} + \frac{1}{Z_2}\frac{u_2}{u_1} + Z_1V'_2 + \frac{Z_1}{qZ_2}\frac{u_2}{u_1}V_1'^{-1}$$

where

$$V'_1 = v_1\frac{1}{1 + \frac{qZ_2^2}{Z_1}\frac{u_1}{u_2}}, \quad V'_2 = \left(1 + \frac{qZ_2^2}{Z_1}\frac{u_1}{u_2}\right)v_2$$

Now we transform V_1' and V_2' to v_1 and v_2 by multiplication operator

$$R_3 = \Phi\left(\frac{Z_2^2}{Z_1}\frac{u_1}{u_2}\right)$$

which leaves K_{12} invariant and transform C_{12}'' into

$$\tilde{C}_{12} = R_3^{-1}C_{12}''R_3 = Z_2\frac{u_1}{u_2} + \frac{1}{Z_2}\frac{u_2}{u_1} + Z_1 v_2 + \frac{Z_1}{qZ_2}\frac{u_2}{u_1}v_1^{-1}$$

Altogether the operator $A = R_1 F_2^{-1}\tilde{R}_2 F_2 R_3$ gives $\tilde{C}_{12}$ from C_{12} and leaves K_{12} invariant,

$$\tilde{C}_{12} = A^{-1}C_{12}A, \quad A^{-1}K_{12}A = K_{12}$$

In more explicit form A^{-1} acts on the function of two variables $f(x_1, x_2)$ as follows:

$$\left[A^{-1}f\right](x_1, x_2) = \frac{1}{c\gamma(x_{12} - s_1 + 2s_2)}\int_{\mathbb{R}}dt e^{2\pi it(s_2 - \omega'')}\frac{\gamma(-t - \omega'' + i0)}{\gamma(x_1 - t - s_1)}f(x_1, t) \tag{5.1}$$

Operator $\tilde{C}_{12}$ is much more simple then C_{12} and the problem of simultaneous diagonalization of $\tilde{C}_{12}$ and K_{12} allows separation of variables. Consider equations for the corresponding eigenfunctions

$$K_{12}\Psi_p(x_1, x_2) = v_1 v_2 \Psi_p(x_1, x_2) = e^{\frac{i\pi p}{\omega}}\Psi_p(x_1, x_2)$$

$$\left(Z_2\frac{u_1}{u_2} + \frac{1}{Z_2}\frac{u_2}{u_1} + Z_1 v_2 + \frac{Z_1}{qZ_2}\frac{u_2}{u_1}v_1^{-1}\right)\Psi_p(x_1, x_2) = \left(Z_3 + Z_3^{-1}\right)\Psi_p(x_1, x_2)$$

where we parameterize the eigenvalues by p and s_3. The first equation allows to exclude v_1^{-1} from the second to get

$$\left[Z_2\frac{u_1}{u_2} + \frac{1}{Z_2}\frac{u_2}{u_1} + Z_1\left(1 + \frac{e^{-\frac{i\pi p}{\omega}}}{qZ_2}\frac{u_2}{u_1}\right)v_2\right]\Psi_p(x_1, x_2) = \left(Z_3 + Z_3^{-1}\right)\Psi_p(x_1, x_2)$$

The general solution of the first equation is given by

$$\Psi_p(x_1, x_2) = e^{-2\pi ipx_1}\Psi_p(x_{21})$$

where $x_{21} = x_2 - x_1$ and after substitution

$$\Psi_p(x_{21}) = \frac{e^{-2\pi is_1 x_{21}}}{\gamma(x_{21} + p - s_2)}\Psi(x_{21})$$

which eliminates the factor in front of v_2, we get

$$\left(Z_2\frac{u_1}{u_2} + \frac{1}{Z_2}\frac{u_2}{u_1} + v_2\right)\Psi(x_{21}) = \left(Z_3 + Z_3^{-1}\right)\Psi(x_{21})$$

Introducing the new operators

$$u = \frac{1}{Z_2}\frac{u_2}{u_1}, \quad v = v_2$$

we rewrite the remaining equation in the form

$$\left(v + u + u^{-1}\right)\Psi = \left(Z_3 + Z_3^{-1}\right)\Psi$$

The operator in the LHS is well known in CFT. It appears as a trace of monodromy of Lax operator in the Liouville model [1]. In quantum Teichmüller thery it got the name of the length operator for geodesics. R. Kashaev has shown [12, 13] that this operator has continues spectrum in the interval $[2, \infty]$ with eigenvalues parameterized in the form $Z + Z^{-1}$ with

$$Z = \exp\left(-\frac{i\pi s}{\omega}\right), \quad s \geq 0$$

and eigenfunctions are given by

$$\phi(x, s) = e^{-i\pi(x-\omega'')^2}\gamma(x + s - \omega'' + i0)\gamma(x - s - \omega'' + i0) \tag{5.2}$$

The latter are even functions of s so that these can be considered for any $s \in \mathbb{R}$. Kashaev proved [12] the orthogonality and completeness for $\phi(x, s)$ in the form

$$\int_{\mathbb{R}} dx \overline{\phi(x, s)}\phi(x, s') = \rho^{-1}(s)\left[\delta(s - s') + \delta(s + s')\right] \tag{5.3}$$

$$\int_0^{+\infty} ds \rho(s)\overline{\phi(x, s)}\phi(y, s) = \delta(x - y) \tag{5.4}$$

with $\rho(s)$ given by

$$\rho(s) = M(s)M(-s) = -4\sin\frac{\pi s}{\omega}\sin\frac{\pi s}{\omega'}$$

where $M(s)$, which can be considered as analogue of the Jost function from scattering theory or Harish-Chandra-Gindikin-Karpelevich function from the theory of representations of $SL(2,\mathbb{R})$, can be taken as

$$M(s) = ce^{-2i\pi s^2 - 2i\pi s\omega''}\gamma(2s + \omega'')$$

One can say that the operator

$$[Uf](s) = \int_{-\infty}^{+\infty} dx M(s)\phi(x, s)f(x) = F(s)$$

acts from $L_2(\mathbb{R})$ into the subspace of $L_2(\mathbb{R})$, defined by condition $F(s) = S(s)F(-s)$ where the reflection coefficient $S(s)$ is given by $S(s) = M(s)/M(-s)$. Incidentally, the same reflection coefficient appears in the discussion of the zero modes in the Liouville model in [15].

It is evident that integral operator P with the kernel

$$P(s, s') = \frac{1}{2}\left[\delta(s - s') + S(s)\delta(s + s')\right]$$

defines a projection and U maps $L_2(\mathbb{R})$ into subspace $PL_2(\mathbb{R})$. However the natural completeness

$$\int_{-\infty}^{+\infty} ds ds' P(s, s')\overline{U(x, s)}U(y, s') = \delta(x - y)$$

reduces to (5.4) due to the fact, that $\phi(x, s)$ is an even function of s and property $\overline{M(s)} = M(-s)$. The inversion $s \to -s$ is evidently connected to the Weyl reflection. The proof of Kashaev results is given in Appendix.

After all we obtain the following expression for the eigenfunction of the undressed Casimir operator

$$\Psi_p(x_1, x_2) = e^{-2\pi i p x_1}\frac{e^{-2\pi i s_1 x_{21}}}{\gamma(x_{21} + p - s_2)}\phi(x_{21} - s_2, s_3) \tag{5.5}$$

and now we can use them to formulate the orthogonality and completeness for the kernel $S(x_1, x_2, x_3)$.

6. Undressing of the eigenfunctions

First of all we have to find out the connection between $\Psi_p(x_1, x_2)$ and undressed eigenfunction $\mathrm{A}^{-1}S(x_1, x_2, x_3)$. The explicit expression for the undressed eigenfunction reads

$$\mathrm{A}^{-1}S(x_1, x_2, x_3) = S_0 c^{-1} e^{-2\pi i(s_1 x_{23} + s_2 x_{31} + s_3 x_{21})} \frac{1}{\gamma(x_{12} - s_1 + 2s_2)} \frac{\gamma(x_{31} - \omega'')}{\gamma(x_{31} + s_1 - s_2 - s_3)} \times$$

$$\times \int_{\mathbb{R}} \mathrm{d}t \, e^{2\pi i t(s_2 - s_1 - s_3 - \omega'')} \frac{\gamma(-t - \omega'' + i0)\gamma(t - x_3 + s_3 - s_2 - \omega'' + i0)}{\gamma(x_1 - t + s_2 + s_3 + \omega'' + i0)\gamma(t - x_3 - s_1)}$$

where undressing operator A^{-1} is given by (5.1). The t-integral is reduced to (3.5c) and can be calculated in explicit form so that we obtain

$$\mathrm{A}^{-1}S(x_1, x_2, x_3) = S_0 e^{2\pi i \omega''2} \frac{e^{-i\pi(s_2 - s_1 + s_3)^2}}{\gamma(s_2 - s_1 - s_3)} \frac{e^{2\pi i(s_1 + s_3)x_{12}}}{\gamma(x_{12} + s_2 + s_3 + \omega'')} \frac{e^{2\pi i(\omega'' - s_3)x_{13}}\gamma(x_{23} + s_3 - s_2 - \omega'')}{\gamma(x_{13} + \omega'')}$$

Next step is the calculation of Fourier transformation with respect variable x_3 using (3.5b):

$$S_p(x_1, x_2) = \int_{\mathbb{R}} \mathrm{d}x_3 \, e^{-2\pi i p x_3} \mathrm{A}^{-1}S(x_1, x_2, x_3)$$

$$= S_0 e^{2\pi i \omega''2} \frac{e^{-i\pi(s_2 - s_1 + s_3)^2}}{\gamma(s_2 - s_1 - s_3)} \frac{e^{2\pi i(s_1 + s_3)x_{12}}}{\gamma(x_{12} + s_2 + s_3 + \omega'')} \times$$

$$\times e^{-2\pi i p x_1} \int_{\mathbb{R}} \mathrm{d}t \, e^{-2\pi i t(p - s_3 + \omega'')} \frac{\gamma(x_{21} + s_3 - s_2 - \omega'' + i0 - t)}{\gamma(\omega'' - i0 - t)}$$

$$= S_0 e^{2\pi i \omega''2} c \frac{e^{-i\pi(s_2 - s_1 + s_3)^2}\gamma(p - s_3)}{\gamma(s_2 - s_1 - s_3)} \frac{e^{-2\pi i p x_1}}{\gamma(x_{21} - s_2 + p)} \frac{e^{2\pi i(s_1 + s_3)x_{12}}\gamma(x_{21} + s_3 - s_2 - \omega'')}{\gamma(x_{12} + s_2 + s_3 + \omega'')}$$

This expression coincides with (5.5),

$$S_p(x_1, x_2) = \mathrm{Z}(s_1, s_2 | s_3, p)\Psi_p(x_1, x_2)$$

up to overall normalization $\mathrm{Z}(s_1, s_2 | s_3, p)$,

$$\mathrm{Z}(s_1, s_2 | s_3, p) = S_0 i e^{2\pi i \omega''2} \frac{e^{-i\pi s_3^2 - 2\pi i s_3(s_2 + \omega'') - i\pi(s_2 - s_1 + s_3)^2}\gamma(p - s_3)}{\gamma(s_2 - s_1 - s_3)}$$

The special choice of the initial normalization constant S_0 given by

$$S_0 = -i e^{-2\pi i \omega''2} e^{i\pi s_3^2 + 2\pi i s_3(s_2 + \omega'') + i\pi(s_2 - s_1 + s_3)^2}\gamma(s_2 - s_1 - s_3) = e^{2\pi i s_3 \omega''} e^{i\phi} \qquad (6.1)$$

where ϕ is real phase, leads to simplification

$$\mathrm{Z}(s_1, s_2 | s_3, p) = \gamma(p - s_3)$$

so that we obtain properly normalized eigenfunctions

$$S_p(x_1, x_2) = e^{2\pi i s_3 p}\gamma(p - s_3) \frac{e^{-2\pi i p x_1}}{\gamma(x_{21} - s_2 + p)} e^{-2\pi i s_1 x_{21}}\phi_{s_3}(x_{21} - s_2) \qquad (6.2)$$

Now we shall prove the orthogonality and completeness of the functions $S(x_1, x_2, x_3)$ in the momentum representation

$$S(x_1, x_2, p) = \int_{\mathbb{R}} dx_3 e^{-2\pi i p x_3} S(x_1, x_2, x_3)$$

It will be sufficient to prove the orthogonality and completeness for the undressed eigenfunctions $S_p(x_1, x_2)$ because the dressed eigenfunction $S(x_1, x_2, p) = \mathrm{A}S_p(x_1, x_2)$ is obtained from the $S_p(x_1, x_2)$ after action of the dressing unitary operator A. Due to unitarity this dressing operator effectively cancels out from considered relations.

Let us begin from orthogonality. There the dressing operator A cancels out on the first step

$$\int_{\mathbb{R}^2} dx_1 dx_2 \overline{S(x_1 x_2 q)} S(x_1 x_2 p) = \int_{\mathbb{R}^2} dx_1 dx_2 \overline{S_q(x_1 x_2)} S_p(x_1 x_2)$$

$$= \frac{\gamma(p-s)}{\gamma(q-s')} \int_{\mathbb{R}^2} dx_1 dx_2 e^{-2\pi i(p-q)x_1} \frac{\gamma(x_{21} - s_2 + q)}{\gamma(x_{21} - s_2 + p)} \overline{\phi(x_{21} - s_2, s')} \phi(x_{21} - s_2, s)$$

$$= \delta(p-q) \frac{\gamma(p-s)}{\gamma(p-s')} \int_{\mathbb{R}} dx \overline{\phi(x, s')} \phi(x, s)$$

$$= \rho^{-1}(s) \delta(p-q) \left[\delta(s-s') + \delta(s+s') \frac{\gamma(p-s)}{\gamma(p+s)} \right]$$

Note that the appearance of the second term containing $\delta(s+s')$ and kernel of the intertwining operator (3.7) is the direct consequence of the equivalence of representations π_s and π_{-s}.

The completeness for the undressed eigenfunction can be proven as follows

$$\int_0^{+\infty} ds \rho(s) \int_{\mathbb{R}} dp \overline{S_p(x_1' x_2')} S_p(x_1 x_2) = \int_{\mathbb{R}} dp e^{-2\pi i p(x_1 - x_1')} \frac{\gamma(x_{21}' - s_2 + p)}{\gamma(x_{21} - s_2 + p)} \times$$

$$\times \int_0^{+\infty} ds \rho(s) \overline{\phi(x_{21}' - s_2, s)} \phi(x_{21} - s_2, s)$$

$$= \int_{\mathbb{R}} dp e^{-2\pi i p(x_1 - x_1')} \frac{\gamma(x_{21}' - s_2 + p)}{\gamma(x_{21} - s_2 + p)} \delta(x_{21}' - x_{21})$$

$$= \delta(x_1' - x_1) \delta(x_2' - x_2)$$

Due to unitarity of the dressing operator the same relation holds for the dressed eigenfunctions,

$$\int_0^{+\infty} ds \rho(s) \int_{\mathbb{R}} dp \overline{S(x_1' x_2' p)} S(x_1 x_2 p) = \delta(x_1' - x_1) \delta(x_2' - x_2)$$

Appendix A. Orthogonality and completeness of Kashaev eigenfunctions
We shall need the generalization of identity (3.5b) in the form

$$\int_{\mathbb{R}} dt e^{-2\pi i t s} \frac{\gamma(t+a)}{\gamma(t+b)} = c e^{2\pi i s(b - \omega'')} \frac{\gamma(a - b + \omega'') \gamma(-s - \omega'')}{\gamma(a - b - s + \omega'')} \tag{A.1}$$

$$= c^{-1} e^{2\pi i s(a + \omega'')} \frac{\gamma(b - a + s - \omega'')}{\gamma(b - a - \omega'') \gamma(s + \omega'')} \tag{A.2}$$

where the integral converges under conditions

$$\mathrm{Im}(s) < 0, \quad \mathrm{Im}(a - b - s) < 0 \tag{A.3}$$

The inverse formula is

$$\frac{\gamma(t+a)}{\gamma(t+b)} = \frac{1}{c\gamma(b-a-\omega'')} \int_{\mathbb{R}} \mathrm{d}s e^{2\pi is(t+a+\omega'')} \frac{\gamma(s+b-a-\omega'')}{\gamma(s+\omega'')} \tag{A.4}$$

and the contour goes below the singularity at $s = 0$.

Appendix A.1. Orthogonality
We take eigenfunctions in the form

$$\phi(x,\lambda) = e^{-i\pi(x-\omega'')^2} \gamma(x+\lambda-\omega'')\gamma(x-\lambda-\omega'')$$

so that

$$\overline{\phi(x,\lambda)}\phi(x,\mu) = e^{4\pi ix\omega''} \frac{\gamma(x+\mu-\omega'')\gamma(x-\mu-\omega'')}{\gamma(x+\lambda+\omega'')\gamma(x-\lambda+\omega'')}$$

and the singularities here have to be understood as $\omega'' \to \omega'' - i0$ We have to calculate

$$I(\lambda,\mu) = \int_{\mathbb{R}} \mathrm{d}x \overline{\phi(x,\lambda)}\phi(x,\mu)$$

This function is even in λ and μ and it is sufficient to calculate it in one quadrant, say $\lambda > 0$ and $\mu < 0$. First we transform the ratio of two γ-functions using (A.4),

$$\frac{\gamma(x+\mu-\omega'')}{\gamma(x+\lambda+\omega'')} = \frac{1}{c\gamma(\lambda-\mu+\omega'')} \int_{\mathbb{R}} \mathrm{d}s e^{2\pi is(x+\mu)} \frac{\gamma(s+\lambda-\mu+\omega'')}{\gamma(s+\omega'')}$$

and calculate the x-integral using (A.2),

$$\int_{\mathbb{R}} \mathrm{d}x e^{2\pi ix(s+2\omega'')} \frac{\gamma(x-\mu-\omega'')}{\gamma(x-\lambda+\omega'')} = c^{-1} e^{2\pi i\mu(s+2\omega'')} \frac{\gamma(\mu-\lambda-s-\omega'')}{\gamma(\mu-\lambda+\omega'')\gamma(-s-\omega'')}$$

and after these two steps arrive to the following expression for $I(\lambda,\mu)$:

$$I(\lambda,\mu) = \frac{1}{c^2\gamma(\lambda-\mu+\omega'')\gamma(\mu-\lambda+\omega'')} \int_{\mathbb{R}} \mathrm{d}s e^{4\pi is\mu} \frac{\gamma(s+\lambda-\mu+\omega'')\gamma(\mu-s-\lambda-\omega'')}{\gamma(s+\omega'')\gamma(-s-\omega'')}$$

The ratio of γ-functions is reduced to the simple exponent

$$\frac{\gamma(s+\lambda-\mu+\omega'')\gamma(\mu-s-\lambda-\omega'')}{\gamma(s+\omega'')\gamma(-s-\omega'')} = e^{i\pi(s+\lambda-\mu+\omega'')^2 - i\pi(s+\omega'')^2}$$

due to reflection relation (3.2) so that we obtain for $\lambda > 0$ and $\mu < 0$

$$I(\lambda,\mu) = \frac{e^{i\pi(\lambda-\mu)^2 + 2i\pi\omega''(\lambda-\mu)}}{c^2\gamma(\lambda-\mu+\omega'')\gamma(\mu-\lambda+\omega'')} \int \mathrm{d}s e^{2i\pi s(\lambda+\mu)}$$

$$= \frac{e^{4i\pi\lambda^2 + 4i\pi\lambda\omega''}}{c^2\gamma(2\lambda+\omega'')\gamma(-2\lambda+\omega'')} \delta(\lambda+\mu)$$

The full answer is restored by the symmetry

$$\int_{\mathbb{R}} \mathrm{d}x \overline{\phi(x,\lambda)}\phi(x,\mu) = \frac{1}{M(\lambda)M(-\lambda)} [\delta(\lambda-\mu) + \delta(\lambda+\mu)]$$

where

$$M(\lambda) = ce^{-2i\pi\lambda^2 - 2i\pi\lambda\omega''} \gamma(2\lambda+\omega'')$$

It is exactly the formula (5.3) with

$$\rho(\lambda) = M(\lambda)M(-\lambda) = e^{-4\pi i\lambda\omega''} \frac{\gamma(2\lambda+\omega'')}{\gamma(2\lambda-\omega'')} = \left(e^{\frac{i\pi\lambda}{\omega}} - e^{-\frac{i\pi\lambda}{\omega}}\right)\left(e^{\frac{i\pi\lambda}{\omega'}} - e^{-\frac{i\pi\lambda}{\omega'}}\right)$$

Appendix A.2. Completeness
Now we have to calculate

$$I(x,y) = \int_0^{+\infty} \mathrm{d}\lambda \rho(\lambda)\overline{\phi(x,\lambda)}\phi(y,\lambda) = \int_{-\infty}^{+\infty} \mathrm{d}\lambda \sigma(\lambda)\overline{\phi(x,\lambda)}\phi(y,\lambda)$$

where we used the symmetry $\lambda \to -\lambda$,

$$\rho(\lambda) = \sigma(\lambda) + \sigma(-\lambda), \quad \sigma(\lambda) = e^{-4\pi i \lambda \omega''} - e^{-4\pi i \lambda(\omega'' - 2\omega')}$$

We introduce regularization and obtain the following expression

$$I(x,y) = e^{i\pi(x^2-y^2)-2i\pi(x-y)\omega''}\int_{\mathbb{R}} \mathrm{d}\lambda \sigma(\lambda)e^{2\pi\lambda\delta}\frac{\gamma(y+\lambda-\omega''+i\epsilon)\gamma(y-\lambda-\omega''+i\epsilon)}{\gamma(x+\lambda+\omega''-i\epsilon)\gamma(x-\lambda+\omega''-i\epsilon)}$$

where $\epsilon > 0, \delta > 0$ and $\delta > 2\epsilon$. First we transform the ratio of γ-functions using (A.4),

$$\frac{\gamma(y+\lambda-\omega''+i\epsilon)}{\gamma(x+\lambda+\omega''-i\epsilon)} = \frac{1}{c\gamma(x-y+\omega''-2i\epsilon)}\int_{\mathbb{R}} \mathrm{d}s e^{2\pi i s(y+\lambda+i\epsilon)}\frac{\gamma(s+x-y+\omega''-2i\epsilon)}{\gamma(s+\omega'')}$$

where the contour goes below the singularity at $s = 0$. Let us consider the λ-integral with the first contribution in $\sigma(\lambda)$ and for convenience make the change of the variable $\lambda \to -\lambda$,

$$I_1(s) = \int \mathrm{d}\lambda e^{-2\pi i \lambda(s-2\omega''+i\delta)}\frac{\gamma(y+\lambda-\omega''+i\epsilon)}{\gamma(x+\lambda+\omega''-i\epsilon)}$$

The second condition in (A.3) is fulfilled due to relation $\delta > 2\epsilon$ and using (A.2) we obtain

$$I_1(s) = c^{-1}e^{2\pi i(y+i\epsilon)(s-2\omega''+i\delta)}\frac{\gamma(x-y+s-\omega''+i(\delta-2\epsilon))}{\gamma(x-y+\omega''-2i\epsilon)\gamma(s-\omega''+i\delta)}$$

The same calculation with the second contribution in $\sigma(\lambda)$ gives

$$I_2(s) = \int \mathrm{d}\lambda e^{-2\pi i\lambda(s-2\omega+2\omega'+i\delta)}\frac{\gamma(y+\lambda-\omega''+i\epsilon)}{\gamma(x+\lambda+\omega''-i\epsilon)}$$

$$= c^{-1}e^{2\pi i(y+i\epsilon)(s-2\omega+2\omega'+i\delta)}\frac{\gamma(x-y+s+\omega''-2\omega+2\omega'+i(\delta-2\epsilon))}{\gamma(x-y+\omega''-2i\epsilon)\gamma(s+\omega''-2\omega+2\omega'+i\delta)}$$

The change of variables $s \to s-2\omega'$ in s-integral containing $I_2(s)$ transforms ratio of s-dependent γ-functions to the form

$$\frac{\gamma(x-y+s+\omega-\omega'-2i\epsilon)}{\gamma(s+\omega-\omega')}\frac{\gamma(x-y+s-\omega+\omega'+i(\delta-2\epsilon))}{\gamma(s-\omega+\omega'+i\delta)}$$

where we restored s-dependent γ-functions from the first stage. In the corresponding s-integral containing $I_1(s)$ there is the following ratio of s-dependent γ-functions

$$\frac{\gamma(x-y+s+\omega''-2i\epsilon)}{\gamma(s+\omega'')}\frac{\gamma(x-y+s-\omega''+i(\delta-2\epsilon))}{\gamma(s-\omega''+i\delta)}$$

The formula

$$\gamma(z+\omega-\omega')\gamma(z-\omega+\omega') = \gamma(z+\omega'')\gamma(z-\omega'')$$

allows to transform one expression to another for $\delta = 0$. It means that in situation when it is possible to skip δ-regularization, we obtain the integral over closed contour. Let us deform the contour in integral with $I_1(s)$: the contour which goes above the singularity at $s = 0$ and a small closed contour around $s = 0$ leading to additional contribution

$$2\pi i \mathrm{Res}_{s=0} = \frac{1}{c} \frac{\gamma(x - y + \omega'' - 2i\epsilon)\gamma(x - y - \omega'' - 2i\epsilon + i\delta)}{\gamma(-\omega'' + i\delta)}$$

In the remaining two integrals it is possible to put $\delta = 0$ and therefore to reduce it to the integral over closed contour without any singularity inside. As a result only the term $2\pi i \mathrm{Res}_{s=0}$ leads to nonzero contribution and restoring all needed factors we obtain

$$\frac{c\gamma(x - y - \omega'' - 2i\epsilon + i\delta)}{\gamma(x - y + \omega'' - 2i\epsilon)\gamma(-\omega'' + i\delta)} \to \frac{1}{2\pi i} \frac{i\delta}{(x - y - 2i\epsilon)(x - y + i\delta - 2i\epsilon)} \to \delta(x - y)$$

so that

$$\int_0^{+\infty} \mathrm{d}\lambda \rho(\lambda) \overline{\phi(x, \lambda)} \phi(y, \lambda) = \delta(x - y)$$

Acknowledgments

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