

3-qubit entanglement: A Jordan algebraic perspective

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Abstract. It is by now well known that three qubits can be totally entangled in two physically distinct ways. Here we review work classifying the physically distinct forms of 3-qubit entanglement using the elegant framework of Jordan algebras, Freudenthal-Kantor triple systems and groups of type E_7 . In particular, it is shown that the four Freudenthal-Kantor ranks correspond precisely to the four 3-qubit entanglement classes: (1) Totally separable $A-B-C$, (2) Biseparable $A-BC$, $B-CA$, $C-AB$, (3) Totally entangled W , (4) Totally entangled GHZ. The rank 4 GHZ class is regarded as maximally entangled in the sense that it has non-vanishing quartic norm, the defining invariant of the Freudenthal-Kantor triple system. While this framework is specific to three qubits, we show here how the essential features may be naturally generalised to an arbitrary number of qubits.

1. Introduction

This contribution relates a talk given at the *QQQ* conference *3Quantum: Algebra Geometry Information* (Tallinn, July 2012) in the sub-theme on Jordan algebras. However, it also incorporated two further sub-themes, quantum information and string theory, and so in some limited sense reflected the interdisciplinary nature of the *QQQ* conference. Given the length constraints, for the sake of clarity we focus here on interplay between Jordan algebras and quantum entanglement. The interested reader may consult [1, 2] and the references therein for the connection to black holes in string theory.

The central idea is that the Freudenthal-Kantor triple system (F-KTS) based on the smallest spin-factor Jordan algebra naturally captures the classification of the physically distinct forms of 3-qubit entanglement as prescribed by the paradigm of *stochastic local operations and classical communication* (SLOCC) [3]. In particular, it is shown that the four *Freudenthal-Kantor ranks* correspond precisely to the four 3-qubit entanglement classes: (1) Totally separable $A-B-C$, (2) Biseparable $A-BC$, $B-CA$, $C-AB$, (3) Totally entangled W , (4) Totally entangled GHZ (Greenberger-Horne-Zeilinger). This agrees perfectly with the SLOCC classification first obtained in [4]. The rank 4 GHZ class is regarded as maximally entangled in the sense that it has non-vanishing quartic norm, the defining invariant of the Freudenthal-Kantor triple system.

Our hope is give an account of this idea which is accessible to both the quantum information and Jordan algebra communities. Accordingly, we begin in section 2 with an elementary introduction to the essential concepts of entanglement and entanglement classification. In particular, a three player non-local game is used to illustrate that three qubits can be totally entangled in two physically distinguishable ways and how the concept of SLOCC is used to classify different forms of entanglement. Then, in section 3 we briskly review the necessary elements of cubic Jordan algebras and the F-KTS before defining the *3-qubit Jordan algebra*



and the corresponding *3-qubit F-KTS*. Having laid the foundations we proceed in section 4 to demonstrate how the 3-qubit F-KTS elegantly characterises the 3-qubit entanglement classes via the F-KTS *ranks*. Finally, in section 5, we illustrate how the key features of the 3-qubit F-KTS may be generalised to an arbitrary number of qubits.

2. How entangled is totally entangled?

The importance of entanglement to quantum computing protocols has precipitated a need to characterise, qualitatively and quantitatively, the “amount” of entanglement contained in given state. Indeed, what is actually meant by “amount” is somewhat ambiguous and there are many possible measures present in the literature. See, for example, [5, 6]. One might ask, for instance, when can two totally entangled, but ostensibly different, states facilitate the same set of non-local quantum protocols. Two states equivalent in this sense could reasonably be considered to have the same degree of entanglement.

Already for pure 3-qubit states this question is non-trivial. Three qubits can be totally entangled in two physically distinct ways [4]. These two entanglement forms are represented by the states

$$|W\rangle = 1/\sqrt{3} (|001\rangle + |010\rangle + |100\rangle), \quad |GHZ\rangle = 1/\sqrt{2} (|000\rangle + |111\rangle) \quad (2.1)$$

To those unfamiliar with entanglement classification, what separates these states is perhaps not entirely obvious; they are both totally entangled and permutation symmetric. Nonetheless, they exhibit distinct non-local/information-theoretic properties. One such difference is nicely drawn out by the three-player non-local game introduced in [7]. This is essentially a rephrasing of Mermin’s elegant presentation [8] of a “Bell-type theorem without inequalities”, the first of which was found by Greenberger, Horne and Zeilinger (GHZ) [9]. The formulation as a non-local game is, however, especially appealing in that it emphasises not only the remarkable non-local properties of these states, but also how they can be used in an information-theoretic sense. More specifically, in the context of cooperative games of incomplete information [10].

A non-local game [10, 7] consists of *players* (Alice, Bob, Charlie...), who act cooperatively in order to win, and a *referee* who coordinates the game. The players may collectively decide on a strategy before the game commences. Once it has begun they may no longer communicate. Whether or not the players win is determined by the referee. To begin the referee randomly selects one question, from a known fixed set \mathcal{Q} , to be sent to each player. The players know only their own questions. Each player must then send back a response from the set of answers \mathcal{A} . The referee determines whether the players win using the set of sent questions and received answers according to some predetermined rules. These rules are known to the players before the game gets under way so that they may attempt to devise a winning strategy.

For the three-player game [7] the questions sent to Alice, Bob and Charlie, denoted respectively by r, s and t , are taken uniformly at random from the set $\mathcal{Q} = \{0, 1\}$. However, the referee ensures that $rst \in \{000, 110, 101, 011\}$ and the players are aware of this. The answers a, b, c , sent back by Alice, Bob and Charlie, are elements of $\mathcal{A} = \{0, 1\}$. See Figure 1. The players win if $r \vee s \vee t = a \oplus b \oplus c$, where \vee and \oplus respectively denote disjunction and addition mod 2, i.e for question sets $rst = 000, 011, 101$ and 110 the answer set abc must satisfy $a \oplus b \oplus c = 0, 1, 1$ and 1 , respectively.

In the quantum version, Alice, Bob and Charlie each possess a qubit, which they may manipulate locally. The 3-qubit state may be entangled and used as a resource to help the players win. However, before examining how this works let us consider how well the players can do classically, i.e. unassisted by entanglement.

A classical strategy amounts to specifying three functions, a, b, c , one for each player, from the question set \mathcal{Q} to the answer set \mathcal{A} . The condition that the players win may then be written

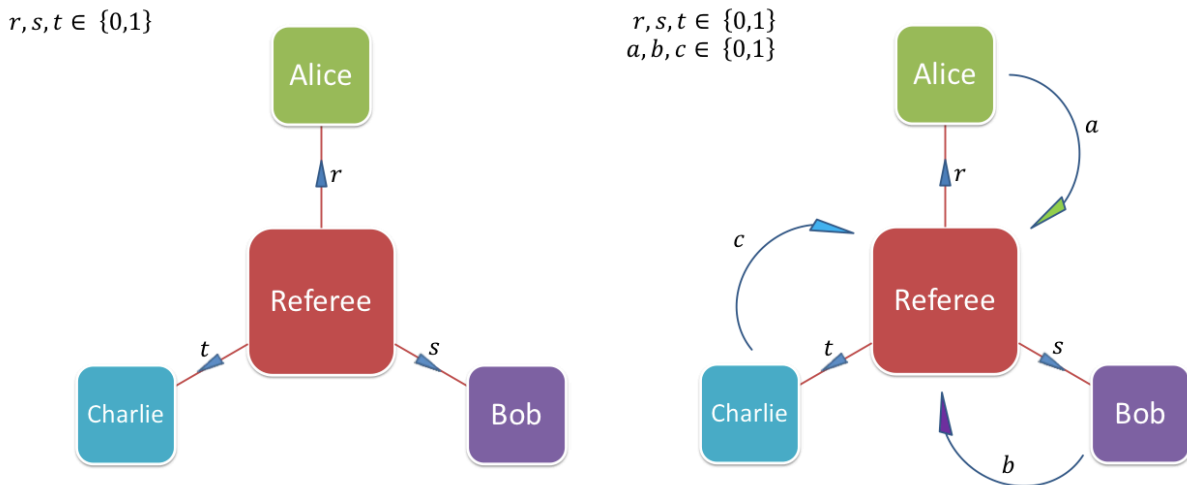


Figure 1. Three player non-local game.

as

$$a(0) \oplus b(0) \oplus c(0) = 0, a(1) \oplus b(1) \oplus c(0) = 1, a(1) \oplus b(0) \oplus c(1) = 1, a(0) \oplus b(1) \oplus c(1) = 1 \quad (2.2)$$

This implies that the best one can do is win 75% of the time; the four equations cannot be simultaneously satisfied as can be seen by adding them mod 2, which yields the contradiction $0 = 1$ [7]. On the other hand, the simple strategy that “everyone always answers 1” satisfies three of the four equations so that the 75% upper bound is actually met.

Can this be bettered when equipped with an entangled resource? The answer is a resounding yes: by sharing the GHZ state,

$$|\Psi\rangle = 1/2 (|000\rangle - |011\rangle - |101\rangle - |110\rangle) \quad (2.3)$$

which is equivalent to (2.1) under a local unitary rotation, they can always win [7].

The winning quantum strategy is remarkably simple. If a player receives the question “0” they measure their qubit in the computational basis $\{|0\rangle, |1\rangle\}$. If a player receives the question “1” they measure their qubit in the Hadamard basis $\{(|0\rangle + |1\rangle)/\sqrt{2}, (|0\rangle - |1\rangle)/\sqrt{2}\}$. Their measurement outcomes are sent back as their answers. By symmetry we need only consider the two cases $rst = 000$ and $rst = 011$. (1) $rst = 000$: All players measure in the computational basis. From (2.3) it is clear that only an odd number of 0’s can appear $\Rightarrow a \oplus b \oplus c = 0$. Always win. (2) $rst = 011$: Alice measures in the computational basis, while Bob and Charlie measure in the Hadamard basis. Consulting the locally rotated state,

$$\mathbb{1} \otimes H \otimes H |\Psi\rangle = 1/2 (|001\rangle + |010\rangle - |100\rangle + |111\rangle)$$

where H is the unitary operator relating the computational basis to the Hadamard basis, it is clear that only an even number of 0’s can appear $\Rightarrow a \oplus b \oplus c = 1$. Always win. Hence, using the GHZ entangled resource (2.3), Alice, Bob and Charlie can win 100% of the time, outdoing the best classical strategy by 25%.

One might naively expect to be able to devise a winning strategy when Alice, Bob and Charlie are equipped with a shared W -state, given it is also totally entangled. However, the W -state cannot be used win with certainty even when allowing Alice, Bob and Charlie to each measure

in any pair of bases they wish [11]. Hence, despite both being totally entangled, the GHZ-state is “more” entangled than the W -state, in this context at least. Certainly, they constitute physically inequivalent forms of entanglement.

We must therefore conclude that it is not enough to simply say a state is totally entangled -one must also specify in what way it is totally entangled. This is achieved using the paradigm of *local operations and classical communication* (LOCC). See, for example, [5]. Roughly, given a composite quantum system we allow purely *local* quantum operations to be performed on the individual components. These local operations may be supplemented by classical communication: the separated experimenters may communicate via a classical channel, e-mail for example. Any number of LOCC rounds may be performed, which makes the class of allowed operations difficult to characterise. For an in-depth account of LOCC protocols, see [6]. In this manner arbitrary classical correlations between the constituent subsystems may be generated. However, no quantum correlations may be established -all information exchanged was classical. LOCC protocols cannot increase the amount of entanglement.

This motivates the concept of *Stochastic* LOCC equivalence, introduced in [12, 4]: two states lie in the same SLOCC-equivalence class if and only if they may be transformed into one another with some *non-zero probability* using LOCC operations. The crucial observation is that since LOCC cannot create entanglement any two SLOCC-equivalent states must necessarily possess the same entanglement, irrespective of the particular measure used. It is this property which makes the SLOCC paradigm so suited to the task of classifying entanglement.

We restrict our attention to pure states. For an n -qubit system, two states

$$|\Psi\rangle = a_{A_1 \dots A_n} |A_1 \dots A_n\rangle, \quad |\Phi\rangle = b_{B_1 \dots B_n} |B_1 \dots B_n\rangle \quad (2.4)$$

are SLOCC-equivalent if and only if they are related by an element of $\text{SL}_1(2, \mathbb{C}) \times \text{SL}_2(2, \mathbb{C}) \times \dots \times \text{SL}_n(2, \mathbb{C})$ [4], which will be referred to as the *SLOCC-equivalence group*. The Hilbert space is partitioned into equivalence classes or orbits under the SLOCC-equivalence group. For the n -qubit system the space of SLOCC-equivalence classes is given by

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \dots \otimes \mathbb{C}^2 / \text{SL}_1(2, \mathbb{C}) \times \text{SL}_2(2, \mathbb{C}) \times \dots \times \text{SL}_n(2, \mathbb{C}) \quad (2.5)$$

This is the space of physically distinct form of entanglement. Hence, the SLOCC entanglement classification amounts to understanding (2.5).

3. The 3-qubit Jordan algebra and Freudenthal-Kantor triple system

3.1. Cubic Jordan algebras

A Jordan algebra \mathfrak{J} is vector space defined over a ground field \mathbb{F} equipped with a bilinear product satisfying

$$X \circ Y = Y \circ X, \quad X^2 \circ (X \circ Y) = X \circ (X^2 \circ Y), \quad \forall X, Y \in \mathfrak{J} \quad (3.1)$$

The class of *cubic* Jordan algebras are constructed as follows [13]. Let V be a vector space equipped with a cubic norm, which is a homogeneous map of degree three, $N : V \rightarrow \mathbb{F}$ s.t.

$$N(\lambda X) = \lambda^3 N(X), \quad \forall \lambda \in \mathbb{F}, \quad X \in V \quad (3.2)$$

such that

$$N(X, Y, Z) := 1/6 [N(X+Y+Z) - N(X+Y) - N(X+Z) - N(Y+Z) + N(X) + N(Y) + N(Z)] \quad (3.3)$$

is trilinear. If V further contains a base point $N(c) = 1, c \in V$ one may define the following three maps

$$\begin{aligned} \text{Tr} : V &\rightarrow \mathbb{F}, & X &\mapsto 3N(c, c, X) \\ S : V \times V &\rightarrow \mathbb{F}, & (X, Y) &\mapsto 6N(X, Y, c) \\ \text{Tr} : V \times V &\rightarrow \mathbb{F}, & (X, Y) &\mapsto \text{Tr}(X) \text{Tr}(Y) - S(X, Y) \end{aligned} \quad (3.4)$$

A cubic Jordan algebra \mathfrak{J} , with multiplicative identity $1 = c$, may be derived from any such vector space if N is *Jordan cubic*. That is: (i) the trace bilinear form (3.4) is non-degenerate (ii) the quadratic adjoint map, $\sharp: \mathfrak{J} \rightarrow \mathfrak{J}$, uniquely defined by $\text{Tr}(X^\sharp, Y) = 3N(X, X, Y)$, satisfies $(X^\sharp)^\sharp = N(X)X$ for all $X \in \mathfrak{J}$. The Jordan product is then given by

$$X \circ Y = 1/2 [X \times Y + \text{Tr}(X)Y + \text{Tr}(Y)X - S(X, Y)1] \quad (3.5)$$

where $X \times Y := (X + Y)^\sharp - X^\sharp - Y^\sharp$.

Definition 3.1 (structure group $\text{Str}(\mathfrak{J})$). Invertible \mathbb{F} -linear transformations σ preserving the cubic norm up to a fixed scalar factor,

$$\text{Str}(\mathfrak{J}) := \{\sigma \in \text{Iso}_{\mathbb{F}}(\mathfrak{J}) | N(\sigma A) = \lambda N(A), \lambda \in \mathbb{F}\} \quad (3.6)$$

The reduced structure group $\text{Str}_0(\mathfrak{J}) \subset \text{Str}(\mathfrak{J})$ preserves the norm exactly.

The exceptional Jordan algebra of 3×3 Hermitian octonionic matrices, denoted \mathfrak{J}_3^0 , is perhaps the most important and well known example. The reduced structure group $\text{Str}_0(\mathfrak{J}_3^0)$ in this case is given by the 78-dimensional exceptional Lie group $E_{6(-26)}$.

Any cubic Jordan algebra element may be assigned a $\text{Str}(\mathfrak{J})$ -invariant Rank [14]. The ranks partition the V .

Definition 3.2 (cubic Jordan algebra rank). A non-zero element $A \in \mathfrak{J}$ has a rank given by

$$\text{Rank } A = 1 \Leftrightarrow A^\sharp = 0, \text{ Rank } A = 2 \Leftrightarrow N(A) = 0, A^\sharp \neq 0, \text{ Rank } A = 3 \Leftrightarrow N(A) \neq 0 \quad (3.7)$$

3.2. The Freudenthal-Kantor triple system

In 1954, Freudenthal [15] found that the 133-dimensional exceptional Lie group E_7 could be understood in terms of the automorphisms of a construction based on the 56-dimensional E_7 -module built from the exceptional Jordan algebra of 3×3 Hermitian octonionic matrices. A key feature of this construction is a triple product. Later, the Freudenthal construction was essentially extended and elaborated, in particular, by Kantor [16], and then also by Yamaguti, Asano, Ono [17, 18] and Kamiya [19], hence the term *Freudenthal-Kantor triple system* (F-KTS) [18] is in use.

Given a cubic Jordan algebra \mathfrak{J} defined over a field \mathbb{F} , one is able to construct an F-KTS by defining the vector space $\mathfrak{F}(\mathfrak{J})$, $\mathfrak{F}(\mathfrak{J}) = \mathbb{F} \oplus \mathbb{F} \oplus \mathfrak{J} \oplus \mathfrak{J}$. An arbitrary element $x \in \mathfrak{F}(\mathfrak{J})$ is conventionally written as a “ 2×2 matrix” $x = \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix}$, where $\alpha, \beta \in \mathbb{F}, A, B \in \mathfrak{J}$, but for notational convenience we will often also write $x = (\alpha, \beta, A, B)$.

The F-KTS comes equipped with a non-degenerate bilinear antisymmetric quadratic form, a quartic form and a trilinear triple product [15, 20]:

- (i) Quadratic form $\{x, y\}: \mathfrak{F} \times \mathfrak{F} \rightarrow \mathbb{F}$

$$\{x, y\} = \alpha\delta - \beta\gamma + \text{Tr}(A, D) - \text{Tr}(B, C) \quad (3.8a)$$

where $x = (\alpha, \beta, A, B)$ and $y = (\gamma, \delta, C, D)$.

(ii) Quartic form $q : \mathfrak{F} \rightarrow \mathbb{F}$

$$q(x) = -2[\alpha\beta - \text{Tr}(A, B)]^2 - 8[\alpha N(A) + \beta N(B) - \text{Tr}(A^\sharp, B^\sharp)] \quad (3.8b)$$

(iii) Triple product $T : \mathfrak{F} \times \mathfrak{F} \times \mathfrak{F} \rightarrow \mathfrak{F}$ which is uniquely defined by

$$\{T(x, y, w), z\} = q(x, y, w, z) \quad (3.8c)$$

where $q(x, y, w, z)$ is the full linearisation of $q(x)$ such that $q(x, x, x, x) = q(x)$.

Definition 3.3 (the automorphism group $\text{Aut}(\mathbb{F} - KTS)$ [20]). Invertible \mathbb{F} -linear transformations preserving the quartic and quadratic forms:

$$\text{Aut}(\mathbb{F} - KTS) := \{\sigma \in \text{Iso}_{\mathbb{F}}(\mathbb{F} - KTS) \mid \{\sigma x, \sigma y\} = \{x, y\}, q(\sigma x) = q(x)\} \quad (3.9)$$

Lemma 3.4 (Brown [20]). *The following transformations generate elements of $\text{Aut}(\mathbb{F} - KTS)$:*

$$\begin{aligned} \varphi(C) : \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} &\mapsto \begin{pmatrix} \alpha + \text{Tr}(B, C) + \text{Tr}(A, C^\sharp) + \beta N(C) & A + \beta C \\ B + A \times C + \beta C^\sharp & \beta \end{pmatrix} \\ \psi(D) : \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} &\mapsto \begin{pmatrix} \alpha & A + B \times D + \alpha D^\sharp \\ B + \alpha D & \beta + \text{Tr}(A, D) + \text{Tr}(B, D^\sharp) + \alpha N(D) \end{pmatrix} \\ \hat{\tau} : \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} &\mapsto \begin{pmatrix} \lambda\alpha & \tau A \\ {}^t\tau^{-1}B & \lambda^{-1}\beta \end{pmatrix} \end{aligned} \quad (3.10)$$

where $C, D \in \mathfrak{J}$ and $\tau \in \text{Str}(\mathfrak{J})$ s.t. $N(\tau A) = \lambda N(A)$. For convenience we also define $\mathcal{Z} = \phi(-\mathbf{1})\psi(\mathbf{1})\phi(-\mathbf{1})$,

$$\mathcal{Z} : (\alpha, \beta, A, B) \mapsto (-\beta, \alpha, -B, A) \quad (3.11)$$

The archetypal example is given by setting $\mathfrak{J} = \mathfrak{J}_3^0$, in which case elements of $\mathbb{F} - KTS(\mathfrak{J}_3^0)$ transform as the **56** of E_7 . Decomposing under reduced structure group $\text{Str}_0(\mathfrak{J}_3^0) = E_6$ gives the branching **56** \rightarrow **1** + **1** + **27** + **27'** (neglecting the $\text{SO}(2)$ weights), where α, β comprise the singlets and A, B transform as the **27**, **27'**.

The conventional concept of matrix rank may be generalised to Freudenthal-Kantor triple systems in a natural and $\text{Aut}(\mathfrak{F})$ -invariant manner.

Definition 3.5 (The F-KTS Rank [21, 22]). The rank of a non-zero element $x \in \mathfrak{F}$ is defined by:

$$\begin{aligned} \text{Rank } x = 1 &\Leftrightarrow \Upsilon_x(y) = 0 \ \forall y \\ \text{Rank } x = 2 &\Leftrightarrow \exists y \text{ s.t. } \Upsilon_x(y) \neq 0, T(x, x, x) = 0 \\ \text{Rank } x = 3 &\Leftrightarrow T(x, x, x) \neq 0, q(x) = 0 \\ \text{Rank } x = 4 &\Leftrightarrow q(x) \neq 0 \end{aligned} \quad (3.12)$$

where $\Upsilon_x(y) := 3T(x, x, y) + x\{x, y\}x$.

3.3. The 3-qubit Freudenthal-Kantor triple system

Definition 3.6 (3-qubit cubic Jordan algebra). We define the 3-qubit cubic Jordan algebra, denoted as \mathfrak{J}_{ABC} , as the complex vector space $\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$ with elements $A = (A_1, A_2, A_3)$ and cubic norm

$$N(A) = A_1 A_2 A_3 \quad (3.13)$$

Using the cubic Jordan algebra construction (3.4), one finds

$$\text{Tr}(A, B) = A_1 B_1 + A_2 B_2 + A_3 B_3 \quad (3.14)$$

so that, using $\text{Tr}(A^\sharp, B) = 3N(A, A, B)$, the quadratic adjoint is given by

$$A^\sharp = (A_2 A_3, A_1 A_3, A_1 A_2) \quad (3.15)$$

and therefore

$$(A^\sharp)^\sharp = (A_1 A_2 A_3 A_1, A_1 A_2 A_3 A_2, A_1 A_2 A_3 A_3) = N(A)A \quad (3.16)$$

It is not hard to check $\text{Tr}(A, B)$ is non-degenerate and so N is Jordan cubic. In fact, it is the smallest degree three spin-factor Jordan algebra, see for example [13, 23]. The Jordan product is given by $A \circ B = (A_1 B_1, A_2 B_2, A_3 B_3)$. The structure and reduced structure groups are given by $[\text{SO}(2, \mathbb{C})]^3$ and $[\text{SO}(2, \mathbb{C})]^2$ respectively.

Definition 3.7 (3-qubit Freudenthal-Kantor triple system). We define the 3-qubit Freudenthal-Kantor triple system, denoted $\mathbb{F} - KTS_{ABC}$, as the complex vector space

$$\mathbb{F} - KTS_{ABC} := \mathbb{C} \oplus \mathbb{C} \oplus \mathfrak{J}_{ABC} \oplus \mathfrak{J}_{ABC} \quad (3.17)$$

We identify the eight components of $\mathbb{F} - KTS_{ABC}$ with the eight three qubit basis vectors $|ABC\rangle$ so that, for $|\psi\rangle = a_{ABC}|ABC\rangle$,

$$\begin{pmatrix} \alpha & (A_1, A_2, A_3) \\ (B_1, B_2, B_3) & \beta \end{pmatrix} \leftrightarrow \Psi := \begin{pmatrix} a_{111} & (a_{001}, a_{010}, a_{100}) \\ (a_{110}, a_{101}, a_{011}) & a_{000} \end{pmatrix} \quad (3.18)$$

Crucially, the automorphism group of $\mathbb{F} - KTS_{ABC}$, as defined in (3.9), is $\text{SL}(2, \mathbb{C}) \times \text{SL}(2, \mathbb{C}) \times \text{SL}(2, \mathbb{C}) \rtimes S_3$, where S_3 is the three-qubit permutation group. On including the permutation group the three biseparable entanglement classes, $A-BC$, $B-CA$ and $C-AB$, are identified.

4. F-KTS entanglement classification

The ranks of the F-KTS are in fact the entanglement classes: all states of a given rank $r = 1, 2, 3$ are SLOCC-equivalent. Rank four states constitute a $\dim_{\mathbb{C}} = 1$ family of equivalent states parametrised by $q(\Psi)$. More specifically, we have: (Rank 1) Totally separable states $A-B-C$, (Rank 2) Biseparable states $A-BC$, (Rank 3) Totally entangled W states, (Rank 4) Totally entangled GHZ states. The rank 4 GHZ class is regarded as maximally entangled in the sense that it has non-vanishing quartic norm. See Table 1. To prove this statement we use the following result, which is an extension of lemma 24 in [22].

Table 1. The entanglement classification of three qubits as according to the F-KTS rank system.

Class	Rank	Representative state	F-KTS rank condition	
			vanishing	non-vanishing
Null	0	0	Ψ	—
$A-B-C$	1	$ 111\rangle$	$\Upsilon_{\Psi}(\Phi)$	Ψ
$A-BC$	2	$ 111\rangle + 100\rangle$	$T(\Psi, \Psi, \Psi)$	$\Upsilon_{\Psi}(\Phi)$
W	3	$ 111\rangle + 001\rangle + 010\rangle$	$q(\Psi)$	$T(\Psi, \Psi, \Psi)$
GHZ	4	$ 111\rangle + 001\rangle + 010\rangle + k 100\rangle$	—	$q(\Psi)$

Lemma 4.1. *Every state Ψ is SLOCC-equivalent to the reduced canonical form:*

$$\Psi_{\text{red}} = (1, 0, A, 0) \longleftrightarrow |111\rangle + a_{001}|001\rangle + a_{010}|010\rangle + a_{100}|100\rangle \quad (4.1)$$

where $\text{Det } a = 4a_{011}a_{101}a_{110}$.

Proof. We start with a generic state (α, β, A, B) . We may always assume B is non-zero. (If $B = 0$ and $A \neq 0$ use \mathcal{Z} . If $B = 0, A = 0$ then we may assume that α non-zero, using \mathcal{Z} if necessary, and apply $\psi(D)$ to get a non-zero B -component, as required.) Apply $\phi(C)$ with $C^\sharp = 0$ s.t. and $\alpha \mapsto \alpha + \text{Tr}(C, B) = 1$, which is always possible since $B \neq 0$, the trace form is non-degenerate and \mathfrak{J}_{ABC} is spanned by its rank 1 elements. We are left with a new state $(1, \beta', A', B')$. Now apply $\psi(D)$ with $D = -B'$, so that $(1, \beta', A', B') \mapsto (1, \beta'', A'', 0)$.

Hence we may assume from the outset that our state is in the reduced form $(1, \beta, A, 0)$. We now show that we may also set $\beta = 0$. Since the Jordan ranks (3.7) partition \mathfrak{J}_{ABC} , there are four subcases to consider: (i) $A = 0$, (ii) $A \neq 0, A^\sharp = 0$, (iii) $A^\sharp \neq 0, N(A) = 0$ and (iv) $N(A) \neq 0$.

(i) Apply $\phi(C)$ with $C \neq 0, C^\sharp = 0$: $(1, \beta, 0, 0) \mapsto (1, \beta, \beta C, 0)$ so that we are now in case (ii).

(ii) We may assume with out loss of generality, $A = (a, 0, 0)$, $a \neq 0$. Let $C = (0, c, 0)$,

$$\phi(C) : (1, \beta, A, 0) \mapsto (1, \beta, A', A \times C)$$

where $A' = (a, \beta c, 0) \Rightarrow A'^\sharp \neq 0, N(A') = 0$. Apply $\psi(D)$ with $D = -A \times C$,

$$\psi(D) : (1, \beta, A', A \times C) \mapsto (1, \beta, A', 0)$$

so that we find ourselves in case (iii).

(iii) Without loss of generality we may assume $A = (0, a_2, a_3)$, $a_2, a_3 \neq 0$. Let $C = (c, 0, 0)$,

$$\phi(C) : (1, \beta, A, 0) \mapsto (1, \beta, A', B')$$

where $A' = (\beta c, a_2, a_3)$ and $B' = (0, ca_3, ca_2)$. Apply $\psi(D)$ with $D = -B'$,

$$\psi(D) : (1, \beta, A', B') \mapsto (1, \beta - 2ca_2a_3, A'', 0)$$

and let $c = \beta/(2a_2a_3)$ to obtain the canonical reduced form $(1, 0, A'', 0)$.

(iv) The augment of case (iii) does not require $N(A) = 0$ and, hence, also applies to the present case. \square

Given Theorem 4.1, the entanglement classifications follows almost trivially. The the rank conditions applied to the reduced canonical form (4.1) dramatically simplify and imply the that each rank (up to permutation) is represented a state corresponding to the classes of Table 1:

$$\begin{array}{llll} \text{Rank } \Psi_{\text{red}} = 1 & \Leftrightarrow & A = 0 & \Rightarrow & \Psi_{\text{red}} = |111\rangle \\ \text{Rank } \Psi_{\text{red}} = 2 & \Leftrightarrow & A^\sharp = 0, A \neq 0, & \Rightarrow & \Psi_{\text{red}} = |111\rangle + |001\rangle \\ \text{Rank } \Psi_{\text{red}} = 3 & \Leftrightarrow & N(A) = 0, A^\sharp \neq 0 & \Rightarrow & \Psi_{\text{red}} = |111\rangle + |001\rangle + |010\rangle \\ \text{Rank } \Psi_{\text{red}} = 4 & \Leftrightarrow & N(A) \neq 0 & \Rightarrow & \Psi_{\text{red}} = |111\rangle + |001\rangle + |010\rangle + k|100\rangle \end{array} \quad (4.2)$$

where $k \neq 0$. To reach the final form of the representative states on the far left of (4.2), we have scaled using

$$\hat{\tau} \in [\text{SO}(2, \mathbb{C})]^2 \cong \text{Str}_0(\mathfrak{J}_{ABC}) \subset \text{Aut}(\mathbb{F} - KTS_{ABC})$$

see (3.10). Since the ranks partition $\mathbb{F} - KTS_{ABC}$, this completes the orbit (entanglement class) classification of (2.5), which is summarised in Table 1. As claimed, elements of rank 1, 2 and 3 belong to a single orbit corresponding to totally separable, biseparable and W states, respectively. Rank 4 elements belong to a one complex dimensional family of orbits, parametrised by $q(\Psi) = -8k$, and correspond to GHZ states. Note, the rank classification places GHZ above W ; they are not merely inequivalent, but ordered, as reflected by three player non-local game [11].

5. Generalising to an n -qubit F-KTS

The success of the F-KTS classification of 3-qubit entanglement naturally raises the question of generalisation. Are there other composite quantum systems amenable to the F-KTS treatment? What about mixed states? Is there an extension to an arbitrary number of qubits? Indeed, it has already been shown that many of the F-KTS based on cubic Jordan algebras provide the pure state SLOCC entanglement classification of composite quantum systems, including mixtures of bosonic and fermionic qudits [24, 1, 25, 26, 2] and [Levay P and Sarosi G arXiv:1206.5066]. Moreover, in the case of three qubits the F-KTS may be used in the mixed state classification [27].

In the subsequent sections we focus on the final question posed above. While there is no arbitrary n -qubit F-KTS *per se*, we may attempt to identify those aspects of the 3-qubit F-KTS which naturally generalise to an arbitrary number of qubits in the hope that these universal features are illuminating. This is the approach taken here.

5.1. The n -qubit state reorganised

Recall, the Jordan algebra formulation of the F-KTS corresponded to decomposing the representation carried by the F-KTS under the $\text{Str}_0(\mathfrak{J}) \subset \text{Aut}(\mathbb{F} - KTS)$. In the case of three qubits we found the state split into the direct sum of four pieces,

$$\alpha = a_{111}, \quad \beta = a_{000}, \quad A = (a_{100}, a_{010}, a_{001}), \quad B = (a_{011}, a_{101}, a_{110}) \quad (5.1)$$

where α, β are $\text{Str}_0(\mathfrak{J})$ singlets. This leads us to the first important observation: α, β, A, B are the closed subsets under the 3-qubit permutation group S_3 . Indeed, if we are only interested in the SLOCC entanglement classification up to permutations, as we are, it is only natural to work with S_n -closed subsets as the basic building blocks. Hence, the 2^n state vector coefficients will be collected into the $n + 1$ subsets closed under S_n . It will prove convenient to represent these subsets using $n + 1$ totally symmetric tensors with ranks ranging from 0 to n ,

$$\mathcal{A}_n := \{A_0, A_{i_1}, A_{i_1 i_2}, \dots, A_{i_1 i_2 \dots i_n}\}, \quad \text{where } i_k = 1, 2, \dots, n \quad (5.2)$$

which are vanishing on any diagonal, i.e. $A_{i_1 i_2 \dots i_n} = 0$ if any two indices are the same. The counting of components goes like p -forms, correctly yielding a total of 2^n independent coefficients. Indeed, we could have just as well defined \mathcal{A}_n as a set of totally antisymmetric tensors of ranks 0 to n , avoiding the need to impose tracelessness. However, the 3-qubit F-KTS structure most naturally transfers over using the symmetric formulation, so we will stay with that convention here.

For three qubits we have (with a slight abuse of notation for A_{ijk})

$$A_0 = a_{000}, \quad A_i = \begin{pmatrix} a_{100} \\ a_{010} \\ a_{001} \end{pmatrix}, \quad A_{ij} = \begin{pmatrix} 0 & a_{110} & a_{101} \\ a_{110} & 0 & a_{011} \\ a_{101} & a_{011} & 0 \end{pmatrix}, \quad A_{ijk} = a_{111} \quad (5.3)$$

Note, numbering the qubits from left to right, the values of the indices on the symmetric tensors determine which indices on its corresponding state vector coefficient take the value 1. For example, $A_1 = a_{100}$, $A_2 = a_{010}$, $A_3 = a_{001}$ and $A_{12} = a_{110}$, $A_{13} = a_{101}$ and so on. I am grateful to Duminda Dahanayake for pointing out this rule.

5.2. The n -qubit algebra

The second feature we might hope to generalise is the set of cubic Jordan algebra maps, $A \times B, \text{Tr}(A, B), N(A)$, see subsection 3.1, which played such a key role in the construction of the various covariants and invariants. Recall, group theoretically these maps correspond to

picking out certain irreps appearing in the tensor product of the $\text{Str}_0(\mathfrak{J})$ -representation carried by $A, B \in \mathfrak{J}$. For example, in the case $\text{Str}_0(\mathfrak{J}_3^0) = E_{6(-26)}$, with A and B transforming as the **27**, $A \times B$ is the **27'** in $\mathbf{27} \times \mathbf{27} = \mathbf{27}'_s + \mathbf{351}_a + \mathbf{351}'_s$ and $N(A)$ is the singlet in $\mathbf{27} \times \mathbf{27} \times \mathbf{27}$. Each of the cubic Jordan algebra maps may be written using the irreducible $E_{6(-26)}$ invariant tensors, d_{ijk} and d^{ijk} , where a downstairs (upstairs) $i = 1, 2, \dots, 27$ transforms as a **27** (**27'**). For example, $(A^\sharp)^i = \frac{1}{2!} d^{ijk} A_j A_k$, and $N(A) = \frac{1}{3!} d^{ijk} A_i A_j A_k$. For the sake of clarity, we will often drop the combinatorial factors in the following. For three qubits, the invariant tensors were simply

$$d_{ijk} = |\epsilon_{ijk}|, \quad d^{ijk} = |\epsilon^{ijk}| \quad (5.4)$$

which naturally suggests the n -qubit generalisation,

$$d_{i_1 \dots i_n} := |\epsilon_{i_1 \dots i_n}|, \quad d^{i_1 \dots i_n} := |\epsilon^{i_1 \dots i_n}| \quad (5.5)$$

This allows us to dualise a rank p tensor,

$$A^{i_1 i_2 \dots i_{n-p}} := 1/p! \, d^{i_1 i_2 \dots i_{n-p} i_{n-p+1} \dots i_n} A_{i_{n-p+1} \dots i_n} \quad (5.6)$$

For an n -qubit state, rank p pairs $A_{i_1 i_2 \dots i_p}, A^{i_1 i_2 \dots i_p}$ are precisely bit-flip related. For example, for three qubits, $A_i = (a_{100}, a_{010}, a_{001})$ and $A^i = (a_{011}, a_{101}, a_{110})$. This is crucial for building S_n invariants. Equipped with $d_{i_1 \dots i_n}, d^{i_1 \dots i_n}$ the n -qubit space \mathcal{A}_n of symmetric tensors may be endowed with a pseudo-algebraic structure: \mathcal{A}_n is closed so long as we compose the tensors by contracting with $d_{i_1 \dots i_n}$ and $d^{i_1 \dots i_n}$.

5.3. The n -qubit generalised F -KTS transformations

The F -KTS transformations (3.10) for three qubits in the our new notation are given by

$$\begin{aligned} \phi(C) : \begin{pmatrix} A^{ijk} \\ A^{ij} \\ A^i \\ A^0 \end{pmatrix} &\mapsto \begin{pmatrix} A^{ijk} \\ A^{ij} + C_k A^{ijk} \\ A^i + C_j A^{ij} + C_j C_k A^{ijk} \\ A^0 + C_i A^i + C_i C_j A^{ij} + C_i C_j C_k A^{ijk} \end{pmatrix} \\ \psi(D) : \begin{pmatrix} A_0 \\ A_i \\ A_{ij} \\ A_{ijk} \end{pmatrix} &\mapsto \begin{pmatrix} A_0 + D^i A_i + D^i D^j A_{ij} + D^i D^j D^k A_{ijk} \\ A_i + D^j A_{ij} + D^j D^k A_{ijk} \\ A_{ij} + D^k A_{ijk} \\ A_{ijk} \end{pmatrix} \\ \hat{\tau}(\lambda) : \begin{pmatrix} A_0 \\ A_i \\ A^i \\ A^0 \end{pmatrix} &\mapsto \begin{pmatrix} d_{lmn} \lambda^l \lambda^m \lambda^n & A_0 \\ \xi^l d_{lmn} \lambda^m \lambda^n & A_i \\ \xi^l \xi^m d_{lmn} \lambda^n & A^i \\ \xi^l \xi^m \xi^n d_{lmn} & A^0 \end{pmatrix} \end{aligned} \quad (5.7)$$

where $\lambda \in \mathbb{C} - \{0\}$ and $\xi = \lambda^{-1}$. For $\phi(C), \psi(D)$ we have made a judicious choice of dualisations in order to make the correct n -qubit generalisation manifest. Under $\phi(C)$ a rank p tensor $A^{i_1 i_2 \dots i_p}$ transforms into the sum of all tensors $A^{i_1 i_2 \dots i_q}$ with $q \geq p$ contracted with the necessary powers of C_i to give back rank p . Explicitly,

$$\begin{aligned} A^{i_1 i_2 \dots i_n} &\mapsto [A^{i_1 i_2 \dots i_n}] \\ A^{i_1 i_2 \dots i_{n-1}} &\mapsto [A^{i_1 i_2 \dots i_{n-1}} + C_{i_n} A^{i_1 i_2 \dots i_n}] \\ A^{i_1 i_2 \dots i_{n-2}} &\mapsto [A^{i_1 i_2 \dots i_{n-2}} + C_{i_{n-1}} A^{i_1 i_2 \dots i_{n-1}} + C_{i_{n-1}} C_{i_n} A^{i_1 i_2 \dots i_{n-1} i_n}] \\ &\vdots \\ A^0 &\mapsto [A^0 + C_i A^i + C_i C_j A^{ij} + \dots + C_{i_1} C_{i_2} \dots C_{i_n} A^{i_1 i_2 \dots i_n}] \end{aligned} \quad (5.8)$$

Similarly, under $\psi(D)$ a rank p tensor $A_{i_1 i_2 \dots i_p}$ transforms into the sum of all $A_{i_1 i_2 \dots i_q}$ with $q \geq p$, contracted with the necessary powers of D^i to give back rank p ,

$$\begin{aligned} A_0 &\mapsto [A_0 + D^i A_i + D^i D^j A_{ij} + \dots + D^{i_1} D^{i_2} \dots D^{i_n} A_{i_1 i_2 \dots i_n}] \\ A_i &\mapsto [A_i + D^j A_{ij} + \dots + D^{i_2} D^{i_3} \dots D^{i_n} A_{i i_2 i_3 \dots i_n}] \\ A_{ij} &\mapsto [A_{ij} + \dots + D^{i_3} D^{i_4} \dots D^{i_n} A_{i j i_3 i_4 \dots i_n}] \\ \vdots &\quad \quad \quad \vdots \\ A_{i_1 i_2 \dots i_n} &\mapsto [A_{i_1 i_2 \dots i_n}] \end{aligned} \quad (5.9)$$

The generalised $\hat{\tau}(\lambda)$ may also be concisely written using this notation,

$$\hat{\tau}(\lambda) : A_{i_1 i_2 \dots i_p} \mapsto \xi^{j_1} \xi^{j_2} \dots \xi^{j_p} d_{j_1 j_2 \dots j_n} \lambda^{j_{p+1}} \dots \lambda^{j_{n-1}} \lambda^{j_n} A_{i_1 i_2 \dots i_p} \quad (5.10)$$

Hence, adopting the notational convention $A_{[p]}$ ($A^{[p]}$) for a rank p tensor with downstairs (upstairs) indices, the n -qubit transformations ϕ, ψ and $\hat{\tau}$ may be summarised as follows,

$$\begin{aligned} \phi(C_{[1]}): A^{[p]} &\mapsto \sum_{k=p}^n C_{[1]}^{(k-p)} A^{[k]} \\ \psi(D^{[1]}): A_{[p]} &\mapsto \sum_{k=p}^n D^{[1](k-p)} A_{[k]} \\ \hat{\tau}(\lambda^{[1]}): A_{[p]} &\mapsto \xi^{[1](p)} d_{[n]} \lambda^{[1](n-p)} A_{[p]} \end{aligned} \quad (5.11)$$

One useful observation that immediately follows is that one can always assume $A_0 = 1, A^i = 0$ under SLOCC. It is also clear that the Jordan ranks (3.7) naturally generalises to a set of rank conditions on $A_{[1]}, A^{[1]}$.

However, we have yet to develop a systematic method for writing covariants/invariants in this scheme. One example, though, defined for n qubits, is given by

$$(\mathcal{A}_n, \mathcal{B}_n) = \sum_{k=0}^n (-1)^k / k! A_{[k]} B^{[k]} \quad (5.12)$$

This is symmetric (antisymmetric) for even (odd) n . It is simply the determinant in the 2-qubit case and the antisymmetric bilinear form of the F-KTS in the 3-qubit case. There are four algebraically independent 4-qubit permutation and SLOCC-equivalence group invariants [28] of order two, six, eight and twelve, (5.12) being the order two example.

2-qubit example. The 2-qubit state corresponds to

$$|\psi\rangle \leftrightarrow \Psi = \{A_0, A_i, A_{ij}\}, \quad i, j = 1, \dots, 2 \quad (5.13)$$

where

$$\left\{ A_0 = a_{00}, \quad A_i = \begin{pmatrix} a_{10} \\ a_{01} \end{pmatrix}, \quad A^0 = a_{11} \right\} \quad (5.14)$$

It is easy to verify in this scheme that every state \mathcal{A}_2 is SLOCC-equivalent to the reduced canonical form: $\mathcal{A}_2^{\text{red}} = \{1, 0, k\} \longleftrightarrow |00\rangle + k|11\rangle$, where $k = (\mathcal{A}_2, \mathcal{A}_2)$. First we may always assume A_i is non-zero. (If $A_i = 0$ then we may assume that A_0 non-zero using \mathcal{Z} if necessary.) Now apply $\phi(C)$ to get a non-zero A_i . Apply $\psi(D)$ with $d_{ij} D^i D^j = 0$ so that $A_0 \mapsto A_0 + D^i A_i$. Choose D s.t. $A_0 + D^i A_i = 1$. Finally, apply $\phi(C)$ with $C_i = -A_i$.

This gives us the well known 2-qubit SLOCC entanglement classification: there are just two classes, separable and entangled, the latter being a family of orbits parametrised by $(\mathcal{A}_2, \mathcal{A}_2) = k$.

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