

The influence of boundary conditions on the excitation of instabilities in magnetohydrodynamic systems

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Abstract. The recent stage of the magnetohydrodynamic energy principle applied to laboratory and space plasmas is briefly reviewed. In detail, the energy principle is presented for an internally homogeneous pinch in a perfectly conducting wall. The plasma is separated from the wall by a vacuum. The principle is applied to ITER-type and lightning systems. Thereat, a system of mathematical equations of motion for fluid elements is derived using a cylindrical coordinate system. But the obtained equations may be also applied to plasmas with disturbances of non-cylindrical symmetry. From the equations of motion, an analytical relation for the radial displacements of the fluid elements is presented, which describes magnetohydrodynamic waves as e.g. sausage and kink ones. The numerical results here presented are, as a first step, only performed for plasma disturbances with cylindrical symmetry and outer azimuthal magnetic fields directed parallel to the conducting wall. Thus, the dispersion relations for sausage instabilities in ITER-type and lightning plasmas are solved. It is shown for which values of the inner and external magnetic fields of the systems instabilities occur. In case of lightnings, the radial displacements in the plasma are estimated.

1. Introduction

In nuclear fusion systems, plasma instabilities getting unstable for different reasons interrupt the fusion process and make the application of nuclear fusion for an energy gain in reactor systems up to now unachievable. In the Earth's magnetosphere, plasma instabilities modify the physical parameters and vary the transport of solar energy and radio signals from vertical sounding stations. Thus, it is essential to study plasma instabilities in both nuclear fusion and magnetospheric systems. Using the energy principle of magnetohydrodynamics (MHD), this all can be done without knowing the exact values of growth rates of unstable waves. According to [1], the growth rates of MHD instabilities in inertial fusion plasmas are, anyway, on the order or smaller than 5 percent of the possible observation times. And if even shorter observation times will be possible in future, the application of the MHD energy principle will often be much easier as a kinetic calculation of the growth rate.

Applying the MHD energy principle means to analyse, if the potential energy of the plasma system may decrease for any of its allowable small displacements. If this is the case, the system is unstable. Thus, applying the MHD energy principle, one has to solve variational problems for the potential energy.



The MHD energy principle is only applicable if the force on the particles is self-adjoint in the plasma. Because of this self-adjointness, a decrease of the potential energy is equivalent to a negative square of a frequency of a wave excited in the system. Thus, the analysis of the self-adjointness of the force is an important problem. Recent applications of the energy principle to systems with different plasma boundaries, especially to space systems, show that the self-adjointness reduces the number of the allowable small displacements which may be taken into account.

The present work considers instabilities in plasmas which are described within the frame of ideal MHD, that means the system has an infinite electrical conductivity. Hence, in section 2 the energy principle of ideal MHD is briefly introduced. Section 3 reviews recently used extended energy principles which are applied for plasmas not bounded by a vacuum and/or perfectly conducting wall, and where the magnetic field at the plasma boundary is not directed parallelly to the boundary. In section 4, the MHD energy principle is applied to an internally homogeneous pinch. There an analytical equation is derived to describe axial fluid displacements in systems with non-cylindrical disturbances. Numerical applications are performed for systems with cylindrically symmetric disturbances in sections 5 and 6 for ITER-type and lightning plasmas, respectively. Some conclusions are made in section 7.

2. Energy principle of ideal MHD

The present work considers instabilities in plasmas which may be described within the frame of ideal magnetohydrodynamics, the basic equations of which are the continuity equation

$$\frac{d\rho}{dt} = -\rho \nabla \vec{v}, \quad (1)$$

the momentum balance

$$\rho \frac{d\vec{v}}{dt} = \vec{j} \times \vec{B} - \nabla p + \rho \vec{g}, \quad (2)$$

the equation of state

$$\frac{d}{dt} \left[\frac{p}{\rho^\gamma} \right] = 0, \quad (3)$$

and the Maxwell equations without displacement current

$$\frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E}, \quad (4)$$

$$\nabla \times \vec{B} = \mu_o \vec{j}. \quad (5)$$

From the Ohm's law

$$\vec{j} = \sigma(\vec{E} + \vec{v} \times \vec{B}) \quad (6)$$

follows in the approximation of ideal MHD

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}). \quad (7)$$

The 14 scalar expressions (1-3, 4-5, 7) form a full system of equations of ideal magnetohydrodynamics to determine the 14 variables plasma pressure p , mass density ρ , fluid velocity \vec{v} , vectors of magnetic induction \vec{B} and electric field \vec{E} , and electrical current \vec{j} .

Besides, the Maxwell equation

$$\nabla \cdot \vec{B} = 0 \quad (8)$$

is valid.

Expressing the momentum balance of the linearized MHD system of equations (2) by the Lagrange displacement $\vec{\xi}$, using the relation between the Euler velocity $\vec{u}(\vec{r}, t)$ and the Lagrange velocity of a fluid element (which was situated at \vec{r}_o at time t_o)

$$\frac{\partial \vec{\xi}(\vec{r}, t)}{\partial t} = \vec{u}(\vec{r}, t) \approx \vec{u}(\vec{r}_o, t) + (\vec{\xi} \cdot \nabla) \vec{u} + \dots \approx \vec{u}(\vec{r}_o, t), \quad (9)$$

the momentum balance may be written in the form

$$\begin{aligned} \rho_o \frac{\partial^2 \vec{\xi}}{\partial t^2} &= \vec{F}(\vec{\xi}). \\ \vec{F}(\vec{\xi}) &= \nabla(\gamma p_o \nabla \cdot \vec{\xi} + (\vec{\xi} \cdot \nabla) p_o) \\ &+ \frac{1}{\mu_o} \left((\nabla \times \nabla \times (\vec{\xi} \times \vec{B}_o)) \times \vec{B}_o + (\nabla \times \vec{B}_o) \times (\nabla \times (\vec{\xi} \times \vec{B}_o)) \right) - \vec{g}((\vec{\xi} \cdot \nabla) \rho_o + \rho_o \nabla \cdot \vec{\xi}) \end{aligned} \quad (10)$$

gives the force on a fluid element in the plasma (\vec{g} - gravity acceleration).

Further it is assumed that the separation of variables

$$\vec{\xi}(\vec{r}_o, t) = \vec{\xi}_k(\vec{r}_o) \tau_k(t) \quad (11)$$

is possible and the function $\tau_k(t)$ describes a plain wave

$$\tau_k \sim \exp[i(\omega_k t + \varphi_k)]. \quad (12)$$

Then, ω_k is the solution of the equation

$$-\rho_o \omega_k^2 \vec{\xi}_k = \vec{F}(\vec{\xi}_k). \quad (13)$$

For the potential energy density W , $\vec{F} = -\nabla W$, it follows

$$W = - \int_0^{\vec{\xi}} \vec{F}(\vec{\eta}) d\vec{\eta} = -\frac{\vec{\xi}}{2} \vec{F}. \quad (14)$$

Thus, for the stability of the MHD system it is necessary and sufficient that the potential energy caused by the plasma displacement $\vec{\xi}$ has no negative values.

Taking the MHD stability condition for ideal plasmas into account,

$$\nabla p_o = \frac{1}{\mu_o} (\nabla \times \vec{B}_o) \times \vec{B}_o, \quad (15)$$

I.B. Bernstein, J.P. Freidberg [2-4], and Spatschek [5] found, that eq. (14) may be used for the determination of instabilities in plasmas, surrounded by a vacuum, which are located in a fixed conducting wall, provided that in addition to the volume contribution to the potential energy

$$W_F = \frac{1}{2} \int_{V_p} \left[\frac{Q^2}{\mu_o} + \frac{1}{\mu_o} (\nabla \times \vec{B}_o) (\vec{\xi} \times \vec{Q}) + (\nabla \cdot \vec{\xi}) (\vec{\xi} \cdot \nabla) p_o + \gamma p_o (\nabla \cdot \vec{\xi})^2 \right] d\vec{r}, \quad (16)$$

a vacuum contribution

$$W_V = \frac{1}{2\mu_o} \int_{V_V} (\delta \vec{B}_V)^2 d\vec{r}, \quad (17)$$

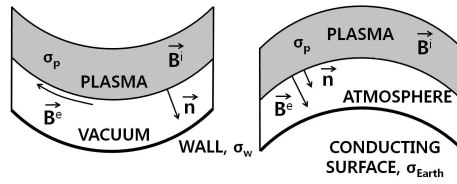


Figure 1. Comparison of the boundary condition of a laboratory plasma at a conducting wall with a vacuum barrier sheet (left side) with the boundary condition of the magnetosphere at the Earth's surface with an ionospheric barrier sheet. σ_P - electrical conductivity of the plasma, \vec{B}^i - magnetic field of the plasma, \vec{B}^e - magnetic field at the plasma-vacuum boundary and in the Earth's atmosphere, respectively. An analogous figure, the work [6] contains.

and a surface contribution

$$W_S = -\frac{1}{2} \int_S (\vec{\xi} \cdot \vec{n})^2 \vec{n} \cdot \nabla \left(p_o + \frac{B_o^2}{2\mu_o} - \frac{B_V^2}{2\mu_o} \right) d\vec{S} \quad (18)$$

are taken into account.

$$W = W_F + W_S + W_V, \quad \vec{Q} = \nabla \times (\vec{\xi} \times \vec{B}_o). \quad (19)$$

3. Boundary conditions

Indeed, eqs. (16-19) may be applied for laboratory fusion systems, where the magnetic field is parallel to the walls. But for magnetospheric systems, with non-parallel magnetic field lines, additional conditions follow for the allowable displacements $\vec{\xi}$. Besides, in case of the Earth's magnetosphere, the plasma is surrounded by a vacuum at lower altitudes, and by an ionospheric or magnetospheric boundary at larger altitudes (see Fig. 1). Such boundary conditions were first studied by Miura [6], who introduced two approximations. First he took only geomagnetic field lines into account, which are perpendicular to the ionospheric boundary, and second, he neglected the neutral atmosphere. In doing so he found that when the vectors $\vec{\xi}$ and $\vec{\eta}$ satisfy the boundary conditions

$$\eta_{\parallel} = 0 \quad \text{or} \quad \nabla \cdot \vec{\xi} = 0 \quad (20)$$

and

$$\eta_{\perp} = 0 \quad \text{or} \quad (\vec{B} \cdot \nabla) \xi_{\perp} = 0 \quad (21)$$

at the unperturbed ionospheric boundaries, the force operator of the plasma of the Earth's atmosphere is self-adjoint. But, taking the neutral atmosphere into account, $\nabla \cdot \vec{\xi} \neq 0$ is valid, and the second condition of eq. (20) is not satisfied.

4. Application to an internally homogeneous pinch

Now a homogeneous plasma is considered. It is assumed that the plasma is located in a cylinder in the equilibrium state. In the plasma, a uniform axial magnetic field $B_z \vec{n}_z$ exists. Further, it is supposed that the plasma is surrounded by a vacuum region, and in the vacuum only an azimuthal magnetic field is located. The whole system consisting of the cylindrical plasma core and the surrounding vacuum is contained in a cylindrical conducting shell. That means, the plasma does not contact the shell at all, there is a vacuum boundary between the plasma and the shell.

In the homogeneous plasma of the core of the pinch, $\nabla \times \vec{B}_z = 0$ and $\nabla p_o \approx 0$. The approximation $\nabla \times \vec{B}_z = 0$ was first introduced by Sturrock [7] and Spatschek [5], and it means that no current exists in the inner part of the plasma (i.e. along the z -axis). Here, the same approximation is used to compare partial results with that of the works [5, 7]. Indeed, in the case of the θ -pinch with an axial magnetic field in the plasma core, the currents are almost negligible along the symmetry axis (i.e. at $r = 0$ in cylindrical coordinates (r, φ, z)), and the pressure gradients vanish (see Figs. 11.12 and 11.13, p. 266, in [7]).

Under such conditions, one finds for the minimum potential energy of the plasma in the cylinder [8]

$$\text{Min } W_F \approx \frac{1}{2} \int \left(\gamma p_o (\nabla \cdot \vec{\xi}) - \frac{(\vec{B}_z \cdot \vec{Q})}{\mu_o} \right) (\vec{\xi} \cdot d\vec{F}). \quad (22)$$

The equation of motion (10) in the plasma reduces to

$$-\omega^2 \rho_o \xi_z = \gamma p_o \frac{\partial}{\partial z} (\nabla \cdot \vec{\xi}), \quad (23)$$

$$-\omega^2 \rho_o \xi_r = \gamma p_o \frac{\partial}{\partial r} \nabla \cdot \vec{\xi} + \frac{B_z^2}{\mu_o} \left[\frac{\partial}{\partial r} \frac{1}{r} \frac{\partial \xi_\varphi}{\partial \varphi} + \frac{\partial^2 \xi_r}{\partial z^2} + \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} (r \xi_r) \right], \quad (24)$$

$$-\omega^2 \rho_o \xi_\varphi = \frac{\gamma p_o}{r} \frac{\partial}{\partial \varphi} \nabla \cdot \vec{\xi} + \frac{B_z}{\mu_o} \left[\frac{1}{r^2} \frac{\partial^2}{\partial \varphi \partial r} (r \xi_r) + \frac{\partial^2 \xi_\varphi}{\partial z^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} (\xi_\varphi) \right]. \quad (25)$$

Further, to find the dispersion relation of the waves in the plasma, one may use the ansatz

$$\vec{\xi}(\vec{r}) = \vec{\xi}_o(r) \exp(im\varphi + ik_z z) \quad (26)$$

and simplify the spatial derivatives with respect to φ and z in eqs. (23-25)

$$(v_s^2 k_z^2 - \omega^2) \xi_z = ik_z \frac{v_s^2}{r} \frac{\partial}{\partial r} (r \xi_r) - k_z m v_s^2 \frac{\xi_\varphi}{r}, \quad (27)$$

$$(v_A^2 k_z^2 - \omega^2) \xi_r = (v_A^2 + v_s^2) \frac{\partial}{\partial r} \left[\frac{1}{r} \frac{\partial}{\partial r} (r \xi_r) \right] + im(v_A^2 + v_s^2) \frac{\partial}{\partial r} \left(\frac{\xi_\varphi}{r} \right) + ik_z v_s^2 \frac{\partial \xi_z}{\partial r}, \quad (28)$$

$$(v_A^2 k_z^2 - \omega^2) \xi_\varphi = (v_A^2 + v_s^2) \frac{im}{r^2} \frac{\partial}{\partial r} (r \xi_r) + \frac{im v_s^2 - m^2 v_A^2}{r} \frac{\xi_\varphi}{r} + \frac{im k_z v_s^2}{r} \xi_z, \quad (29)$$

$$v_A^2 = \frac{B_z^2}{\mu_o \rho_o}, \quad v_s^2 = \frac{\gamma p_o}{\rho_o}. \quad (30)$$

With

$$X = \frac{1}{r} \frac{\partial}{\partial r} (r \xi_r), \quad Y = \frac{\xi_\varphi}{r}, \quad (31)$$

one finds from eqs. (27, 29)

$$X = \frac{(v_s^2 k_z^2 - \omega^2)}{ik_z v_s^2} \xi_z + \frac{m}{i} Y, \quad Y = \frac{r(v_A^2 k_z^2 - \omega^2)}{(m^2 + im)v_s^2} \xi_\varphi - \frac{[ik_z^2 v_s^4 + (v_s^2 k_z^2 - \omega^2)(v_A^2 + v_s^2)]}{k_z(m+i)v_s^4} \xi_z \quad (32)$$

Substituting X and Y into eq. (28), and acting on (28) with the operator $(1/r)(\partial/\partial r)r$, one obtains

$$(v_A^2 k_z^2 - \omega^2) X = \frac{v_A^2 v_s^2 k_z^2 - \omega^2(v_A^2 + v_s^2)}{ik v_s^2} \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \xi_z}{\partial r} \right), \quad (33)$$

$$X = \frac{-iv_s^2 \omega^2 + iv_s^4 k_z^2 + m[v_s^2 v_A^2 k_z^2 - ik_z^2 v_s^4 - \omega^2(v_A^2 + 2v_s^2)]}{k_z(im-1)v_s^4} + \frac{v_A^2 k_z^2 - \omega^2}{(im-1)v_s^2} r \xi_\varphi. \quad (34)$$

Considering only equilibrium plasma systems with cylindrical symmetry ($\partial/\partial\varphi = 0$) and neglecting eq. (25), which then describes pure Alfvén waves in the plasma ($\omega^2 = k^2 v_A^2$ at any ξ_φ), one finds from (23, 24) at locally constant magnetic fields B_z (wave number $k = k_z$)

$$\frac{\partial^2 \xi_z}{\partial r^2} + \frac{1}{r} \frac{\partial \xi_z}{\partial r} - \alpha^2 \xi_z = 0, \quad \alpha^2 = \frac{(v_s^2 k^2 - \omega^2)(v_A^2 k^2 - \omega^2)}{v_s^2 v_A^2 k^2 - (v_s^2 + v_A^2) \omega^2}, \quad v_A^2 = \frac{B_z^2}{\mu_o \rho_o}, \quad v_s^2 = \frac{\gamma p_o}{\rho_o}, \quad (35)$$

which was already derived by Sturrock [9] studying the sausage instability. Thus, one has

$$\xi_z(r) = \frac{\text{const}}{\alpha^2} I_o(\alpha r), \quad (36)$$

where I_o is the modified Bessel function of first kind of order zero.

The equations (23, 24, 36) result then into a solution for $\xi_r(z)$,

$$\xi_r = \frac{v_s^2 (v_A^2 k^2 - \omega^2) - v_A^2 \omega^2}{i k v_s^2 (k^2 v_A^2 - \omega^2)} \frac{\partial \xi_z}{\partial r} = - \frac{i \text{const}}{\alpha k} \frac{[v_s^2 (v_A^2 k^2 - \omega^2) - v_A^2 \omega^2]}{v_s^2 (k^2 v_A^2 - \omega^2)} \frac{\partial I_o(\alpha r)}{\partial(\alpha r)}. \quad (37)$$

The analysis of α^2 described by eq. (35) gives the following maximum and minimum values:

$$\text{Max}(\alpha^2) = \omega^2 (v_s - v_A)^2 / (v_A^2 v_s^2) \text{ at } \omega = \text{const}, k^2 = \omega^2 (v_s^2 + v_A^2 - v_s v_A) / (v_s^2 v_A^2),$$

$$\text{Max}(\alpha^2) = k^2 (v_s^2 - v_A^2)^2 / (v_s^2 + v_A^2)^2 \text{ at } k = \text{const}, \omega^2 = 2 v_s^2 v_A^2 k^2 / (v_s^2 + v_A^2),$$

$$\text{Min}(\alpha^2) = \omega^2 (v_s + v_A)^2 / (v_A^2 v_s^2) \text{ at } \omega = \text{const}, k^2 = \omega^2 (v_s^2 + v_A^2 + v_s v_A) / (v_s^2 v_A^2),$$

$$\text{Min}(\alpha^2) = k^2 \text{ at } k = \text{const}, \omega^2 = 0.$$

Thus, at constant k or ω the α^2 -values of the minima are larger than the values of the maxima. At $\omega^2 = v_s^2 v_A^2 k^2 / (v_s^2 + v_A^2)$, α^2 diverges (see the application for ITER, Fig. 2).

The perturbation of the vector potential $\delta \vec{A}$ between the plasma surface and the outer shell, that means within the vacuum region, satisfies the vacuum wave equation. The perturbation of the electric field within the vacuum equals $\delta \vec{E} = -\partial(\delta \vec{A})/\partial t$, and the magnetic field perturbation is $\delta \vec{B} = \nabla \times (\delta \vec{A})$.

Further, combining the equation of motion (27-29)) with the boundary condition for the electromagnetic field at the plasma-vacuum interface (see [9]: eq. (15.3.3), the index i describes the plasma region and the index e designates the vacuum region, $B^{(i)} = B_z$, $B^{(e)} = B_\varphi$)

$$\vec{n} \times [\delta \vec{E}^{(i)} + \vec{v}^{(i)} \times \vec{B}^{(i)}] = \vec{n} \times [\delta \vec{E}^{(e)} + \vec{v}^{(e)} \times \vec{B}^{(e)}], \quad (38)$$

one finds the dispersion relation for the existing magnetohydrodynamic waves $\omega(k)$ under the influence of a plasma boundary (see [9]: $\partial/\partial\varphi = 0$ and $B_z = \text{const}$, eq. (15.5.14) without k^2 in the numerator of the second term on the right side)

$$\omega^2 = \frac{B_z^2 k^2}{\mu_o \rho_o} - \frac{B_\varphi^2}{\mu_o \rho_o R^2} \frac{\alpha R \partial I_o(\alpha R) / \partial(\alpha R)}{I_o(\alpha R)}. \quad (39)$$

R is the radius of the plasma-vacuum boundary. According to the dispersion relation (39), there would be no plasma instability, but a normal Alfvén wave, in case that the outer azimuthal magnetic field would vanish. That means, the plasma-vacuum interface determines the stability of the plasma. The plasma may become unstable at sufficiently large values of $\kappa = B_\varphi^2 / B_z^2$.

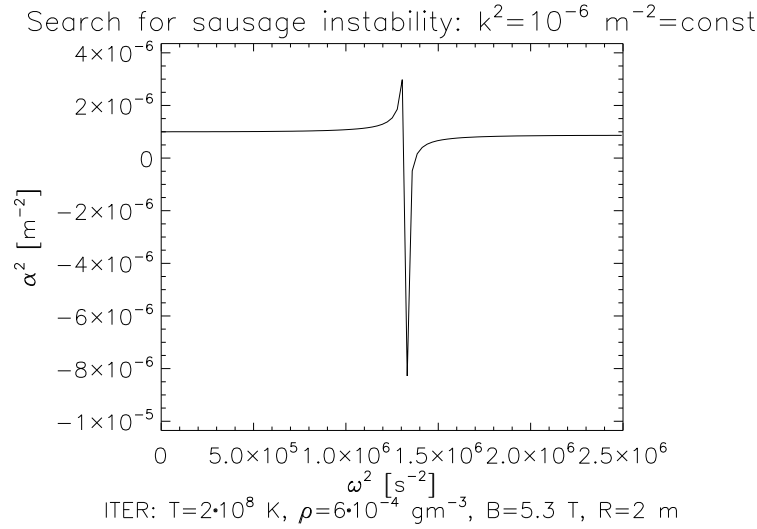


Figure 2. Function $\alpha^2(\omega^2)$ for an ITER-type plasma.

5. ITER

Since 2005 in Caderache, France, the International Thermonuclear Experimental Reactor ITER is under construction. It is a project of Europe, Japan, the former Soviet Union and the United States, joint in 2003 by China and South Korea. ITER is a Tokamak-type reactor with a total radius of 10.7 m and a height of 30 m.

ITER is designed to deliver a fusion power of 500 MW. It contains 18 superconducting toroidal and 6 poloidal field coils. The strength of its toroidal magnetic field will be about 5.3 T (here locally described by B_z). It can reach up to 11.8 T. A device volume of 840 m³ will contain a plasma of 0.5 g at mean temperature of $2 \cdot 10^8$ K. Thus, in case of ITER, one has to consider a plasma with a mass density of about $\rho = 6 \cdot 10^{-4}$ gm⁻³. The Alfvén and sound velocities of the plasma amount to $v_A = 6 \cdot 10^6$ m/s and $v_s = 1.2 \cdot 10^6$ m/s, respectively.

Studying the possibility of magnetohydrodynamic instabilities of an ITER-type plasma, one has first to determine the maximum and minimum values of the α -function (eq. (35)), as well as the location of its divergency. In Fig. 2, α^2 is presented for the ITER plasma as function of ω^2 . It is to be seen, that the α^2 -value at small ω^2 is also small, so that the argument of the Bessel function $R\alpha$ is smaller than one, and one may approximate $I_o(\alpha R)$ by

$$I_o(\alpha R) \approx 1 + \frac{\alpha^2 R^2}{4}, \quad \frac{\partial I_o(\alpha R)}{\partial(\alpha R)} = \frac{\alpha R}{2}. \quad (40)$$

Substituting eq. (40) for the Bessel function into eq. (39), one has

$$\omega^2 - k^2 v_A^2 = -\kappa v_A^2 \frac{\alpha^2}{(2 + \alpha^2 R^2/2)}. \quad (41)$$

Neglecting the $\alpha^2 R^2$ dependence on the right side of eq. (41) and expressing α^2 by eq. (35), two solutions of eq. (41) follow for ω^2 ,

$$\omega_1^2 = k^2 v_A^2 \quad (42)$$

and

$$\omega_2^2 = \frac{v_s^2 v_A^2 k^2 (2 - \kappa)}{2v_s^2 + v_A^2 (2 - \kappa)}. \quad (43)$$

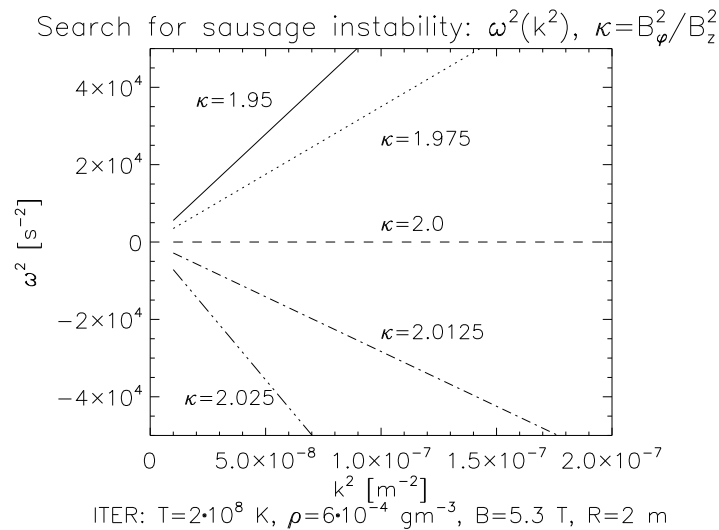


Figure 3. Dispersion relation of waves in an ITER-type plasma at different ratios of azimuthal to axial magnetic field B_ϕ/B_z . Solution for $\alpha R < 1$. At $\omega^2 < 0$, the sausage instability is excited.

The relation (42) describes a stable Alfvén wave. Equation (43) is the dispersion relation of an additional wave which is unstable in the case of $2 \leq \kappa < 2 + 2v_s^2/v_A^2$. At $\kappa = 2 + 2v_s^2/v_A^2$, the applied approximation eq. (40) is not valid anymore, ω_2^2 - and thus also α^2 - diverge. Further, using the approximation of the Bessel function eq. (40), it follows from the dispersion relation eq. (39) at $\omega^2 = 0$ (i.e. $\alpha^2 = k^2$) that $k^2 R^2 = \kappa - 2$. Thus, one obtains for $\alpha R < 1$ that $\kappa < 3$. Consequently, studying plasmas with $\kappa \geq 3$, the formulae (40) cannot be used at frequencies $\omega^2 \approx 0$.

Results for ω_2^2 are presented in Fig. 3 for the case of the ITER-type plasma. It is to be seen that the strength of the instability grows monotonically with increasing κ at small wave numbers k . Having found an instability for a given κ and a special wave number, the system is obtained to be unstable for the κ at any wave number. But the wave numbers studied in Fig. 3 are not of real importance for ITER. They are too small. Nevertheless, the obtained results are very helpful as limiting cases for numerical analyses, when the nonlinear dynamics of ITER-type plasma instabilities at smaller wavelengths is studied.

Considering large $\alpha R > 1$, the Bessel function may be approximated by Hankels expansion

$$I_o(\alpha R) \approx \frac{\exp(\alpha R)}{\sqrt{2\pi\alpha R}} \left(1 + \frac{1}{8\alpha R}\right). \quad (44)$$

Therefrom follows

$$\frac{\partial I_o(\alpha R)/\partial(\alpha R)}{I_o(\alpha R)} \approx \frac{8\alpha R - 3}{8\alpha R + 1} \approx 1 \quad (45)$$

and one finds

$$\omega^2 = \frac{2v_s^2 v_A^2 k^2 + v_A^4 k^2 \pm \sqrt{\beta^2 - 4\beta v_s^4 k^2}}{2(v_s^2 + v_A^2)}, \quad \beta = \frac{\kappa^2 v_A^4}{R^2}. \quad (46)$$

As ω^2 is real, one has $k^2 < \beta/(4v_s^4)$. or $k^2 < 46\kappa^2$ under ITER-like conditions. From eq. (46) follows for $\omega^2 < 0$ (sign minus), that the relation

$$k^2 < \frac{2\kappa^2}{R^2(2 + \chi)^2} \left[-1 + \sqrt{1 + \chi^2 + \chi^3 + 0.25\chi^4} \right], \quad \chi = \frac{v_A^2}{v_s^2}. \quad (47)$$

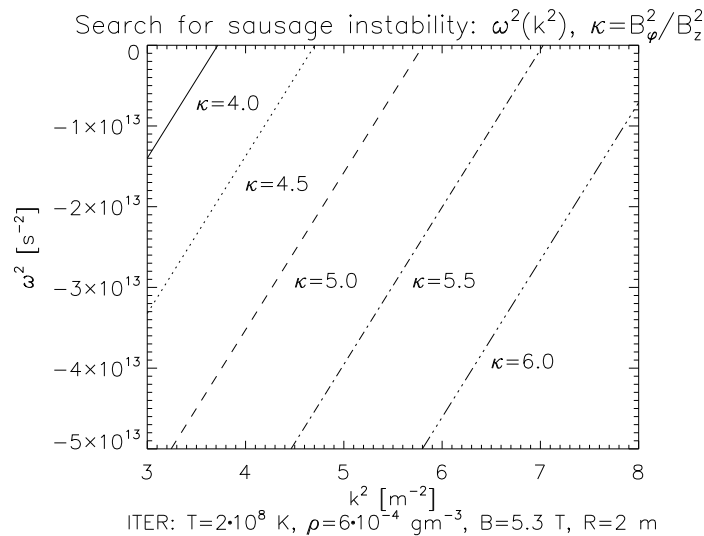


Figure 4. Dispersion relation of waves in an ITER-type plasma at different ratios of azimuthal to axial magnetic field B_ϕ/B_z . Solution for $\alpha R > 1$. At $\omega^2 < 0$, the sausage instability is excited.

has to be satisfied. In case of the ITER plasma with $v_A > v_s$, eq. (47) may be approximated by

$$k^2 < \frac{\kappa^2}{R^2} = \frac{\beta}{v_A^4}. \quad (48)$$

Thus instabilities in ITER-type plasmas at $\alpha R > 1$ should occur at wavelengths $\lambda > 2\pi R/\kappa = 4\pi/\kappa$ m. For $\kappa = 2$ (as studied before for $\alpha R < 1$) unstable waves have much smaller wavelengths, but they are larger than 6 m. And if κ increases, the wavelength will further decrease. In Figure 4, results of the numerical calculation of ω^2 as function of the square of the wave number k^2 are presented for different values of the parameter κ . To describe smaller wavelengths than that considered in Fig. 3 for $\alpha R < 1$, much larger k^2 -values are chosen. They correspond to wavelengths of about 3 m. Results are only shown for $\omega^2 < 0$, as then also $\alpha^2 > 0$. Indeed for the k^2 -interval studied, $\alpha R = 1$ is satisfied at $\omega^2 \approx 10^{12} - 10^{15} \text{ s}^{-2}$. For stable plasmas with $\omega^2 > 0$, values of α^2 below zero occur. From Figure 4 follows, that, in contradiction to the case $\alpha R < 1$, under the condition of $\alpha R > 1$ instabilities may be obtained for all considered κ -parameters, but they do not occur at any wave number k . The smaller the wave number the stronger the instability. The transition from the stable to the unstable ITER-type plasmas at large αR has to be described using Bessel functions of complex arguments. This will be done in a future work.

6. Lightnings

In nature, pinches occur very often in connection with the appearance of lightnings. Here, temperatures up to some 30000 K are found, and electrical currents are of the order of 40 kA [10]. The lifetime of lightnings amounts to 1 ms. The discharge radii with values of a few centimeters are rather small. This results into magnetic fields of about 0.4 T and a magnetic pressure of 0.6 atm (60795 Pa). Thus, at MHD pressure balance, the pinch pressure is about 1.6 atm (162120 Pa) and the pinch Alfvén speed has a value of 4000 m/s [10].

Concerning the plasma model considered in the present work, the \vec{B}_z in the inner plasma core, i.e. the hot lightning plasma, equals 0.4 nT. Of course, the hot lightning plasma is not surrounded

by a vacuum, but by a plasma with much lower ionization intensity, which is here approximated by a vacuum. The magnetic field of the surrounding plasma is the geomagnetic one with an intensity of $5 \cdot 10^{-5}$ T. In the Earth's lower atmosphere, always an electrical field exists, which is directed vertically to the Earth's surface. It causes field-aligned electrical currents which may generate azimuthal magnetic fields. It is known that thunderclouds mainly help to conserve the vertical electrical field of the Earth's atmosphere. But lightnings may be indeed directed to the Earth or in the opposite direction, so they may increase or decrease the normal electric field locally. In the Earth's atmosphere also horizontal electric fields are observed. But these fields are not taken into account in the present model. Besides, the gravitational acceleration is neglected.

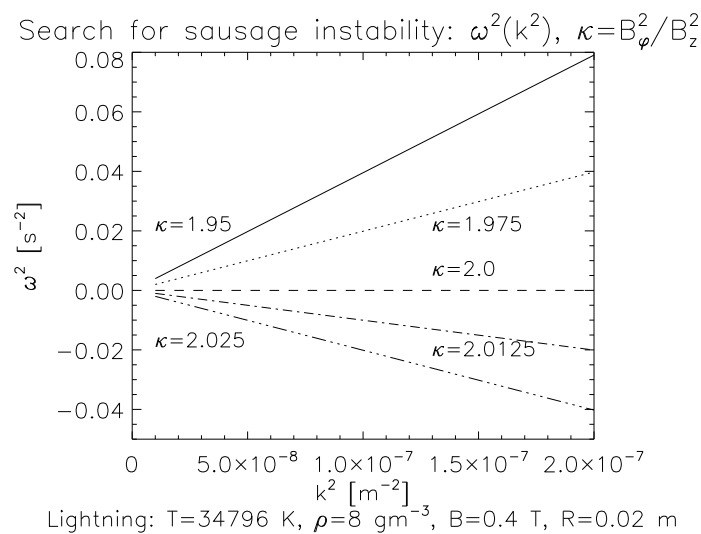


Figure 5. Dispersion relation of waves in a lightning plasma at different ratios of azimuthal to axial magnetic field B_ϕ/B_z . At $\omega^2 < 0$, the sausage instability is excited.

For the above mentioned plasma parameters in a lightning, the dispersion equation (43) is solved and the results are presented in Fig. 5 for the same κ -parameters and wave numbers k as have been chosen for the ITER study at $\alpha R < 1$ in section 5. It is to be seen, that again under the condition $\kappa > 2$, the plasma becomes unstable. But the square of the frequencies of the waves ω^2 is about 6 orders of magnitude smaller than in the case of the ITER plasma. Besides, ω^2 increases more slowly with the wave number than in the ITER case. The parameter αR of the Bessel function is smaller than unity in the considered plasma region. Thus, the radial displacement of the plasma fluid in the lightning was found using eq. (37). The results are shown in Fig. 6. One may conclude that the amplitudes of the radial plasma displacements grow with increasing wave number k . In the studied plasma parameter region they have a maximum value of 10 m.

7. Conclusions

The magnetohydrodynamic energy principle is presented for an internally homogeneous pinch and applied to ITER-type and lightning systems. In doing so, a system of mathematical equations of motion for fluid elements (eqs. (23-25)) is derived. The expressions are written in a cylindrical coordinate system, but they may be also applied to plasmas with disturbances of non-cylindrical symmetry. From the equations of motion, analytical formulae to be solved to

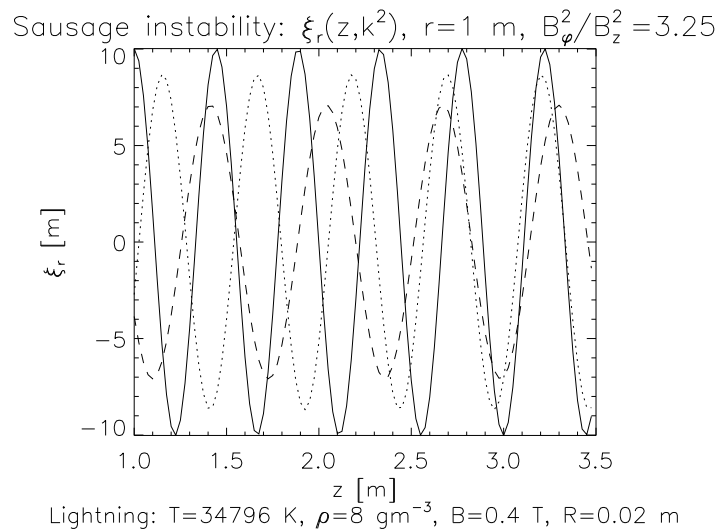


Figure 6. Radial displacement in a lightning as function of the coordinate in direction of the local axial magnetic field. The ratio of azimuthal to axial magnetic field equals $\kappa = 1.8$, that means a sausage instability is excited. $\omega^2 = -2.8 \cdot 10^9$ s $^{-2}$. — $k^2 = 200$ m $^{-1}$, $k^2 = 150$ m $^{-1}$, - - - $k^2 = 100$ m $^{-1}$.

obtain the radial fluid displacements and the wave dispersion are presented, which describe the excitation of magnetohydrodynamic waves, e.g. of sausage and kink ones.

To simplify the first numerical solutions presented in the present work, the obtained dispersion relation is only solved for plasma disturbances with cylindrical symmetry and homogeneous axial magnetic fields in the inner plasma core. Using the model of a pinch plasma separated from the outer conducting wall by a vacuum, values of the ratio of the inner axial to the external azimuthal magnetic fields are found, at which a sausage instability will occur. Numerical solutions of the dispersion equation are presented for stable and unstable systems within the limit of small values of αR , where α is a function of the wave frequency and wave number, and R describes the radial dimension of the plasma system. In the case of ITER-type plasmas, small αR correspond to rather large wavelengths in unstable systems. Thus, to describe more realistic technical problems, the calculations were further extended to the region of $\alpha R > 1$. Under such conditions, in dependence on the wave number, unstable waves with wavelengths of the order of 2-3 m occur. But therefore the external azimuthal magnetic field at the outer wall has to be two times stronger than the inner axial magnetic field. Thus the performed numerical analysis shows the importance of the outer boundary conditions of a plasma for the excitation of instabilities in the system. Besides, the results may help to chose such magnetic field ratios that sausage instabilities will not appear in ITER-type plasmas.

Further, the model of the pinch plasma separated from the outer conducting wall by a vacuum is applied to lightnigs in the Earth's atmosphere. In this case, the mathematical approximations made for $\alpha R < 1$ are more appropriate, and it was possible to calculate the radial displacements in the unstable plasma. But in cases of planetary atmospheres, the lightning plasma with the larger degree of ionization is not bounded to a vacuum, but to other material with a lower degree of ionization. Besides, at lightning altitudes particle collisions have to be taken into account in the plasma model, which was not done here. Therefore, the magnetospheric energy principle has to be further developed for weakly-collisional, partially ionized systems in future. Some steps to determine instabilities in the Earth's atmosphere taking also weak collisions between charged

and neutral particles into account were already performed by the authors and presented in [11] and [12] (misprints occur in eqs. (35-37) of [12], Fig. 2 of [12] is correct). There the MHD energy principle was not used.

Last not least, consequences of an extended magnetospheric energy principle derived by A. Miura [6], that means of a principle which one recently tries to apply for much larger scales than the lightning ones in the Earth's magnetosphere, are briefly reviewed in the present work.

8. Acknowledgments

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