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Theory for Quartet Condensation in Fermi Systems with Applications to Nuclei and Nuclear Matter

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Abstract. The theory of quartet condensation is further developed. The onset of quartetting in homogeneous fermionic matter is studied with the help of an in-medium modified four fermion equation. It is found that at very low density quartetting wins over pairing. At zero temperature, in analogy to pairing, a set of equations for the quartet order parameter is given. Contrary to pairing, quartetting only exists for strong coupling and breaks down for weak coupling. Reasons for this finding are detailed. In an application to nuclear matter, the critical temperature for α particle condensation can reach values up to around 8 MeV. The disappearance of α-particles with increasing density, i.e. the Mott transition, is investigated. In finite nuclei the Hoyle state, that is the ${0_2}^+$ of ^{12}C is identified as an 'α-particle condensate' state. It is conjectured that such states also exist in heavier $n\alpha$-nuclei, like ^{16}O, ^{20}Ne, etc. The sixth 0^+ state in ^{16}O is proposed as an analogue to the Hoyle state. The Gross-Pitaevski equation is employed to make an estimate of the maximum number of α particles a condensate state can contain. Possible quartet condensation in other systems is discussed briefly.

1. Introduction

One of the most amazing phenomena in quantum many-particle systems is the formation of quantum condensates. At present, the formation of condensates is of particular interest in strongly coupled fermion systems in which the crossover from Bardeen-Cooper-Schrieffer (BCS) pairing to Bose-Einstein condensation (BEC) may be investigated. Among very different quantum systems such as the electron-hole exciton and bi-exciton systems in excited semiconductors, atoms in traps at extremely low temperatures, etc., nuclear matter is especially well suited for the study of correlation effects in a quantum liquid. However, more exotic systems like the dense hydrogen gas may also feature condensation phenomena of electron-electron,

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proton-proton, and electron-proton pairs [1] where the strongly bound hydrogen molecule can be thought of as a candidate for condensation in the same context. Coming to nuclear systems, neutron matter, nuclear matter, but also finite nuclei are superfluid. However, at low density, nuclear matter will not cluster into pairs, i.e. deuterons, but rather into α -particles which are much more stable. Also heavier clusters, starting with Carbon, may be of importance but are presently not considered for condensation phenomena. Therefore, one may ask the question whether there exists quartetting, i.e. α-particle condensation, in nuclei, analogous to nuclear pairing. The only nucleus which in its ground state has a pronounced α -cluster structure is ^{8}Be. In section 5 we will show a figure of ^{8}Be in the laboratory frame and in the intrinsic deformed frame. We will see that ^{8}Be is formed out of two almost free α - particles roughly 4 fm apart, only weakly overlapping with their surface tails. Actually ^{8}Be is slightly unstable and the two α's only hold together via the Coulomb barrier. Because of the large distance of the two α-particles, the 0^+ ground state of ^{8}Be has, in the laboratory frame, a spherical density distribution whose average is very low: about 1/3 of ordinary saturation density ρ_0. ^{8}Be is, therefore, a very large object with an rms radius of about 3.7 fm to be compared with the nuclear systematics of $R = r_0 A^{1/3} = 2.44$ fm. Definitely ^{8}Be is a rather unusual and, in its kind, unique nucleus. One may ask the question what happens when one brings a third α-particle alongside of ^{8}Be. We know the answer: the 3α system collapses to the ground state of ^{12}C which is much denser than ^{8}Be and can not accommodate, with its small radius of 2.4 fm, three more or less free α-particles barely touching one another. One nevertheless may ask the question whether the dilute three α configuration ^{8}Be- α, or $\alpha - \alpha - \alpha$, may not form an isomeric or excited state of ^{12}C. That such a state indeed exists will be one of the main subjects of our considerations. Once one accepts the idea of the existence of an α-gas state in ^{12}C, there is no reason why equivalent states at low density should not also exist in heavier $n\alpha$-nuclei, like ^{16}O, ^{20}Ne, etc. In a mean field picture, i.e. all α's being ideal bosons (in this context remember that the first excited state of an α-particle is at ~ 20 MeV, by factors higher than in all other nuclei), all α's will occupy the lowest 0S-state, i.e. they will condense into this state. This forms, of course, not a macroscopic condensate but it can be understood in the same sense as we know that nuclei are superfluid because of the presence of a finite number of Cooper pairs. These α condensate states are generally close to the α disintegration threshold and, therefore, at higher and higher excitation energy as the number of α's increases. One may think that, as a consequence, these states decay very fast but due to their unusual structure, they couple little to ordinary excited states and will, thus, have an unusual long life time. On the other hand, for example during the cooling process of compact stars [2], where one predicts the presence of α-particles [3], a real macroscopic phase of condensed α's may be formed. In the present contribution we will mainly concentrate on nuclear systems but we also can think about the possibility of quartetting in other Fermi-systems, as mentioned above. One should, however, keep in mind that a pre-requisite for its existence is, as in nuclear physics, that there exists a bound quartet in free space. This is facilitated with the existence of four different types of fermions, all attracting one another. For example to form quartets with cold atoms one could try to trap fermions in four different magnetic substates, a task which eventually seems possible [4]. In the next section we will outline the general theory for quartet condensation and in section 3 we investigate how the binding energy of various nuclear clusters changes with density as well as the critical temperature of α-particle condensation in infinite matter via an in- medium four-nucleon equation (Thouless criterion). In section 4, we give some results concerning α condensation in infinite nuclear matter, and in section 5 we treat α-particle condensation in finite nuclei. Finally in section 6 we conclude with an outlook and further discussions.

2. General Theory for Quartetting

The aim will be to develope a theory for quartet condensation which in many aspects is analogous to the BCS approach for the condensation of pairs. It is, however, evident that a microscopic theory for quartet condensation, involving a highly correlated four fermion cluster, is at least by an order of magnitude more complicated than the pairing case. It also will turn out that the physics is quite different, that is, we will find that quartet condensation only exists in the BEC phase and no quartet condensation in weak coupling with a very long coherence length, the analogue to the BCS phase of pairs, is possible. We will dwell on this aspect in quite some detail. In order to set the frame of our approach, let us start to repeat the BCS approach for pairing. It is well known that the BCS ground state wave function can be written as a coherent state [5]

$$|\mathrm{BCS}\rangle = e^{\sum_{k<k'} z_{kk'} c_k^+ c_{k'}^+} |\mathrm{vac}\rangle \tag{1}$$

with c^+, c fermion creation and annihilation operators and indices k implying momentum vector, spin, and, eventually other quantum numbers. Standard singlet pairing considers $k' = \bar{k}$, i.e. the pair is at rest and the two fermions occupy time reversed states k and $\bar{k}$. The BCS ground state is the vacuum to the quasiparticle destructors $\beta_k = u_k c_k - v_k c_{\bar{k}}^+$ (with $u_k^2 + v_k^2 = 1$), that is

$$\beta_k |\mathrm{BCS}\rangle = 0 \tag{2}$$

where the standard relation $z_{k\bar{k}} = v_k/u_k$ is to be used. The BCS equations can then be obtained by minimising the following mean single particle energy (which can be identified with an energy weighted sum rule [5])

$$e_k = \frac{\langle \mathrm{BCS}|\{\beta_k, [H - \mu\hat{N}, \beta_k^+]\}\mathrm{BCS}\rangle}{\langle \mathrm{BCS}|\{\beta_k, \beta_k^+\}|\mathrm{BCS}\rangle} \tag{3}$$

where $\{.,.\}$ and $[.,.]$ are anticommutators and commutators, respectively and $\hat{N}$ and μ are particle number operator and chemical potential. The Hamiltonian is given by

$$H = \sum \varepsilon_k^0 c_k^+ c_k + \frac{1}{4} \sum_{k_1 k_2 k_3 k_4} \bar{v}_{k_1 k_2 k_3 k_4} c_{k_1}^+ c_{k_2}^+ c_{k_4} c_{k_3} \tag{4}$$

with $\varepsilon_k^0 = k^2/(2m)$ the kinetic energy and $\bar{v}_{k_1 k_2 k_3 k_4}$ the antisymmetrised matrix element of the two body interaction. Since the quasiparticle operators β^+, β are obtained from a canonical transformation among the c^+, c operators, the transformation can be inverted and the c^+, c expressed in terms of β^+, β. Using the killing condition, all expectation values contained in the variational equations $\delta e_k = 0$ can then be expressed in terms of the u, v amplitudes leading directly to the usual coupled BCS equations for u, v. For later reasons of comparison with the quartet case, let us give the BCS in a particular form eliminating the v-amplitudes

$$\xi_k u_k + \frac{|\Delta_k|^2}{E_k + \xi_k} u_k = E_k u_k \tag{5}$$

where $E_k = \sqrt{\xi_k^2 + \Delta_k^2}$ is the quasi-particle energy, $\Delta_k = \sum_{k'} V_{kk'} u_{k'} v_{k'}$ the usual gap function with $V_{kk'}$ the angle averaged two body force in momentum space, and $\xi_k = \varepsilon_k - \mu$ with

$$\varepsilon_k = \varepsilon_k^0 + \sum_{k'} \bar{v}_{kk'kk'} n_{k'} \tag{6}$$

being the Hartree-Fock (HF) single particle energies where $n_k = v_k^2 = 1 - u_k^2$ are the single particle occupation numbers. We also want to give the equation for the pairing order parameter $\kappa_k \equiv \langle c_{\bar{k}} c_k \rangle = u_k v_k$ which is equivalent to the standard gap equation

$$2\varepsilon_k \kappa_k - (1 - 2n_k) \sum_{k'} V_{kk'} \kappa_{k'} = 2\mu \kappa_k \tag{7}$$

It is seen that (7) looks similar to a two-body Schrödinger equation in momentum space, with eigenvalue 2μ and the interaction modified by the Pauli blocking factor $(1 - 2n_k)$. We will see that in the case of quartetting, we will get two coupled equations, analogous to (5) and (7).

So far for the BCS approach concerning pairs. Let us try to set up an analogous procedure for quartets. Obviously we should write for the wave function

$$|Z\rangle = e^{\frac{1}{4!}\sum_{k_1k_2k_3k_4} Z_{k_1k_2k_3k_4} c^+_{k_1} c^+_{k_2} c^+_{k_3} c^+_{k_4}} |\text{vac}\rangle \tag{8}$$

where the quartet amplitudes Z are fully antisymmetric (symmetric) with respect to an odd (even) permutation of the indices. The task will now be to find a killing operator for this quartet condensate state. Whereas in the pairing case the partitioning of the pair operator into a linear combination of a fermion creator and a fermion destructor is unambiguous, in the quartet case there exist two ways to partition the quartet operator, that is into a single plus a triple or into two doubles. Let us start with the superposition of a single and a triple. As a matter of fact it is easy to show that (in the following, we always will assume that all amplitudes are real)

$$q_\nu = u^\nu_{k_1} c_{k_1} - \frac{1}{3!} \sum v^\nu_{k_2k_3k_4} c^+_{k_1} c^+_{k_2} c^+_{k_3} \tag{9}$$

kills the quartet state under the condition

$$Z_{k_1k_2k_3k_4} = \sum_\nu (u^{-1})^\nu_{k_1} v^\nu_{k_2k_3k_4} \tag{10}$$

However, so far, we barely have gained anything, since above quartet destructor contains a non-linear fermion transformation which, a priory, cannot be handled. Therefore, let us try with a superposition of two fermion pair operators which is, in a way, the natural extension of the Bogoliubov transformation in the pairing case, i.e. with $Q = \sum [XP - YP^+]$ where $P^+ = c^+c^+$ is a fermion pair creator. We will, however, find out that such an operator cannot kill the quartet state of Eq. (8). In analogy to the so-called Self-Consistent RPA (SCRPA) approach [6], we will introduce a slightly more general operator, that is

$$Q_\nu = \sum_{k<k'} [X^\nu_{kk'} c_k c_{k'} - Y^\nu_{kk'} c^+_{k'} c^+_k] - \sum_{k_1<k_2<k_3k_4} \eta^\nu_{k_1k_2k_3k_4} c^+_{k_1} c^+_{k_2} c^+_{k_3} c_{k_4} \tag{11}$$

with X, Y antisymmetric in k, k'. Applying this operator on our quartet state, we find $Q_\nu|Z\rangle = 0$ where the relations between the various amplitudes turn out to be

$$\sum_{k<k'} X^\nu_{kk'} Z_{kk'll'} = Y^\nu_{ll'} \qquad \text{and} \qquad \eta^\nu_{l_2l_3l_4;k'} = \sum_k X^\nu_{kk'} Z_{kl_2l_3l_4} \tag{12}$$

These relations are quite analogous to the ones which hold in the case of the SCRPA approach [6]. One also notices that the relation between X, Y, Z amplitudes is similar in structure to the one of BCS theory for pairing. As with SCRPA, in order to proceed, we have to approximate the additional η-term. The quite suggestive recipe is to replace in the η-term of Eq. (11) the density operator $c^+_{k'} c_k$ by its mean value $\langle Z|c^+_{k'} c_k|Z\rangle / \langle Z|Z\rangle \equiv \langle c^+_{k'} c_k \rangle = \delta_{kk'} n_k$,

i.e. $c^+_{k_1}c^+_{k_2}c^+_{k_3}c_{k_4} \to c^+_{k_1}c^+_{k_2}n_{k_3}\delta_{k_3k_4}$ where we supposed that we work in the basis where the single particle density matrix is diagonal, that is, it is given by the occupation probabilities n_k. This approximation, of course, violates the Pauli principle but, as it was found in applications of SCRPA [6], we suppose that also here this violation will be quite mild (of the order of a couple of percent). With this approximation, we see that the η-term only renormalises the Y amplitudes and, thus, the killing operator boils down to a linear super position of a fermion pair destructor with a pair creator. This can then be seen as a Hartree-Fock-Bogoliubov (HFB) transformation of fermion pair operators, i.e., pairing of 'pairs'. Replacing the pair operators by ideal bosons as done in RPA, would lead to a standard bosonic HFB approach [7] (see also, [5],ch.9). Here, however, we will stay with the fermionic description and elaborate an HFB theory for fermion pairs. For this, we will suppose that we can use the killing property $Q_\nu|Z\rangle = 0$ even with the approximate Q-operator. As with our experience from SCRPA, we assume that this violation of consistency is weak.

Let us continue with elaborating our just defined frame. We will then use for the pair-killing operator

$$Q_\nu = \sum_{k<k'}[X^\nu_{kk'}c_kc_{k'} - Y^\nu_{kk'}c^+_{k'}c^+_k]/N^{1/2}_{kk'} \tag{13}$$

with (the approximate) property $Q|Z\rangle = 0$ and the first relation in (12). The normalisation factor $N_{kk'} = |1 - n_k - n_{k'}|$ has been introduced so that $< [Q, Q^+] >= \frac{1}{2}\sum(X^2 - Y^2) = 1$, i.e., the quasi-pair state $Q^+|Z\rangle$ and the X, Y amplitudes being normalised to one. In analogy with (3), we now will minimise the following energy weighted sum rule

$$\Omega_\nu = \frac{\langle Z|[Q_\nu, [H - 2\mu\hat{N}, Q^+_\nu]]|Z\rangle}{\langle Z|[Q_\nu, Q^+_\nu]|Z\rangle} \tag{14}$$

The minimisation with respect to X, Y amplitudes leads to

$$\begin{pmatrix} \mathbf{H} & \mathbf{\Delta}^{(22)} \\ -\mathbf{\Delta}^{(22)+} & -\mathbf{H}^* \end{pmatrix} \begin{pmatrix} X^\nu \\ Y^\nu \end{pmatrix} = \Omega_\nu \begin{pmatrix} X^\nu \\ Y^\nu \end{pmatrix} \tag{15}$$

with (we eventually will consider a symmetrized double commutator in $\mathbf{H}$)

$$\begin{aligned} \mathbf{H}_{k_1k_2,k'_1k'_2} &= \langle[c_{k_2}c_{k_1}, [H - 2\mu\hat{N}, c^+_{k'_1}c^+_{k'_2}]]\rangle/(N^{1/2}_{k_1k_2}N^{1/2}_{k'_1k'_2}) \\ &= (\xi_{k_1} + \xi_{k_2})\delta k_1k_2, k'_1k'_2 + N^{-1/2}_{k_1k_2}N^{-1/2}_{k'_1k'_2}\{N_{k_1k_2}\bar{v}_{k_1k_2k'_1k'_2}N_{k'_1k'_2} \\ &+ [(\frac{1}{2}\delta_{k_1k'_1}\bar{v}_{l_1k_2l_3l_4}C_{l_3l_4k'_2l_1} + \bar{v}_{l_1k_2l_4k'_2}C_{l_4k_1l_1k'_1}) - (k_1 \leftrightarrow k_2)] - [k'_1 \leftrightarrow k'_2]\} \end{aligned} \tag{16}$$

where

$$C_{k_1k_2k'_1k'_2} = \langle c^+_{k'_1}c^+_{k'_2}c_{k_2}c_{k_1}\rangle - n_{k_1}n_{k_2}\delta_{k_1k_2k'_1k'_2} \tag{17}$$

is the two body correlation function and

$$\begin{aligned} \mathbf{\Delta}^{(22)}_{k_1k_2,k'_1k'_2} &= -\langle[c_{k_2}c_{k_1}, [H - 2\mu\hat{N}, c_{k'_1}c_{k'_2}]]\rangle/(N^{1/2}_{k_1k_2}N^{1/2}_{k'_1k'_2}) \\ &= N^{-1/2}_{k_1k_2}[(\Delta_{k_1k'_2;k'_1k_2} - k_1 \leftrightarrow k_2) - (k'_1 \leftrightarrow k'_2)]N^{-1/2}_{k'_1k'_2} \end{aligned} \tag{18}$$

with

$$\Delta_{k_1k_2';k_1'k_2} = \sum_{l<l'} \bar{v}_{k_1k_2'll'} \langle c_{k_1'} c_{k_2} c_{l'} c_l \rangle \tag{19}$$

In (15) the matrix multiplication is to be understood as $\sum_{k_1'<k_2'}$ for restricted summation (or as $\frac{1}{2}\sum_{k_1'k_2'}$ for unrestricted summation) . We see from (18) and (19) that the bosonic gap $\mathbf{\Delta}^{(22)}$ involves the quartet order parameter quite in analogy to the usual gap field in the BCS case. The $\mathbf{H}$ operator in (15) has already been discussed in [8] in connection with SCRPA in the particle-particle channel. Equation (15) has the typical structure of a bosonic HFB equation but, here, for fermion pairs, instead of bosons. It remains the task to close those HFB equations in expressing all expectation values involved in the $\mathbf{H}$ and $\mathbf{\Delta}^{(22)}$ fields by the X, Y amplitudes. This goes in the following way. Because of the HFB structure of (15), the X, Y amplitudes obey the usual orthonormality relations, see [5]. Therefore, one can invert relation (13) to obtain

$$c^+_{k'} c^+_k = N^{1/2}_{kk'} \sum_\nu [X^\nu_{kk'} Q^+_\nu + Y^\nu Q_\nu] \qquad (k < k') \tag{20}$$

and by conjugation the expression for cc. With this relation, we can calculate all two body correlation functions in (18) and (16) in terms of X, Y amplitudes. This is achieved in commuting the destruction operators Q to the right hand side and use the killing property. For example, the quartet order parameter in the gap-field (19) is obtained as $\langle c_{k_1'} c_{k_2} c_{l'} c_l \rangle = N^{1/2}_{k_1'k_2} \sum_\nu X^\nu_{k_2k_1'} Y^\nu_{ll'} N^{1/2}_{ll'}$. Remains the task how to link the occupation numbers $n_k = \langle c^+_k c_k \rangle$ to the X, Y amplitudes. Of course, that is where our partitioning of the quartet operator into singles and triples comes into play. Therefore, let us try to work with the operator (9). First, as a side-remark, let us notice that if in (9) we replace $c^+_{k_1} c^+_{k_2}$ by its expectation value which is the pairing tensor, we are back to the standard Bogoliubov transformation for pairing. Here we want to consider quartetting and, thus, we have to keep the triple operator fully. Minimising, as in (3) an average single particle energy, we arrive at the following equation for the amplitudes u, v in (9)

$$\begin{pmatrix} \xi & \mathbf{\Delta}^{(13)} \\ \mathbf{\Delta}^{(13)+} & -\mathcal{N}\mathcal{H}^* \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = E \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{N} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{21}$$

with (we disregard pairing, i.e., $\langle cc \rangle$ amplitudes)

$$\mathbf{\Delta}^{(13)}_{k;k_1k_2k_3} = \Delta_{kk_3;k_2k_1} - [(k_2 \leftrightarrow k_3) - (k_1 \leftrightarrow k_2)] \tag{22}$$

and

$$(\mathcal{N}\mathcal{H}^*)_{k_1k_2k_3;k_1'k_2'k_3'} = \langle \{ c^+_{k_3} c^+_{k_2} c^+_{k_1}, [H - 3\mu\hat{N}, c_{k_1'} c_{k_2'} c_{k_3'}] \} \rangle \tag{23}$$

$$\mathcal{N}_{k_1k_2k_3;k_1'k_2'k_3'} = \langle \{ c^+_{k_3} c^+_{k_2} c^+_{k_1}, c_{k_1'} c_{k_2'} c_{k_3'} \} \rangle \tag{24}$$

We will not give $\mathcal{H}$ in full because it is a very complicated expression involving self-consistent determination of three-body densities. To lowest order in the interaction it is given by

$$\mathcal{H}_{k_1k_2k_3;k_1'k_2'k_3'} = (\xi_{k_1} + \xi_{k_2} + \xi_{k_3}) \delta_{k_1k_2k_3,k_1'k_2'k_3'} + [(1 - n_{k_1} - n_{k_2}) \bar{v}_{k_1k_2k_1'k_2'} \delta_{k_3k_3'} + \text{ permutations}] \tag{25}$$

where $\delta_{k_1k_2k_3,k_1'k_2'k_3'}$ is the fully antisymmetrised three fermion Kronecker symbol. Even this operator is still rather complicated for numerical applications and mostly one will replace the

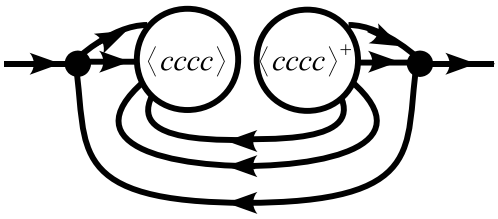


Figure 1. Schematic representation of self-energy for α particle condensation in an uncorrelated nucleon gas.

correlated occupation numbers by their free Fermi- Dirac steps, i.e., $n_k \to n_k^0$. To this order the three body norm in (24) is given by

$$\mathcal{N}_{k_1k_2k_3;k_1'k_2'k_3'} \simeq [\bar{n}_{k_1}^0 \bar{n}_{k_2}^0 \bar{n}_{k_3}^0 + n_{k_1}^0 n_{k_2}^0 n_{k_3}^0]\delta_{k_1k_2k_3,k_1'k_2'k_3'} \tag{26}$$

with $\bar{n}^0 = 1 - n^0$. In principle this effective three-body Hamiltonian leads to three-body bound and scattering states. In our application to nuclear matter given below, we will make an even more drastic approximation and completely neglect the interaction term in the three-body Hamiltonian. Eliminating under this condition the v-amplitudes from (21), one can write down the following effective single particle equation

$$\xi_k u_k^{(\nu)} + \sum_{k_1<k_2<k_3k'} \frac{\mathbf{\Delta}_{k,k_1k_2k_3}^{(13)}(\bar{n}_{k_1}^0 \bar{n}_{k_2}^0 \bar{n}_{k_3}^0 + n_{k_1}^0 n_{k_2}^0 n_{k_3}^0)\mathbf{\Delta}_{k_3k_2k_1k'}^{(13)^*}}{E_\nu + \xi_{k_1} + \xi_{k_2} + \xi_{k_3}} u_{k'}^{(\nu)} = E_\nu u_k^{(\nu)} \tag{27}$$

The occupation numbers are given by

$$n_k = 1 - \sum_\nu |u_k^{(\nu)}|^2 \tag{28}$$

The effective single particle field in (27) is grapphically interpreted in Fig. 1. The gap-fields in (27) are then to be calculated as in (22) and (19) with (20) and the system of equations is fully closed. This is quite in parallel to the pairing case. In cases, where the quartet consists out of four different fermions and in addition is rather strongly bound, as this will be the case for the α particle in nuclear physics, one still can make a very good but drastic simplification: one writes the quartic order parameter as a translationally invariant product of four times the same single particle wave function in momentum space. We will see, how this goes below when we apply our theory to α particle condensation in nuclear matter. Comparing the effective single particle field in (27) with the one of standard pairing, Eq. (5), we find strong analogies but also several structural differences. The most striking is that in the quartet case Pauli factors figure in the numerator of (27) whereas this is not the case for pairing. In principle in the pairing case, they are also there, but since $\bar{n}_k + n_k = 1$, they drop out. This difference has quite dramatic consequences between the pairing and the quartetting case. Namely when the chemical potential μ changes from negative (binding) to positive, the implicit three hole level density

$$g_{3h}(\omega) = \sum_{k_1k_2k_3} (\bar{n}_{k_1}^0 \bar{n}_{k_2}^0 \bar{n}_{k_3}^0 + n_{k_1}^0 n_{k_2}^0 n_{k_3}^0)\delta(\omega + \xi_{k_1} + \xi_{k_2} + \xi_{k_3}) \tag{29}$$

passes through zero at $\omega - 3\mu = 0$ because phase space constraints and energy conservation cannot be fullfilled simultaneously at that point. This is not at all the case for the single particle level density

$$g_{1h}(\omega) = \sum_k (\bar{n}_k^0 + n_k^0)\delta(\omega + \xi_k) \quad = \quad \sum_k \delta(\omega - \xi_k) \tag{30}$$

entering implicitly in the pairing case, since, as indicated $\bar{n}_k^0 + n_k^0 = 1$. Therefore, for positive μ, in the case of quartetting essentially no correlations around the Fermi energy ($\omega - 3\mu = 0$) can build up and, therefore, there is no quartet condensation for positive chemical potential. Quartetting only exists on the BEC side where μ is negative and, thus, $n_k^0 = 0$ and no structural difference exists between the pairing case and the quartetting one for $\mu < 0$. We will show explicit examples of level densities in section 4.

So far, we have discarded possible existence of ordinary pairing mixed with quartetting. Formally, this can easily be achieved in writing everywhere quasiparticle operators instead of particle ones. For example the ground state (8) is then given by

$$|Z\rangle = e^{\frac{1}{4!}\sum Z_{k_1k_2k_3k_4}\beta^+_{k_1}\beta^+_{k_2}\beta^+_{k_3}\beta^+_{k_4}}|\mathrm{BCS}\rangle \tag{31}$$

and the killing operator

$$Q_\nu = \sum_{k<k'} [X^\nu \beta_{k'}\beta_k - Y^\nu_{kk'}\beta^+_k\beta^+_{k'}]/N_{kk'} \tag{32}$$

and analogously for the superposition of singles with triples. Of course, if the amplitudes Z in (31) are zero, we are back to ordinary BCS theory of pairs. All the algebra goes through as before, only we have to deal in the equations of motion with the Hamiltonian in the quasi-particle representation (see [5], App. E). This description of quartetting with quasiparticles shall be worked out in the future. It may help to understand the precise nature of the transition from quartetting to pairing when μ changes from negative to positive values. It is worth noticing that (32) is exactly the ansatz used to derive quasi-particle RPA (QRPA) [5] what implies a linearised equation of motion. Extending to self-consistent QRPA [6] will describe pairing and quartetting consistently within the same formalism.

Of course, the description of quartetting can also be obtained from an equivalent Gorkov-type or Green's function approach. Since this may shed a complementary light on the theory, we very briefly want to sketch how this goes. Let us intruduce two types of matrix Green's functions

$$\mathcal{G}^{(13)} = \begin{pmatrix} G^{(11)} & G^{(13)} \\ G^{(13)+} & G^{(33)} \end{pmatrix} \tag{33}$$

with the usual definition of time ordered Green's functions at zero temperature to be found, e.g., in [9],[5]

$$G^{(11)}_{kk'} = -i\langle T c_k(t) c^+_{k'}(t')\rangle; \qquad G^{(13)}_{k;k'_1k'_2k'_3} = -i\langle T c_k(t)(c_{k'_1}c_{k'_2}c_{k'_3})_{t'}\rangle$$

$$G^{(13)}_{k_1k_2k_3;k'} = i\langle T(c^+_{k_3}c^+_{k_2}c^+_{k_1})_t c^+_{k'}(t')\rangle; \qquad G^{(33)}_{k_1k_2k_3;k'_1k'_2k'_3} = i\langle T(c^+_{k_3}c^+_{k_2}c^+_{k_1})_t(c_{k'_1}c_{k'_2}c_{k'_3})_{t'}\rangle$$

and

$$\mathcal{G}^{(22)} = \begin{pmatrix} G^{(22)}_{-+} & G^{(22)}_{--} \\ G^{(22)}_{++} & G^{(22)}_{+-} \end{pmatrix} \tag{34}$$

with

$$G^{(22)}_{-+,\ k_1k_2;k'_1k'_2} = -i\langle P^{(-)}_{k_1k_2}(t) P^{(+)}_{k'_1k'_2}(t')\rangle$$

and $P^{(-)}_{k_1k_2} \equiv c_{k_1}c_{k_2}$, and $P^{(+)} = P^{(-)^+}$. The other Green's functions in the matrix of (34) are defined analogously. With the equation of motion method for fermion cluster Green's functions, see [8], one then can write down a Dyson equation for these matrix Green's functions

$$\mathcal{G} = \mathcal{G}^{(0)} + \mathcal{G}^{(0)}\Sigma^{(0)}\mathcal{G} \tag{35}$$

where $\mathcal{G}$ is either the matrix Green's function (33) or (34) and $\Sigma^{(0)}$ is a corresponding instantaneous part of the exact self-energy. $\mathcal{G}^{(0)}$ is the free Green's function, as usual. The self-energies $\Sigma^{(0,22)}$ and $\Sigma^{(0,13)}$ can easily be read off from the effective 2x2 Hamiltonians in (15) and (21). The one and two body density matrices entering these Hamiltonians are then given by the residua of the various Green's functions and the system of equations is again closed. This is completely equivalent to the system of equations we established before, using the killing operators. The Green's function formalism has the advantage of easily being generalizable to finite temperature.

This ends the formal aspects of quartetting theory. We will give some further insight when we describe the applications to nuclear systems in the next sections.

3. Nuclear clusters in the medium and critical temperature of quartetting in nuclear matter

As mentioned in the preceding sections, we will apply our theory for quartetting to nuclear matter and also to finite nuclei. In this context it may be useful to study first the behaviour of a single fermion pair, e.g. the deuteron, and a single quartet, i.e., the α particle in a gas of uncorrelated nucleons. For example the in-medium deuteron equation is given by

$$[\varepsilon_{k_1} + \varepsilon_{k_2} - E_{d,P}]\,\psi_{d,P}(12) + \sum_{k_1'<k_2'} [1 - f_{k_1} - f_{k_2}]\;\bar{v}_{k_1k_2k_1'k_2'}\psi_{d,P}(1'2') = 0. \tag{36}$$

with P the total momentum, f_k the Fermi-Dirac distribution at finite temperature (equivalent to the previously defined Fermi step n_k^0 at zero temperature, and ε_k the HF single particle energies defined in (6). This *effective wave equation* describes bound states as well as scattering states. The onset of pair condensation is achieved when the binding energy $E_{d,P=0}$ coincides with 2μ.

Similar equations have been derived from the Green function approach for the case of nucleon numbers $A = 3$ and $A = 4$, describing triton/helion (^{3}He) nuclei as well as α-particles in nuclear matter. The effective wave equation contains in mean field approximation the Hartree-Fock self-energy shift of the single-particle energies as well as the Pauli blocking of the interaction. We give the effective wave equation for the α particle,

$$\begin{aligned}[\varepsilon_{k_1} + \varepsilon_{k_2} + \varepsilon_{k_3} + \varepsilon_{k_4} - E_{\alpha,P}]\psi_{\alpha,P}(1234) \quad &+ \quad \sum_{k_1'<k_2'} [1 - f_{k_1} - f_{k_2}]\bar{v}_{k_1k_2k_1'k_2'}\psi_{\alpha,P}(1'2'34) \\ &+ \quad \text{permutations} = 0.\end{aligned} \tag{37}$$

A similar equation is obtained for $A = 3$.

The effective wave equation has been solved using separable potentials for $A = 2$ by integration. For $A = 3, 4$ we can use a *Faddeev approach* [10]. The shifts of binding energy can also be calculated approximately via perturbation theory. In Fig. 2 we show the shift of the binding energy of the light clusters ($d, t/h$ and α) in symmetric nuclear matter as a function of density for temperature $T = 10$ MeV. It is found that the cluster binding energy decreases with increasing density. Finally, at the *Mott density* $\rho^{\text{Mott}}_{A,n,P}(T)$, the bound state is dissolved. The clusters are not present at higher densities, merging into the nucleonic medium. For a given cluster type characterized by A, n, we can also introduce the Mott momentum $P^{\text{Mott}}_{A,n}(\rho, T)$ in terms of the ambient

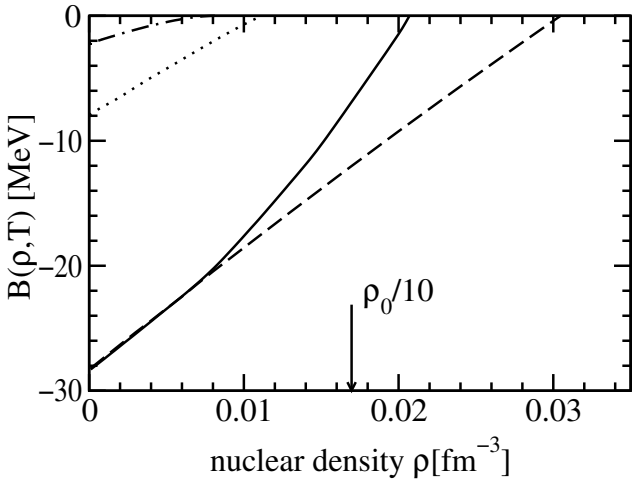


Figure 2. Shift of binding energies of the light clusters (d - dash dotted, t/h - dotted, and α - dashed: perturbation theory, full line : non-perturbative Faddeev-Yakubovski equation) in symmetric nuclear matter as a function of density for given temperature $T = 10$ MeV.

temperature T and nucleon density ρ, such that the bound states exist only for $P \geq P_{A,n}^{\rm Mott}(\rho, T)$.

In general, it is necessary to take account of *all bosonic clusters* to gain a complete picture of the onset of superfluidity. As is well known, the deuteron is weakly bound compared to other nuclei. Higher A-clusters can arise that are more stable. In this section, we will consider the formation of α-particles, which are of special importance because of their large binding energy per nucleon ($\sim$7 MeV). We will not include tritons or helions, which are fermions and not so tightly bound. Moreover, we will not consider nuclei in the iron region, which have even larger binding energy per nucleon than the α-particle and thus constitute, in principle, the dominant component at low temperatures and densities. However, the latter are complex structures of many particles and are strongly affected by the medium as the density increases, since their excitation spectrum starts at much lower energy than the one of the α particle, for instance. So they are assumed not to be of relevance in the density region considered here. Given the medium-modified bound-state energy $E_{4,P}$, the bound-state contribution to the EOS is

$$\rho_4(\beta, \mu) = \sum_P \left[e^{\beta(E_{4,P} - 2\mu_p - 2\mu_n)} - 1 \right]^{-1} . \tag{38}$$

We will not include the contribution of the excited states or that of scattering states. Because of the large specific binding energy of the α particle, low-density nuclear matter is predominantly composed of α particles. This observation underlies the concept of α matter and its relevance to diverse nuclear phenomena.

As exemplified by Eq. (37), the effect of the medium on the properties of an α particle in mean-field approximation (i.e., for an uncorrelated medium) is produced by the Hartree-Fock self-energy shift and Pauli blocking. The shift of the α-like bound state has been calculated using perturbation theory as well as by solution of the Faddeev-Yakubovski equation [10]. It is found that this bound state merges with the continuum of scattering states at a Mott density $\rho_\alpha^{\rm Mott} \approx \rho_0/10$, see Fig. 2. The bound states of clusters d, t, and h with $A < 4$ are already dissolved at the density $\rho_\alpha^{\rm Mott}$. Consequently, if we neglect the contribution of the four-particle scattering phase shifts in the different channels, we can now construct an equation of state $\rho(T, \mu) = \rho^{\rm free}(T, \mu) + \rho^{{\rm bound},d}(T, \mu) + \rho^{{\rm bound},\alpha}(T, \mu)$ so that α-particles determine the behavior of symmetric nuclear matter at densities below $\rho_\alpha^{\rm Mott}$ and temperatures below the binding energy per nucleon of the α-particle. The formation of deuteron clusters alone gives an incorrect description because the deuteron binding energy is small, and the abundance of d-clusters is small compared with that of α-clusters. In the low density region of the phase diagram, α-

matter emerges as an adequate model for describing the nuclear-matter equation of state. With increasing density, the medium modifications – especially Pauli blocking – will lead to a deviation of the critical temperature $T_c(\rho)$ from that of an ideal Bose gas of α-particles (the analogous situation holds for deuteron clusters, i.e., in the isospin-singlet channel). At a critical density which more or less coincides with the point where the chemical potential turns from negative to positive value, the quartet will be quite abruptly dissolved and no α particle survives. This is in line with the arguments given above concerning the level densities.

Symmetric nuclear matter is characterized by the equality of the proton and neutron chemical potentials, i.e., $\mu_p = \mu_n = \mu$. Then an extended Thouless condition based on the relation for the four-body in medium wave function, Eq. (37), at eigenvalue 4μ serves to determine the onset of Bose condensation of α-like clusters, noting that existence of a solution of this relation signals a divergence of the four-particle correlation function. An approximate solution has been obtained by a variational approach, in which the wave function is taken of the following projected mean field form [11] (see also [12] for another variational ansatz):

$$\psi(1234) = \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4 - \mathbf{K})\varphi(\mathbf{k}_1)\varphi(\mathbf{k}_2)\varphi(\mathbf{k}_3)\varphi(\mathbf{k}_4)\chi(1234) \tag{39}$$

where the φ's are single particle wave functions in momentum space to be determined variationally out of the in medium four body wave equation and $\chi(1234)$ is the scalar spin-isospin function. The delta function is a projector on total momentum $\mathbf{K}$ which, for condensates at rest, is to be taken at $\mathbf{K} = 0$.

The results for the critical temperature of symmetric and asymmetric nuclear matter with a separable force which reproduces binding energy and radius of a free α particle are presented in Fig. 3. The crosses on the figure indicate an exact solution of the in medium four body equation with a realistic nucleon-nucleon force [11] [13]. The excellent agreement with the approximate solution justifies the choice of our variational wave function. An important feature is that at the lowest temperatures, Bose-Einstein condensation occurs for α particles rather than for deuterons. As the density increases within the low-temperature regime, the chemical potential μ first reaches -7 MeV, where the α's Bose-condense. By contrast, Bose condensation of deuterons would not occur until μ rises to -1.1 MeV. In the asymmetric case, it is interesting to note that at strong asymmetry α particle condensation wins over the deuteron condensation because of the strong binding of the α's.

The *"quartetting"* transition temperature sharply drops as the rising density approaches the critical Mott value ($\mu \simeq 0$) at which the four-body bound states disappear. The deeper reason for this abrupt disappearence is again the same as we discussed above concerning the non-existence of quartet condensation for positive chemical potential, namely the in medium four fermion level density

$$g_4(\omega) = \sum(\bar{n}^0_{k_1}\bar{n}^0_{k_2}\bar{n}^0_{k_3}\bar{n}^0_{k_4} - n^0_{k_1}n^0_{k_2}n^0_{k_3}n^0_{k_4})\delta(\omega - \xi_{k_1} - \xi_{k_2} - \xi_{k_3} - \xi_{k_4}) \tag{40}$$

goes, for positive μ through zero at $\omega - 4\mu = 0$. At that point, pair formation in the isospin-singlet deuteron-like channel comes into play, and a deuteron condensate will exist below the critical temperature for BCS pairing up to densities above the nuclear-matter saturation density ρ_0. The reason why the critical temperature for deuterons does not break down for positive μ can again be given via the level density. Namely the two particle in medium level density, for pairs *at rest*, equals, up to a factor, the one for single particle states. The critical density at which the α condensate disappears is estimated to be around $\rho_0/10 - \rho_0/5$. Therefore, α-particle condensation primarily only exists in the Bose-Einstein-Condensed (BEC) phase and there does not seem to exist a phase where the quartets acquire a large extension as Cooper pairs do in the weak coupling regime. However, our variational approach on which this conclusion is based represents only a first attempt at the description of the transition from quartetting

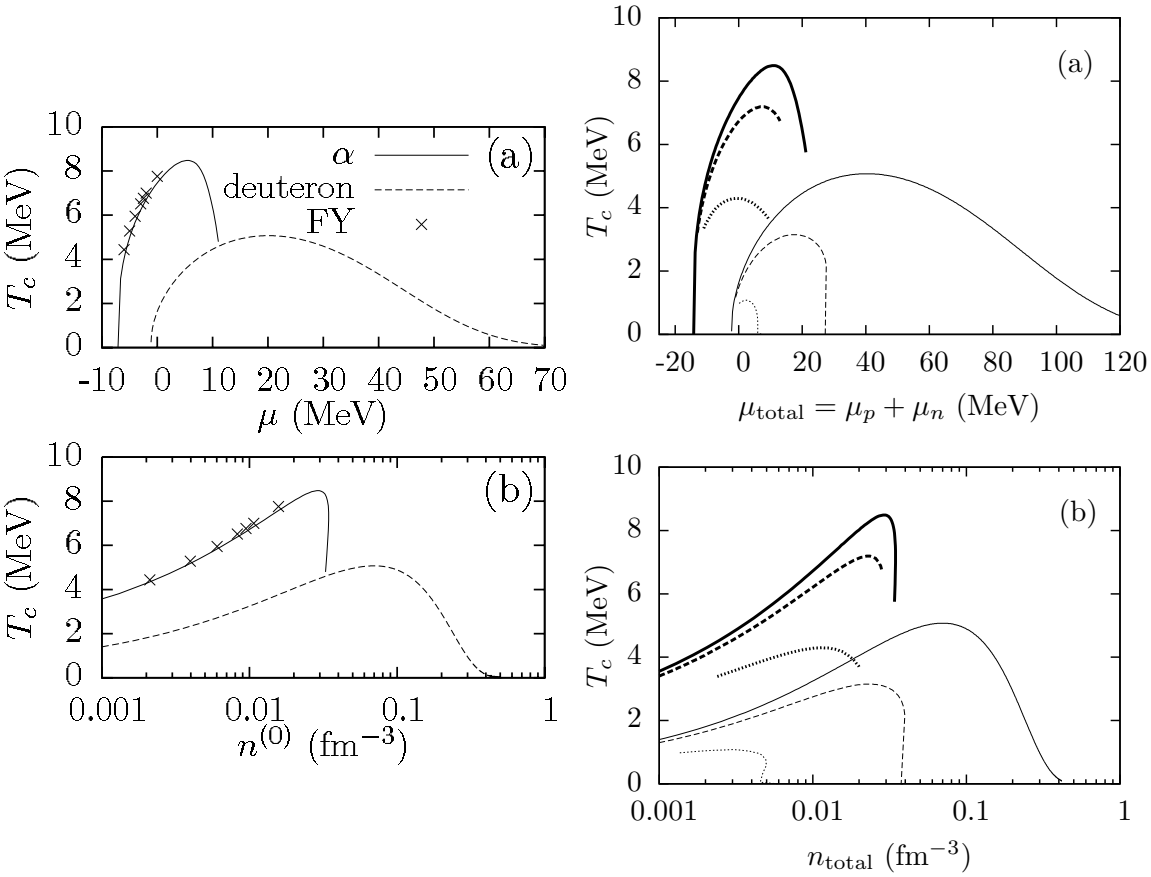


Figure 3. (Left) Critical temperatures for α particle and deuteron condensation in symmetric nuclear matter as a function of μ (a) and density $n^{(0)}$ (b). (Right) Same as in (Left) but for asymmetric nuclear matter for various asymmetry parameters $\delta = (n_n - n_p)/(n_n + n_p)$. $\delta = 0$ (full line); $\delta = 0.5$ (broken line); $\delta = 0.9$ (dotted line).

to pairing. The detailed nature of this fascinating transition remains to be clarified. Further discussions on this phenomenon can be found in [14]. We also should mention that the estimate of our critical temperature for α particle condensation which is based on a generalisation of the Thouless criterion for pairing, is only valid at the upper end of densities for which condensation occurs. For very low densities, the α's should form an ideal Bose gas. For the description of this limit, a theory should be developed which is analogous to the one of Nozières-Scmitt-Rink [15] for pairing. For quartets, this is more involved and has not been worked out so far.

A very intriguing question in relation with non existence of an α condensate at higher densities can be asked: is it possible that in heavy nuclei an α particle condensate exists in the nuclear surface, at least in its fluctuating form? The preformation assumption of α particles in the surface to explain α decay may give a hint to this.

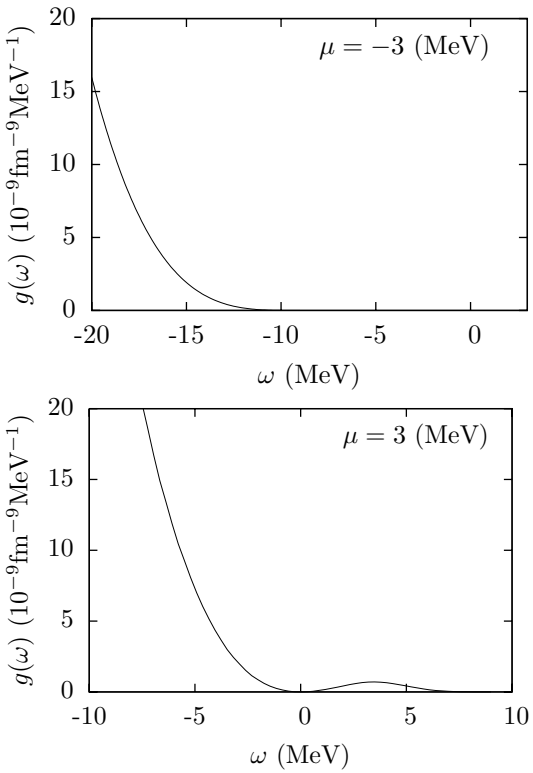
4. Alpha-Particle Condensation in Infinite Nuclear Matter at Zero Temperature

The equation for the four-body order parameter $K_{k_1k_2k_3k_4} \equiv \langle c_{k_1}c_{k_2}c_{k_3}c_{k_4}\rangle$ figuring in (18),(19),(22), obeys, to lowest order in the interaction, formally the same equation as the the one which determines the critical temperature, namely

$$(\varepsilon_{k_1}+\varepsilon_{k_2}+\varepsilon_{k_3}+\varepsilon_{k_4})K_{k_1k_2k_3k_4} \quad - \quad \sum_{k_1'<k_2'}[1-n_{k_1}-n_{k_2}]\bar{v}_{k_1k_2k_1'k_2'}K_{k_1'k_2'k_3k_4} \tag{41}$$

$$+ \quad \text{permutations} = 4\mu K_{k_1k_2k_3k_4} \tag{42}$$

where the occupation numbers n_k should be calculated self-consistently from Eq. (28). These two coupled equations are analogous to (7) and (5) in the case of pairing. Of course, with an un-approximated 4-body order parameter K, this would be a tremendously complicated self-consistent system of in medium 4-body equations to be solved. However, as before, for the critical temperature of α particle condensation, we very effectively approximate the order parameter K by our projected mean field ansatz in Eq. (39). Therefore the only unknown is now the single particle 0S wave function $\varphi(k)$. This constitutes a strong simplification of the problem. It has been solved in [14] and we refer the reader to that reference for further details. We want, however, to show the $3h$ level density from that paper, see Fig. 4, to demonstrate its vanishing behavior around the Fermi-energy for positive μ. The explicit solution of the single particle wave function φ, see Fig. 5 . In this latter figure, we also give the values of the occupation number distribution n_k for various (negative) chemical potentials. It is very interesting to see that even for a slightly positive value of μ (i.e., just before the break down of a solution), the n_k distribution is far from saturation, i.e., from one at $k=0$. Its maximum value is about 0.35 indicating that one is still far in the BEC side when α particle condensation starts to end.



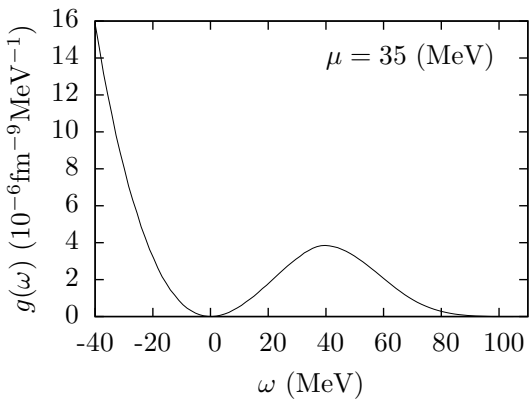


Figure 4. The 3h level density defined in Eq. (29)) for various values of the chemical potential μ at zero temperature. The zero on horizontal axis is at 3μ.

Before we will come to the description of α particle condensation in finite nuclei, let us make some further remarks concerning the approach we have worked out so far. This concerns for instance the approximate treatment of the single particle self energy leading to quartet condensation shown in Fig. 1 . This process implies that α particles are directly formed out

of an uncorrelated nucleon gas. That the scattering between those particles is neglected may not be the worst approximation. On the other hand, at least in nuclear physics, there exist three nucleon bound states like the triton or ^{3}He. In a gas of nucleons, they will be present and then a scattering process of a nucleon on one of those 'trions' may lead to an α particle (or the other way round). In addition, there may be deuterons around and two deuterons may again form an α particle. All these processes, depicted in Fig. 6, are neglected in our present application and should be included in future studies. One will then have a coherent description of a hot mixture of nucleons, Cooper pairs, deuterons, trions, and α-particles which awaits future solution. Another feature which distinguishes the quartet case from pairing is that, as seen in Fig. 1, there are three hole lines and not just one as for pairing. Because one has to sum over the relative momenta of those three holes, the effective single particle field acquires an imaginary part and no sharp quasi-particle pole developes. More on this has been worked out in [14].

5. Alpha-Particle Condensate States in Self-Conjugate 4n Nuclei

Let us discuss the possibility of quartetting in nuclei. The only nucleus having a pronounced α-cluster structure in its ground state is ^{8}Be. In Fig. 7(a), the result of an exact calculation of the density distribution of ^{8}Be in the laboratory frame is shown. In Fig. 7(b), for comparison, the result of the same calculation in the intrinsic, deformed frame is displayed. We see a pronounced two α-cluster structure where the two α's are $\sim$ 4 fm apart, giving rise to a very low average density $\rho \sim \rho_0/3$ as seen in Fig. 7(a). As already discussed in the introduction, ^{8}Be is a rather unusual and unique nucleus. One may be intrigued by the question, already raised earlier, whether loosely bound α- particle configurations may not also exist in heavier nα-nuclei, at least in excited states, naturally close to the nα disintegration threshold. Since α-particles are rather inert bosons (first excited state at $\sim$ 20 MeV), they then would all condense in the lowest S-wavefunction, very much in the same way as do bosonic atoms in magneto-optical traps [16]. This question and exploring related issues of quartetting in finite nuclei will consume most of the rest of the present outline on α particle condensation. In fact, we will be able to offer strong arguments that the 0_2^+ state of ^{12}C at 7.654 MeV is a state of α-condensate nature. First, it should be understood that the 0_2^+ state in ^{12}C is in fact hadronically unstable (as ^{8}Be), being situated about 300 keV above the three α-break up threshold. This state is stabilized only by the Coulomb barrier. It has a width of 8.7 eV and a corresponding lifetime of 7.6×10^{-17} s. As well known, this state is of paramount astrophysical (and biological!) importance due to its role in the creation of ^{12}C in stellar nucleosynthesis. Its existence was predicted in 1953 by the astrophysicist Fred Hoyle [17]; his prediction was confirmed experimentally a few years later by Willy Fowler and coworkers at Caltech [18]. It is also well known that this *Hoyle state*, as it is called, is a notoriously difficult state for any nuclear theory to explain. For example, the most modern no-core shell-model calculations predict the 0_2^+ state in ^{12}C to lie at around 17 MeV above the ground state – more than twice the actual value [19]. This fact alone tells us that the Hoyle state must have a very unusual structure. It is easy to understand that, should it indeed have the proposed loosely bound three α-particle structure, a shell-model type of calculation would have great difficulties in reproducing its properties.

An important development bearing on this issue took place some thirty years ago. Two Japanese physicists, M. Kamimura [20] and K. Uegaki [21], along with their collaborators, almost simultaneously reproduced the Hoyle state from a microscopic theory. They employed a twelve-nucleon wave function together with a Hamiltonian containing an effective nucleon-nucleon interaction. At that time, their work did not attract the attention it deserved; the true importance of their achievement has been appreciated only recently. The two groups started from practically the same ansatz for the ^{12}C wave function, which has the following three α-cluster structure:

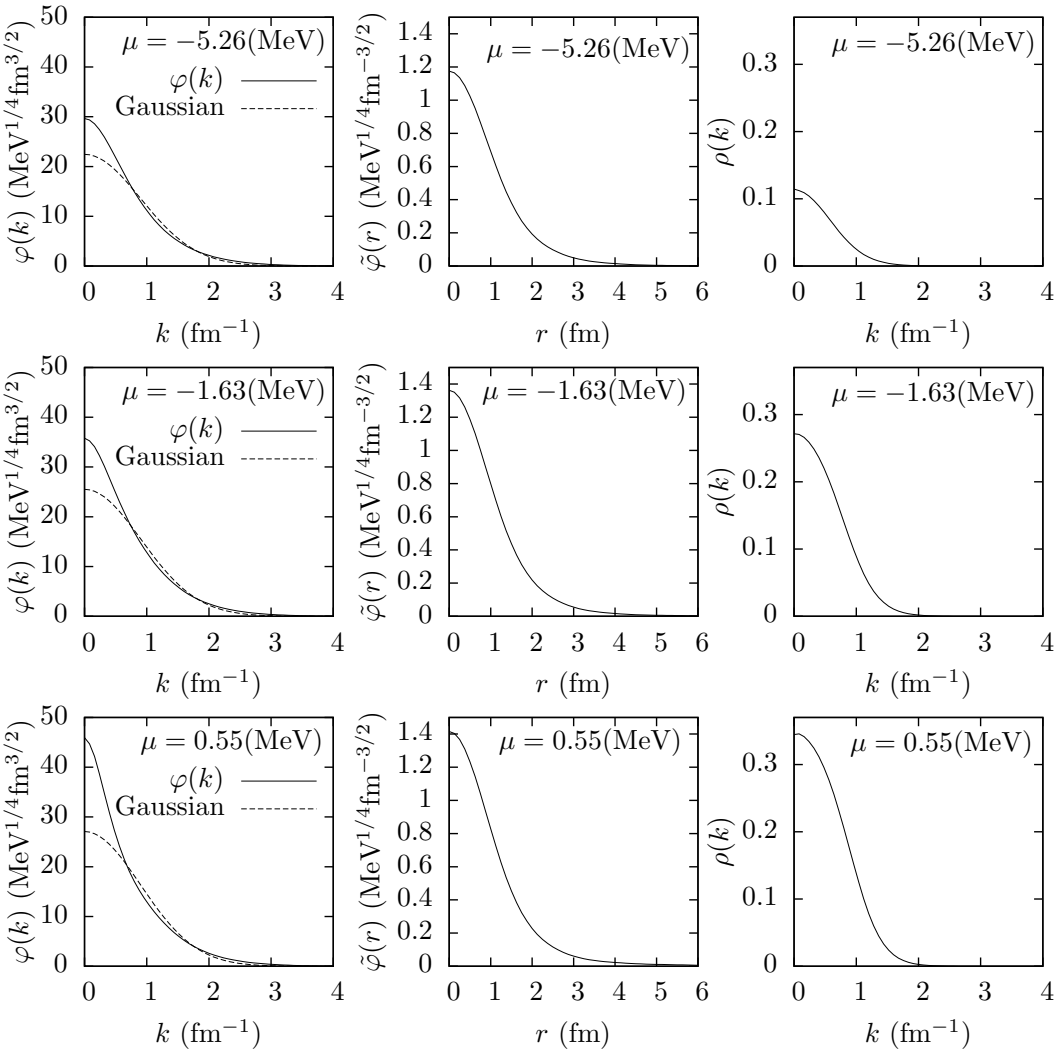


Figure 5. Single particle wave function $\varphi(k)$ for k-space (left), for r-space $\tilde{\varphi}(r)$ (middle), and occupation numbers $\rho(k) \equiv n_k$ (right) at $\mu = -5.26$ (top), -1.63 (middle) and 0.55 (bottom). The r-space wave function $\tilde{\varphi}(r)$ is derived from the Fourier transform of $\varphi(k)$ by $\tilde{\varphi}(r) = \int d^3k e^{i\mathbf{k}\cdot\mathbf{r}}\varphi(k)/(2\pi)^3$. The dashed line in the left figure correspond to the Gaussian approximation with same norm and rms momentum as $\varphi(k)$.

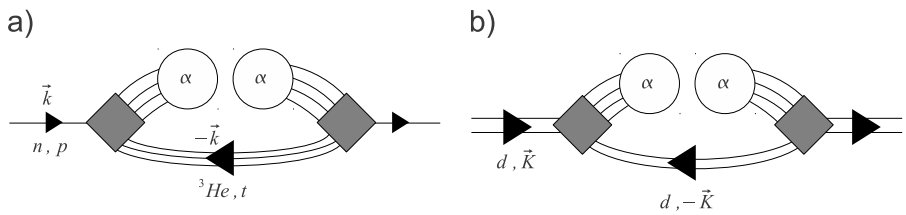


Figure 6. Alpha particle creation by scattering of a nucleon on a triton (^{3}He) (left) or of two deuterons (right).

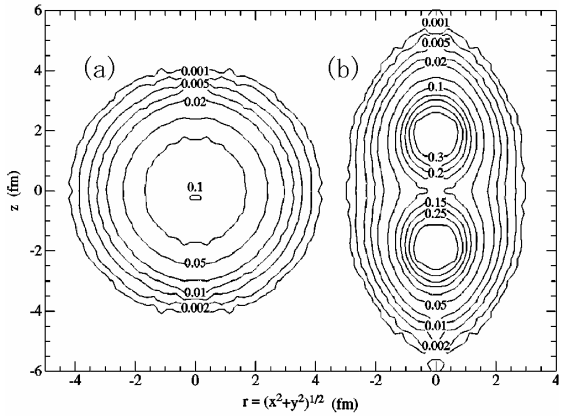


Figure 7. Contours of constant density (taken from Ref. [22]), plotted in cylindrical coordinates, for ^{8}Be(0^+). The left side (a) is in the 'laboratory" frame while the right side (b) is in the intrinsic frame.

$$\langle \mathbf{r}_1 ... \mathbf{r}_{12} | ^{12}\mathrm{C} \rangle = \mathcal{A} \left[\chi(\mathbf{R}, \mathbf{s}) \phi_1 \phi_2 \phi_3 \right] . \tag{43}$$

In this expression, the operator $\mathcal{A}$ imposes antisymmetry in the nucleonic degrees of freedom and ϕ_i, with $i = 1, 2, 3$ for the three α's, is an intrinsic α-particle wave function of prescribed Gaussian form,

$$\phi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4) = \exp \left\{ - \left[(\mathbf{r}_1 - \mathbf{r}_2)^2 + (\mathbf{r}_1 - \mathbf{r}_3)^2 + ... \right] / 2b^2 \right\} , \tag{44}$$

where the size parameter b is adjusted to fit the rms of the free α-particle, and $\chi(\mathbf{R}, \mathbf{s})$ is a yet-to-be determined three-body wave function for the c.o.m. motion of the three α's, their corresponding Jakobi coordinates being denoted by $\mathbf{R}$ and $\mathbf{s}$. The unknown function χ was determined via calculations based on the Generator Coordinate Method [21] (GCM) and the Resonating Group Method [20] (RGM) calculations using the Volkov I and Volkov II effective nucleon-nucleon forces, which fit α-α phase shifts. The precise solution of this complicated three body problem, carried out three decades ago, was truly a pioneering achievement, with results fulfilling expectations. The position of the Hoyle state, as well as other properties including the inelastic form factor and transition probability, successfully reproduced the experimental data. Other states of ^{12}C below and around the energy of the Hoyle state were also successfully described. Moreover, it was already recognized that the three α's in the Hoyle state form sort of a gas-like state. In fact, this feature had already been noted by H. Horiuchi [23] prior to the appearance of Refs. [20, 21], based on results from the orthogonality condition model (OCM) [24]. All three Japanese research groups concluded from their studies that the linear-chain state of three α-particles, postulated by Morinaga many years earlier [25], had to be rejected.

Although the evidence for interpreting the Hoyle state in terms of an α gas was stressed in the cited papers from the late 1970's, two important aspects of the situation were missed at that time. First, because the three α's move in identical S-wave orbits, one is dealing with an α-condensate state what may, in fact, be a general feature in $n\alpha$ nuclei. The second and most important point is that the complicated three-body wave function $\chi(\mathbf{R}, \mathbf{s})$ for the c.o.m. motion of the three α's can be replaced by a structurally and conceptually very simple microscopic three-α wave function of the condensate type, which has practically 100 percent overlap with

the previously constructed ones [26] [27] (see also Ref. [28]). We now describe this condensate wave function.

We start by examining the BCS wave function of ordinary fermion pairing, obtained by projecting the familiar BCS ground-state ansatz onto an N-particle subspace of Fock space. In the position representation, this wave function is

$$\langle \mathbf{r}_1 ... \mathbf{r}_N | \mathrm{BCS} \rangle = \mathcal{A} \left[\phi(\mathbf{r}_1, \mathbf{r}_2) \phi(\mathbf{r}_3, \mathbf{r}_4) ... \phi(\mathbf{r}_{N-1} \mathbf{r}_N) \right] , \tag{45}$$

where $\phi(\mathbf{r}_1, \mathbf{r}_2)$ is the Cooper-pair wave function (including spin and isospin), which is to be determined variationally through the familiar BCS equations. The condensate character of the BCS ansatz is born out by the fact that within the antisymmetrizer $\mathcal{A}$, one has a product of $N/2$ times the same pair wave function ϕ, with one such function for each distinct pair in the reference partition of $\{1, 2, \ldots, N-1, N\}$. Formally, it is now a simple matter to generalize (45) to quartet or α-particle condensation. We write

$$\langle \mathbf{r}_1, \ldots, \mathbf{r}_N | \Phi_{n\alpha} \rangle = \mathcal{A} \left[\phi_\alpha(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4) \phi_\alpha(\mathbf{r}_5, \ldots, \mathbf{r}_8) \cdots \phi_\alpha(\mathbf{r}_{N-3}, \ldots, \mathbf{r}_N) \right] , \tag{46}$$

where ϕ_α is the wave function common to all condensed α-particles. It may be considered as the number projected quartet wave function introduced in (8). Of course, finding the variational solution for this function is, in general, extraordinarily more complicated than finding the Cooper pair-wave function ϕ of Eq. (45). Even so, in the present case that the α-particle is the four-body cluster involved, and for applications to relatively light nuclei, the complexity of the problem can be reduced dramatically. This possibility stems from the fact that an excellent variational ansatz for the intrinsic wave function of the α-particle is provided [as in Eq. (44)], by a Gaussian form with only the size parameter b to be determined. The new aspect in [26] was that in addition even the center-of-mass motion of the system of α-particles can be described very well by a Gaussian wave function with, this time, a size parameter $B \gg b$ to account for the motion over the whole nuclear space. This is a strong technical simplification and, at the same time, underlines the boson condensate character of the wave function. We therefore write

$$\phi_\alpha(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4) = e^{-2\mathbf{R}^2/B^2} \phi(\mathbf{r}_1 - \mathbf{r}_2, \mathbf{r}_1 - \mathbf{r}_3, \cdots) , \tag{47}$$

where $\mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \mathbf{r}_4)/4$ is the c.o.m. coordinate of one α-particle and $\phi(\mathbf{r}_1 - \mathbf{r}_2, ...)$ is the same intrinsic α-particle wave function of Gaussian form as already used in Refs. [20, 21] and given explicitly in Eq. (44). This wave function has a close relation to the one we used in the infinite matter case. To this end, let us Fourier transform the infinite matter ansatz (39) into real space (with a projection onto a finite $\mathbf{K}$)

$$\phi^\infty_{\alpha,\mathbf{K}}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4) = e^{i\mathbf{KR}} \int d^3R' e^{-i\mathbf{KR}'} \tilde{\varphi}(\mathbf{r}_1 - \mathbf{R}') \tilde{\varphi}(\mathbf{r}_2 - \mathbf{R}') \tilde{\varphi}(\mathbf{r}_3 - \mathbf{R}') \tilde{\varphi}(\mathbf{r}_4 - \mathbf{R}') \tag{48}$$

where $\tilde{\varphi}$ is the Fourier transform of φ. Taking for φ a Gaussian, one obtains

$$\phi^\infty_{\alpha,\mathbf{K}}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4) = e^{i\mathbf{KR}} \exp \left\{ - \left[(\mathbf{r}_1 - \mathbf{r}_2)^2 + (\mathbf{r}_1 - \mathbf{r}_3)^2 + ... \right] / 2b^2 \right\} \tag{49}$$

Comparing with (47), we, therefore, see that our variational ansatz for the α particle condensate wave function is of the same spirit as the one we used already in the homogeneous case, only the plane wave c.o.m. wave function has, naturally, been replaced by a Gaussian. For a small number of α particles, it is, of course, important to work with a wave function with a definite number of particles and not with the coherent state (8). However, for a larger number of α's where the handling of (46) becomes more and more difficult because of the explicit antisymmetrisation of

all nucleons, it may be worth to use (47) also in our quartet coherent state (8) as a variational wave function with the two parameters B, b.

Naturally, in Eq. (46) the center of mass $\mathbf{X}_{\rm cm}$ of the three α's, i.e., of the whole nucleus, should be eliminated; this is easily achieved by replacing $\mathbf{R}$ by $\mathbf{R} - \mathbf{X}_{\rm cm}$ in each of the α wave functions in Eq. (46). The α-particle condensate wave function specified by Eqs. (46) and (47), proposed in Ref. [26] and called the THSR wave function, now depends on only two parameters, B and b. The expectation value of an assumed microscopic Hamiltonian H,

$$\mathcal{H}(B,b) = \frac{\langle \Phi_{n\alpha}(B,b)|H|\Phi_{n\alpha}(B,b)\rangle}{\langle \Phi_{n\alpha}|\Phi_{n\alpha}\rangle}\,, \tag{50}$$

can be evaluated, and the corresponding two-dimensional energy surface can be quantized using the two parameters B and b as Hill-Wheeler coordinates.

Before presenting the results, let us discuss the THSR wave function in somewhat more detail. This innocuous-looking variational ansatz, namely Eq. (46) together with Eq. (47), is actually more subtle than it might at first appear. One should realize that two limits are incorporated exactly. One is obtained by choosing $B = b$, for which Eq. (46) reduces to a standard Slater determinant with harmonic-oscillator single-nucleon wave functions, leaving the oscillator length b as the single adjustable parameter. This holds because the right-hand-side of expression (47), with $B = b$, becomes a product of four identical Gaussians, and the antisymmetrization creates all the necessary P, D, etc. harmonic oscillator wave functions automatically [26]. On the other hand, when $B \gg b$, the density of α-particles is very low, and in the limit $B \to \infty$, the average distance between α's is so large that the antisymmetrisation between them can be neglected, i.e., the operator $\mathcal{A}$ in front of Eq. (46) becomes irrelevant and can be removed. In this limiting case, our wave function then describes an ideal gas of independent, condensed α-particles – it is a pure product state of α's! An elucidating study on this aspect is given in Ref. [29].

Evidently, in realistic cases the antisymmetrizer $\mathcal{A}$ cannot be neglected, and evaluation of the expectation value (50) becomes a nontrivial task. The Hamiltonian in Eq. (50) was taken to be the one used in Ref. [30], which features an effective nucleon-nucleon force of the Gogny type, with parameters fitted to α-α scattering phase shifts as available about fifteen years ago. This force also leads to very reasonable properties of ordinary nuclear matter. Our theory is therefore free of any adjustable parameters. The energy landscapes $\mathcal{H}(B,b)$ for various $n\,\alpha$ nuclei are interesting in themselves [31], but for the sake of brevity they are not shown here.

As we discussed already, the variational wave function constructed from the Hill-Wheeler equation based on Eqs. (46), (47), and (50) has practically 100 percent overlap with the RGM and GCM wave functions constructed in Refs. [20] and [21], once the same Volkov force is used [27]. However, one can take fixed optimised values for b and B parameters and, then, the corresponding *single* THSR wave function still has 98 percent squared overlap with the RGM or GCM solutions. It is, thus, not astonishing that our results are very similar to the RGM and GCM ones. For ^{12}C we obtain two eigenvalues: the ground state and the Hoyle state. Theoretical values for positions, rms values, and transition probabilities are given in Table 1 and compared to the data. Inspecting the rms radii, we see that the Hoyle state has a volume 3 to 4 larger than that of the ground state of ^{12}C. This is the primary aspect of the dilute-gas state we highlighted above. Constructing a pure-state α-particle density matrix $\rho(\mathbf{R},\mathbf{R}')$ from our wave function, integrating out of the total density matrix all intrinsic α-particle coordinates, and diagonalizing this reduced density matrix, we find that the corresponding $0S$ α-particle orbit is occupied to 70 percent by the three α-particles [29, 32] whereas the occupation of all other states is down by at least a factor of ten, see Fig. 8.

This is a huge percentage, giving vivid support to the view that the Hoyle state is an almost

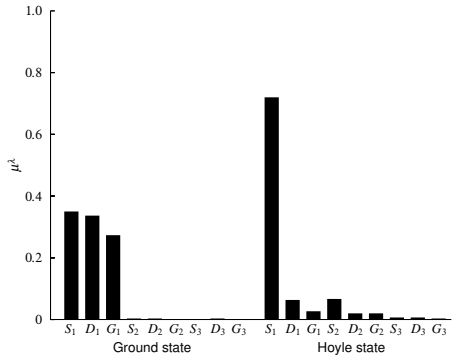


Figure 8. α occupation numbers in the ground state (left) and Hoyle state (right) of ^{12}C.

		condensate w.f. (Hill-Wheeler)	RGM [20]	Exp.
E(MeV)	0_1^+	-89.52	-89.4	-92.2
	0_2^+	-81.79	-81.7	-84.6
$R_{\text{r.m.s.}}$(fm)	0_1^+	2.40	2.40	2.44
	0_2^+	3.83	3.47	
$M(0_2^+ \to 0_1^+)(\text{fm}^2)$		6.45	6.7	5.4

Table 1. Comparison of the binding energies, rms radii ($R_{\text{r.m.s.}}$), and monopole matrix elements ($M(0_2^+ \to 0_1^+)$) for ^{12}C given by solving Hill-Wheeler equation based on Eq. (46) and by Ref. [20]. The effective two-nucleon force Volkov No. 2 was adopted in the two cases for which the 3α threshold energy is calculated to be -82.04 MeV.

ideal α-particle condensate. By way of contrast, we should also mention that in the ground state of ^{12}C, the α-particle occupation is about equally shared between the $0S$, $0D$, and $0G$ orbits, i.e. yielding the shell model limit, clearly invalidating a condensate picture of the ground state, see Fig. 8 (it is important to note that the ground-state energy of ^{12}C is also reasonably reproduced by our theory). We, thus, think that the investigation of the bosonic occupancies is the most adequate way to demonstrate whether a given state can, in nuclei, be qualified as an α condensate or not.

Let us now discuss what to our mind is the most convincing evidence that our description of the Hoyle state is the correct one. Like the authors of Ref. [20], we reproduce very accurately the inelastic form factor $0_1^+ \to 0_2^+$ of ^{12}C, as shown in Fig. 9. As such, the agreement with experiment is already quite impressive in view of the fact that we did not use a single adjustable parameter. Additionally, however, the following study was made, results from which are presented in Fig. 10. We artificially varied the extension of the Hoyle state and examined the influence on the form factor. It was found that the overall shape of the form factor shows little variation, for example in the position of the minimum. On the other hand, we found a strong dependence of the absolute magnitude of the form factor; Fig. 10 illustrates this behavior with a plot showing the variation of the height of the first maximum of the inelastic form factor as a function of the percentage change of the rms radius of the Hoyle state [27]. It can be seen that a 20 percent *increase of the rms radius* produces a remarkable decrease of the maximum – by a factor of two! This strong sensitivity of the magnitude of the form factor to the size of the Hoyle state enhances our firm belief that the agreement with the actual measurement

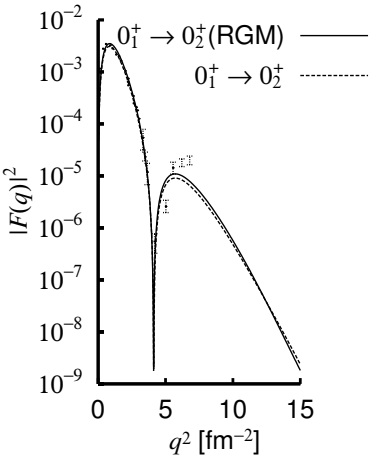


Figure 9. Experimental values of inelastic form factor in ^{12}C to the Hoyle state are compared with our values and those given by Kamimura et al. in Ref. [20] (RGM). In our result, the Hoyle-state wave function is obtained by solving the Hill-Wheeler equation based on Eq. (46).

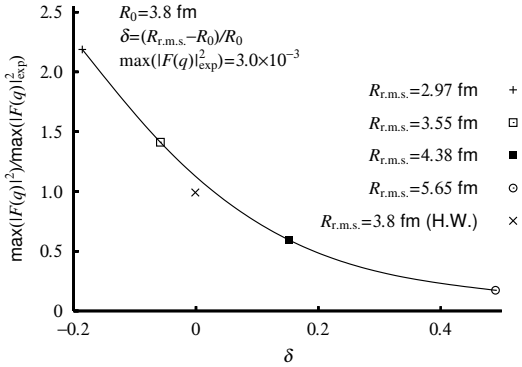


Figure 10. The ratio of the value of the maximum height, theory versus experiment, of the inelastic form factor, i.e. $\max|F(q)|^2/\max|F(q)|^2_{\rm exp}$, is plotted as a function of $\delta = (R_{\rm r.m.s.} - R_0)/R_0$. Here $R_{\rm r.m.s.}$ and R_0 are the rms radii corresponding, respectively, to the wave function of Eq. (46) and that obtained by solving the Hill-Wheeler equation based on Eq. (46).

is tantamount to a proof that the calculated wide extension of the Hoyle state corresponds to reality. We thus advocate and support the view that the Hoyle state can be regarded as the ground state of an α-particle condensate [33]. We should, however, be aware of the fact that it is not an ideal condensate and that the α's occupy the lowest S-state only with 70 percent, as already discussed. The major part of the correlations comes from the Pauli principle and intermediate ^{8}Be formation. It is by the way not clear whether a gas of α particles condenses as such or as 'molecules' of ^{8}Be. The latter are also bosons, of course. We furthermore performed a deformed calculation to investigate the structure of the 2_2^+ state in ^{12}C. The state came at the right energy. Our analysis showed that this state essentially corresponds to exciting one α-particle out of the condensate and putting it into the $0D$ orbit. Without going into details, we also affirm that the width of this state is correctly reproduced [34]. It should also be mentioned that this 2_2^+ state has in our calculations an enormous extension with an rms radius of 4.3 fm what corresponds approximately to eight times the ground state volume of ^{12}C or also to the size of ^{40}Ca. As a matter of fact the properties of this state have been subject of a vivid debate among the experimentalists in the recent past. The situation seems clarified now [35][36][37]. It is for instance, the very nice experiment by Moshe Gai [37] which confirms the properties of the 2_2^+ state beyond any doubt. In that reference it also is given an estimate of the radius which agrees with our value. One can talk about an α- halo state. Further experimental verification of this giant α-gas state would be very welcome.

It is tempting to imagine that the 0_3^+ state which – experimentally – is almost degenerate with the 2_2^+ state, is obtained by lifting one α-particle into the $1S$ orbit. Initial theoretical studies

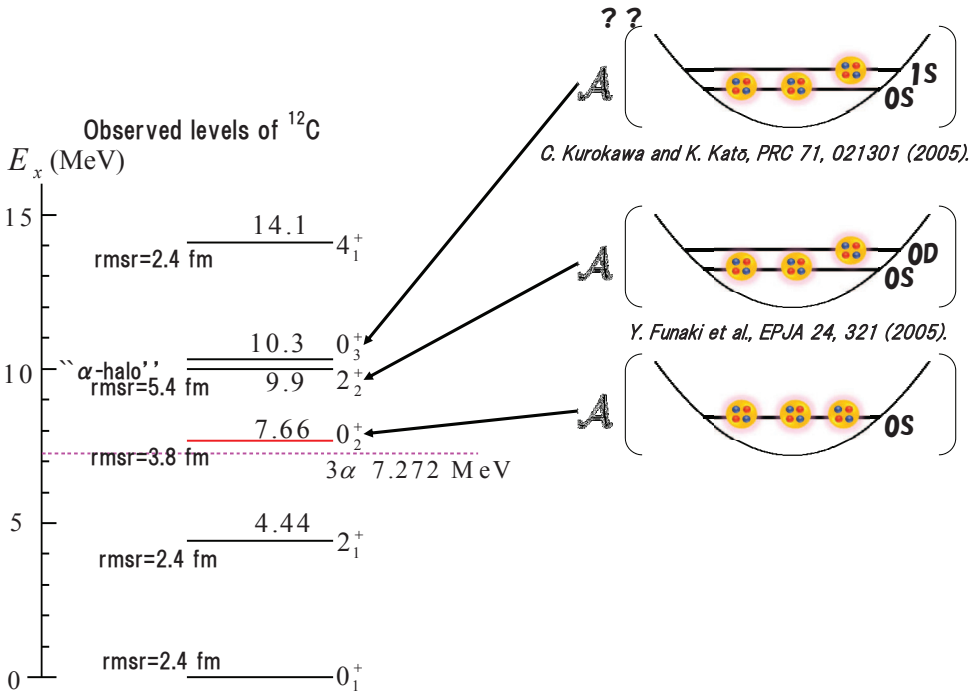


Figure 11. Spectrum of ^{12}C and its interpretation via an analysis with the THSR wave function concerning α cluster states.

[38] indicate that this scenario might indeed apply. However, the width of the 0_3^+ state (~ 3 MeV) is very broad, rendering a theoretical treatment rather delicate. Further investigations are necessary to validate or reject this picture. At any rate, it would be quite satisfying if the triplet of states ($0_2{}^+$, $2_2{}^+$, $0_3{}^+$) could all be explained from the α-particle perspective, since those three states are *precisely* the ones which cannot be reproduced within a (no core) shell-model approach [19]. In Fig. 11 we represent this scenario.

Summarizing our inquiry into the possible role of α clustering in ^{12}C, we have accumulated enough facts to be convinced that the Hoyle state is, indeed, what one may call to first approximation an α-particle condensate state. At the same time, we acknowledge that referring to only three particles as a "condensate" constitutes a certain abuse of the word. However, in this regard it should be remembered that also in the case of nuclear Cooper pairing, only a few pairs are sufficient to obtain clear signatures of superfluidity in nuclei!

What about α-particle condensation in heavier nuclei? Once one accepts the idea that the Hoyle state is essentially a state of three free α-particles held together only by the Coulomb barrier, it is hard to see why analogous states would not also exist in heavier $n\,\alpha$ nuclei like ^{16}O, ^{20}Ne, ^{24}Mg, etc. In this respect, it is important to recognize the argument that the successively higher excitation energies of α condensate states do not necessarily imply very short life times because of their very unusual structure, having little in common with ordinary nuclear states. Our calculations on such nuclei systematically yield a 0^+-state close to the α-particle disintegration threshold. For example in ^{16}O we obtain three 0^+-states [26]: the ground state at $E_0 = -124.8$ MeV (experimental value: 127.62 MeV), a second state at excitation energy $E_{0_2{}^+} = 8.8$ MeV, and a third one at $E_{0_3{}^+} = 14.1$ MeV. The threshold in ^{16}O is at 14.4 MeV. Unfortunately, the relevant experimental information in ^{16}O is not nearly so complete as in ^{12}C. In particular, no measurements are available for transition probabilities of 0^+-states near

the threshold or for inelastic form factors.

In contrast to the situation for ^{12}C, the THSR wave function is certainly not able to describe the structure of all 0^+-states in ^{16}O lying below the disintegration threshold. In ^{12}C knocking loose one α particle, the other two are also loosely bound, since what remains is ^{8}Be. However, in ^{16}O this is not the case. Before reaching a four α particle gas state, there will appear configurations where one α particle orbits around a ^{12}C core in its ground state or in excited states of particle- hole type. A case in point is the first excited state in ^{16}O, i.e., the $0_2{}^+$-state at 6.06 MeV, which is believed to have a structure corresponding to an α-particle orbiting in an S wave around a ^{12}C core in its ground state. Such a configuration is clearly missing from our wave function (46). As a matter of fact a calculation for the first six 0^+ states has been performed in the meanwhile employing a somewhat different, not completely microscopic approach [39]. This method is the so-called 'orthogonality condition model (OCM)' which generally works also quite well for the description of cluster states. The novelty with respect to former calculations of ^{16}O states with this method was that the configuration space was strongly enlarged. In Fig. 12, we show the comparison of the calculated with the experimental sprectrum. In view of the fact that the states are complicated cluster states, the agreement between experiment and theory is very good. The interpretation goes as follows: the first four excited 0^+ states have a ^{12}C + α structure where the α orbits in 0S, 0D, 1S waves around the ground state core of ^{12}C and also in a 0P wave orbiting around the first 1^- of ^{12}C. It is only the first state above the four α particle threshold at 15.1 MeV which is interpreted as an α gas state or a condensate. This state has, indeed, some analogies with the Hoyle state: it is just some hundreds of keV above threshold, it is strongly excited by inelastic electron scattering what means that monopole transition is quite large. Unfortunately the inelastic form factor has not been measured so far.

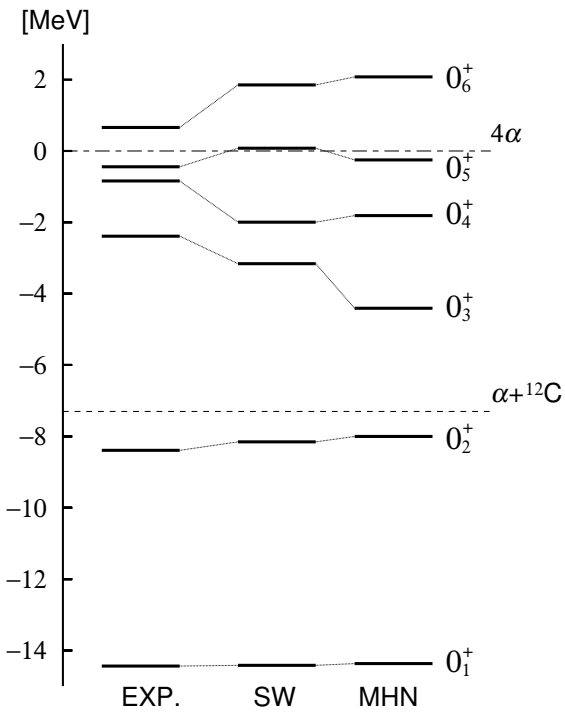


Figure 12. Spectrum of first six 0^+ states in ^{16}O. Left: experiment; middle: theory with SW force; right: theory with MHN force; see [39] for more details.

One interesting question that can be asked at this point is: How many α's can maximally exist in a self-bound α-gas state? Seeking an answer, we performed a schematic investigation using an effective α-α interaction of the Ali-Bodmer form [40] within an α-gas mean-field calculation of the Gross-Pitaevskii type [41]. The parameters of the force were slightly adjusted to reproduce

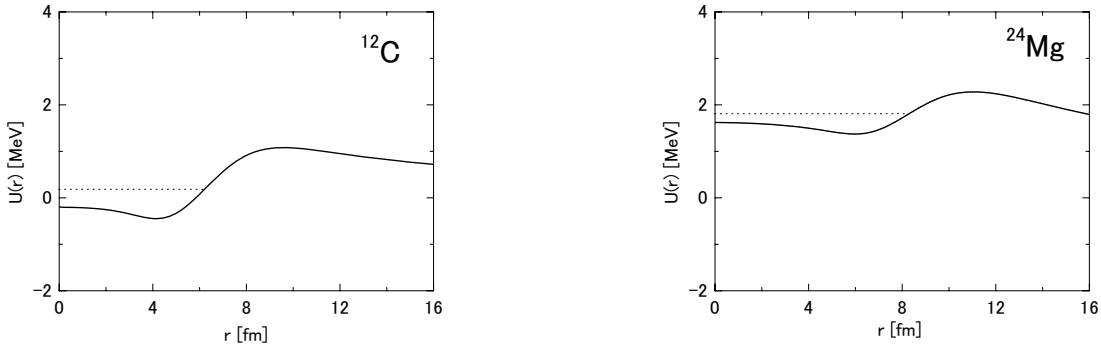


Figure 13. Alpha-particle mean-field potential for three α's in ^{12}C and six α's in ^{24}Mg. Note the lower Coulomb barrier for ^{24}Mg (from Ref. [42]).

our microscopic results for ^{12}C. The corresponding α mean-field potential is shown in Fig. 5. One sees the $0S$-state lying slightly above threshold but below the Coulomb barrier. As more α-particles are added, the Coulomb repulsion drives the loosely bound system of α-particles farther and farther apart, so that the Coulomb barrier fades away. According to our estimate [42], a maximum of eight to ten α-particles can be held together in a condensate. However, there may be ways to lend additional stability to such systems. We know that in the case of ^{8}Be, adding one or two neutrons produces extra binding without seriously disturbing the pronounced α-cluster structure. Therefore, one has reason to speculate that adding a few of neutrons to a many-α state may stabilize the condensate. But again, state-of-the-art microscopic investigations are necessary before anything definite can be said about how extra neutrons will influence an α-particle condensate.

Another interesting idea concerning α-particle condensates was put forward by von Oertzen and collaborators [43, 44]. Adding more and more α-particles to the, e.g., ^{40}Ca core, one will arrive sooner or later at the point of α-particle drip. Therefore minimal further excitation may be sufficient to shake loose some α-particles, so that an $n\,\alpha$-condensate could be created on top of an inert ^{40}Ca core. Similar ideas also have been advanced by Ogloblin [45], who envisions a three-α-particle condensate on top of ^{100}Sn, and earlier by Brenner and Gridnev, who have presented evidence of experimental detection of gaseous α-particles in ^{28}Si and ^{32}S on top of an inert ^{16}O core [46].

Interesting new theoretical developments generalizing the THSR wave function [26] are presently going on, putting nuclear cluster physics on a completely new basis [47]. For example, it was found that the rotational parity doublet ground state bands in ^{20}Ne can be described with a slightly generalised THSR wave function. Also the intriguing question about the intrinsic cluster structure inherent to the THSR wave function is further elucidated in [47]. In this respect it is also interesting that Hartree-Fock-Bogoliubov calculations for expanding $n\alpha$ nuclei show clusterisation into geometrical arrangements of the α's, as, e.g., a tetrahedron for ^{16}O [48].

6. Conclusions, Discussion, Outlook

We have investigated the role that pairing and multiparticle correlations may play in nuclear matter existing in dense astrophysical objects and in finite nuclei. A complete and quantitative description of nuclear matter must allow for the presence of clusters of nucleons, bound or metastable, possibly forming a quantum condensate. In particular, quartetting correlations, responsible for the emergence of α-like clusters, are identified as uniquely important in determining the behavior of nuclear matter in the limiting regime of low density and low

temperature. We have calculated the transition temperature for the onset of quantum condensates made up of α-like and deuteron-like bosonic clusters, and considered in considerable detail the intriguing example of Bose-Einstein condensation of α particles. It turns out that contrary to pairing, quartet condensation primarily exists in the BEC phase at low density. In which way quartet condensation is lost by increasing the density is still an open question. It may be similar to a liquid-gas phase transition. Anyway, it is clear that there can not exist a condensate of quartets with a long coherence length for arbitrarily small attraction as this is the case for pairing in the BCS phase. Arguments for this based on many fermion level densities have been presented. It is inevitable that under increasing density or pressure, the bound α, d, or other nuclidic clusters, present at low density, experience significant modification due to the background medium (and eventually merge with it). We have shown how self-energy corrections and Pauli blocking alter the properties of cluster states, and we have formulated a cluster mean-field approximation to provide an initial description of this process. One result of special interest is the suppression of the α-like condensate, which is dominant at lower densities, as the density reaches and exceeds the Mott value, allowing the pairing transition to occur. Even at lower densities α-particle condensation may be influenced by neutron excess, i.e. in the case of asymmetric nuclear matter, see Fig. 3. A genuine theory for the quartet order parameter in homogeneous infinite matter, similar to BCS theory, is demanded. First results in this direction have been published [14]. A theory for quartet condensation on firmer grounds which parallels the one of pairing is presented in this contribution in section 2.

A truly remarkable manifestation of α-particle condensation seems to be present in finite nuclei. Indeed, the so-called Hoyle state ($0_2{}^+$) in ^{12}C at 7.654 MeV is very likely a dilute gas of three α-particles, held together only by the Coulomb barrier. This view is encouraged by the fact that we can explain all the experimental data in terms of a conceptually simple wave function of the quartet-condensate type. Within the same model, we also systematically predict such states in heavier $n\,\alpha$ nuclei, and the search is on for their experimental identification. With the more phenomenological OCM method, we found that in ^{16}O the sixth 0^+ state at 15.1 MeV should be the candidate for an α condensed state. It is quite natural that such states should exist up to some maximum number of α particles inspite of their increasing excitation energy: these α-gas states are almost orhtogonal to the rest of nuclear states so that decay is strongly hindered. We estimate that the phenomenon will terminate at about eight to ten α's as the confining Coulomb barrier fades away. However, there is the possibility that larger condensates could be stabilized by addition of a few neutrons. Indeed, consider ^{9}Be, which, contrary to ^{8}Be, is bound by $\sim$ 1.5 MeV, still showing a pronounced two α-structure similar to the one of Fig. 7 (b). One could imagine ten α's or more, stabilised by two or four extra neutrons in a low density phase. However, even without being stabilised, if a compressed hot nuclear blob as e.g. produced in a central Heavy Ion collision expands and cools, it may turn on its way out, at a certain low density, into an expanding α condensed state where all α's are in relative S-waves. One may also, in one way or the other (photons?) excite, e.g., ^{40}Ca to the 10 α threshold at about 60 MeV where a slow Coulomb explosion of an α gas would then take place. This would be an analogous situation to an expanding Bose condensate of atoms, once the trapping potential has been switched off. Future dedicated experiments with high resolution multiparticle detectors will tell whether such scenarios can be realised. Intriguing news in this respect come from GANIL where one may have achieved the disintegration of ^{56}Ni into 14 α's (seven α's have been detected with high yield, the other seven may not have been seen because of the detectors were not sensitive to very low energy α particles [49][50]). Other possibilities of loose α-gas states may exist on top of particularly stable cores, like ^{16}O or ^{40}Ca. Indeed in adding α's to e.g. ^{40}Ca, one will reach the α-particle drip line. Compound states of heavy N = Z nuclei of this kind may be produced in heavy ion reactions and an enhanced α-decay rate may reveal the existence of an α-particle condensate. Ideas of this type have been promoted by von Oertzen

[44], and also M. Brenner [46], and A. Ogloblin [45]. However, coincidence measurements of multiple α's of decaying lighter nuclei like ^{16}O may also be very useful [51][52] [53] to detect at least one additional α- condensate state beyond the only one that has been identified so far, namely the $0_2{}^+$-state in ^{12}C.

Another issue which may be raised in the context of α-particle condensation is the question, also discussed in condensed matter physics [54], whether α's condense as singles or as doubles, i.e. as ^{8}Be. In microscopic studies of ^{12}C one, indeed, can see that in the $0_2{}^+$-state two of the three α's are slightly more closer to one another than to the third one [55]. The question is definitely very interesting and desserves future investigation, for instance in what concerns the identification of the α structure of the Hoyle state with the THSR wave function in the intrinsic frame. However, quantitatively, this constitutes probably only a slight modification over the present formulation of α-condensation.

Very recent triple α coincidence experiments [56] have shown that there exists a decay channel of the Hoyle state where the three α's share democratically the available energy, that is, each α carries away one third of the initial energy. This very rare three body decay may give further credit to the idea that the three α particles occupy the same 0S-orbit , that is, they are condensed in this state. Same results have been confirmed by a second group [57] and earlier, with more yield, with heavy ion reactions [58].

What about 'ab initio' calculations of the Hoyle state? Presently several groups are on the track [59][60]. The Los Alamos group has achieved to calculate the density of the Hoyle state, though it is not yet converged in the far tail [60]. It agrees very well with the density from the THSR wave function besides in the far tail.

In conclusion, we see that the idea of α-particle condensation in nuclei and nuclear systems has triggered many new ideas , calculations, and experiments, in spite of the fact that, so far, a compelling case for such a state has only been made in ^{12}C. Even so, the possible existence of a completely new nuclear phase in which α-particles play the role of quasi-elementary constituents is surely fascinating. Hopefully, many more α-particle states of nuclei will be detected in the near future, bringing deeper insights into the role of clustering and quantum condensates in systems of strongly interacting fermions.

Let us mention in the end that a more elaborate report on α particle condensation can be found in [61]

7. Acknowledgements

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