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One loop integration with hypergeometric series by using recursion relations

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Abstract. General one-loop integrals with arbitrary mass and kinematical parameters in d-dimensional space-time are studied. By using Bernstein theorem, a recursion relation is obtained which connects $(n+1)$-point to n-point functions. In solving this recursion relation, we have shown that one-loop integrals are expressed by a newly defined hypergeometric function, which is a special case of Aomoto-Gelfand hypergeometric functions.

We have also obtained coefficients of power series expansion around 4-dimensional space-time for two-, three- and four-point functions. The numerical results are compared with "LoopTools" for the case of two- and three-point functions as examples.

1. Introduction

For discovery of the beyond standard model, we need to know the precise theoretical prediction of standard model. For Large Hadron Collider at CERN and the international linear collider, at least next-to-leading order(NLO) electroweak corrections are necessary. However, it is not easy to calculate Feynman integrations with highly accuracy even for one-loop level. There appear many kinematical parameters including masses, momenta of particles, and space-time dimension. This means that the loop integrals are analytic functions with singularities on a multiple dimensional complex vector space. It is difficult to obtain the numerical accuracy in all kinematical region by a simple numerical approach. Since the numerical stability of an expression is connected to its analytic properties, suitable analytic expressions of one-loop integral are still important.

It is known that any loop integrals are expressed by GKZ-hypergeometric functions[1]. However, theses functions are so general extension of hypergeometric function that it is not easy to obtain numerical values. It is desirable to find a subset of GKZ-hypergeometric functions which corresponds to specific loop integrals to be calculated.

The analytic properties of hypergeometric functions, such as position of singularities, have been investigated for many hypergeometric functions. Since these singularities correspond to physical singularities or large cancellations in numerical calculations, information about singularities helps us to obtain accurate numerical results.

There are various methods to express one-loop integrals by hypergeometric functions[2, 3, 4, 5, 6]. In this article, we show a method to obtain analytic expressions of n-point functions with arbitrary kinematical parameters. Our method is based on Bernstein theorem [7] (see also [8]). This theorem implies that for given polynomial $\mathcal{D}$ of variables $x = (x_1, ..., x_n)$, there exist

differential operator $\mathcal{P}$ and polynomial $b(s)$ of parameter s such that

$$\mathcal{P}(\partial, x, s)\mathcal{D}^{s+1}(x) = b(s)\mathcal{D}^{s}(x), \tag{1}$$

where $\mathcal{P}$ is a polynomial of x, differential operator $\partial = (\partial_1, ..., \partial_n)$, and parameter s. Applying this theorem to the integrands of one-loop integrals, we obtain a recursion relation. In solving this relation, we show that one-loop integrals are expressed by newly introduced hypergeometric function G_n. This function is found to be one of Aomoto-Gelfand hypergeometric functions (or general hypergeometric functions on complex Grassmannians) [9, 10, 11] which make a subset of GKZ-hypergeometric functions.

Starting from our general expression of one-loop integral, two- and three-point function are re-expressed in a linear combination of Gauss hypergeometric function F and Appell's function F_3[12], respectively. For the case of scalar integral of four-point function, we have expanded the analytic expression around 4-dimensional space-time. The result is expressed by Lauricella's function F_D[12] up to the finite order of space-time dimension $d = 4 - 2\epsilon$.

We have numerically calculated two- and three-point function as examples and have compared the results with `LoopTools`[13].

2. Formulation

Let us consider the one-loop $(n+1)$-point function denoted by I_{n+1}. After performing integration of δ-function, the Feynman parameter integration of I_{n+1} is written as:

$$I_{n+1}(s) = \int_{\Delta_n} d^n x \; \mathcal{D}_n^s, \tag{2}$$

$$\begin{aligned}\mathcal{D}_n(x_1, ..., x_n) &= -\frac{1}{2}\sum_{j,k=0}^{n} q_{jk}^2 x_j x_k + \frac{1}{2}\sum_{j,k=0}^{n}(m_j^2 + m_k^2)x_j x_k, \\ &= \frac{1}{2}\sum_{i,j=1}^{n} A_{ij}x_i x_j + \sum_i B_i x_i + C \\ &= \frac{1}{2}(Ax, x) + (B, x) + C, \end{aligned} \tag{3}$$

where m_i are masses of the propagators, $q_{jk} \equiv -\sum_{i=j+1}^{k} p_i$ with external momenta p_i, x_i are Feynman parameters with $x_0 = 1 - \sum_{j\geq 0} x_j$. Here, Δ_n refers to n-dimensional simplex and $A_{ij} \equiv \partial_i\partial_j\mathcal{D}_n$, $B_i \equiv \partial_i\mathcal{D}_n(0)$ and $C \equiv \mathcal{D}_n(0)$. Parameter s depends on the space-time dimension d. If $d = 4 - 2\epsilon$ are chosen, we find $s = 1 - n - \epsilon$ for the standard scalar model.

Let us define operator $\mathcal{P}$ by

$$\mathcal{P} \equiv -(s+2)\frac{1}{E_n} + \frac{1}{2E_n}(A^{-1}\partial\mathcal{D}_n, \partial), \tag{4}$$

where $E_n = (A^{-1}B, B)/2 - C$. We can find the following relation:

$$\mathcal{P}\mathcal{D}_n^{s+1} = (s+1)\mathcal{D}_n^s. \tag{5}$$

This is the explicit expression of Bernstein theorem for scalar one-loop integral and polynomial b-function is found to be $s + 1$. Applying Eq.(5) to Eq.(2) iteratively with partial integrations,

we obtain

$$I_{n+1}(s) = J_{n+1,m}(s) + K_{n+1,m}(s), \tag{6}$$

$$J_{n+1,m}(s) = \frac{1}{2(s+1)E_n}\sum_{j=0}^{m-1}\frac{(s+n/2+1)_j}{(s+2)_j} \times \int_{\Delta_n} d^n x \sum_k \partial_k \left\{ (A^{-1}\partial \mathcal{D}_n)_k \mathcal{D}_n^{s+1}\left(-\frac{\mathcal{D}_n}{E_n}\right)^j \right\}, \tag{7}$$

$$K_{n+1,m}(s) = \frac{(s+n/2+1)_m}{(s+1)_m}\int_{\Delta_n} d^n x\, \mathcal{D}_n^s \left(-\frac{\mathcal{D}_n}{E_n}\right)^m, \tag{8}$$

where $(a)_j \equiv a(a+1)\cdots(a+j-1)$ is Pochhammer's symbol.

If we choose the appropriate parameter region, the second term $K_{n+1,m}(s)$ goes to be zero in the limit $m \to \infty$. The first term is surface integration and can be integrated once easily. The remaining integrations are expressed by n-point functions. We obtain a recursion relation between $(n+1)$- and n-point functions:

$$I_{n+1}(s) = \frac{1}{2}\sum_{j=0}^{\infty}\frac{(s+n/2+1)_j}{(s+1)_{j+1}}\left(-\frac{1}{E_n}\right)^{j+1}\sum_{k=0}^{n} h_{\rho(n),k} I_{n,\rho(n;k)}(s+j+1), \tag{9}$$

where coefficients $h_{\rho(n),k}$ are rational functions of kinematical variables. The suffix $\rho(n) = \Delta_n$ and $\rho(n;k)$ represents a $(n-1)$-dimensional simplex which is obtained by eliminating k-th vertex and faces attaching to the vertex from the original n-dimensional simplex, which appears as a part of the boundary of the original integration domain. In a similar way, we define $\rho(n;k_1,k_2) \equiv \rho(\rho(n;k_1),k_2)$. Using Eq.(9) repetitively, Eq.(2) eventually depend only on the remaining vertex, in which no integration is left. The final formula of $(n+1)$-point function is

$$\begin{aligned} I_{n+1}(s) = {} & \frac{1}{2^n}\sum_{k_n=0}^{n}\sum_{k_{n-1}=0}^{n-1}\cdots\sum_{k_1=0}^{1}\mathcal{D}(n;k_n,k_{n-1},...,k_1)^s h_{\rho(n),k_n} h_{\rho(n;k_n),k_{n-1}}\cdots h_{\rho(n;k_n,...,k_2),k_1} \\ & \times\sum_{j_n=0}^{\infty}\sum_{j_{n-1}=0}^{\infty}\cdots\sum_{j_1=0}^{\infty}\frac{(s+n/2+1)_{j_n}}{(s+1)_{j_n+1}}\frac{(s+j_n+(n-1)/2+2)_{j_{n-1}}}{(s+j_n+2)_{j_{n-1}+1}}\cdots \\ & \quad \frac{(s+j_n+\cdots+j_2+1/2+n)_{j_1}}{(s+j_n+\cdots+j_2+n)_{j_1+1}} \\ & \times\left(-\frac{\mathcal{D}(n;k_n,k_{n-1},...,k_1)}{E_{\rho(n)}}\right)^{j_n+1}\left(-\frac{\mathcal{D}(n;k_n,k_{n-1},...,k_1)}{E_{\rho(n;k_n)}}\right)^{j_{n-1}+1}\cdots \\ & \quad \left(-\frac{\mathcal{D}(n;k_n,k_{n-1},...,k_1)}{E_{\rho(n;k_n,...,k_2)}}\right)^{j_1+1}. \end{aligned} \tag{10}$$

Here, $\mathcal{D}(n;k_n,k_{n-1},...,k_1)$ is the value at the vertex k_{n+1} which does not appear in the list $(k_n,k_{n-1},\cdots,k_1)$. The right-hand side of Eq.(10) shows that it is expressed by a kind of hypergeometric series. We call this hypergeometric series function G_n, which is defined by Eq.(13) in the next section. Using this function, one-loop scalar integral becomes

$$\begin{aligned} I_{n+1}(s) = {} & \frac{1}{2^n(s+1)_n}\sum_{k_n=0}^{n}\sum_{k_{n-1}=0}^{n-1}\cdots\sum_{k_1=0}^{1}\frac{\mathcal{D}(n;k_n,k_{n-1},...,k_1)^{s+n}}{E_{\rho(n)}\cdots E_{\rho(n;k_n,...,k_2)}} \\ & \times h_{\rho(n),k_n} h_{\rho(n;k_n),k_{n-1}}\cdots h_{\rho(n;k_n,...,k_2),k_1} \\ & \times G_n\left(\alpha,\beta;\gamma;\left(-\frac{\mathcal{D}(n;k_n,k_{n-1},...,k_1)}{E_{\rho(n)}}\right),\cdots,\left(-\frac{\mathcal{D}(n;k_n,k_{n-1},...,k_1)}{E_{\rho(n;k_n,...,k_2)}}\right)\right), \end{aligned} \tag{11}$$

where

$$\alpha = (\underbrace{1, \cdots, 1}_{n}), \quad \beta = (\underbrace{1/2, \cdots, 1/2}_{n-1}, s + n/2 + 1), \quad \gamma = s + n + 1. \tag{12}$$

3. G_n-functions

In this section, we discuss the properties of G_n function. This function is defined by:

$$G_n(\alpha, \beta; \gamma; x) = \sum_{j_n=0}^{\infty} \sum_{j_{n-1}=0}^{\infty} \cdots \sum_{j_1=0}^{\infty} \frac{\prod_{i=1}^{n} (\alpha_i)_{j_i} \prod_{k=1}^{n} (\sum_{\ell=k}^{n} \beta_\ell)_{\sum_{\ell=k}^{n} j_\ell}}{(\gamma)_{\sum_{i=1}^{n} j_i} \prod_{k=1}^{n} (\sum_{\ell=k}^{n} \beta_\ell)_{\sum_{\ell=k+1}^{n} j_\ell} \prod_{i=1}^{n} j_i!} \prod_{i=1}^{n} x_i^{j_i}, \tag{13}$$

where $x = (x_1, ..., x_n) \in \mathbb{C}^n$ are variables, $\alpha = (\alpha_1, ..., \alpha_n)$ and $\beta = (\beta_1, ..., \beta_n)$ are complex vectors, and γ is a complex parameter.

Euler type integral representation of G_n is obtained as:

$$\begin{aligned} G_n(\alpha, \beta; \gamma; x) &= \frac{\Gamma(\gamma)}{\prod_{j=1}^{n} \Gamma(\alpha_j) \Gamma(\gamma - \sum_{k=1}^{n} \alpha_k)} \int_{\Delta_n} d^n u \prod_{k=1}^{n} u_k^{\alpha_k - 1} \\ &\quad \times \Big(1 - \sum_{j=1}^{n} u_j\Big)^{\gamma - \sum_{k=1}^{n} \alpha_k - 1} \prod_{k=1}^{n} \Big(1 - \sum_{j=1}^{k} x_j u_j\Big)^{-\beta_k}. \end{aligned} \tag{14}$$

The integrand is a product of powers of linear factors of integration variables, while the original integrand of I_{n+1} is a power of quadratic term. This representation shows that this function is a member of a class of hypergeometric functions, which are called Aomoto-Gelfand hypergeometric functions or hypergeometric functions on complex Grassmannians. This fact and their analytic properties give us important information for numerical calculation. One can easily show the following formulae:

- differentiation (with k-th unit vector e_k)

$$\frac{\partial}{\partial x_\ell} G_n(\alpha, \beta; \gamma; x) = \sum_{k=\ell}^{n} \frac{\alpha_\ell \beta_k}{\gamma} G_n(\alpha + e_\ell, \beta + e_k; \gamma + 1; x). \tag{15}$$

- recursion relation

$$\begin{aligned} G_n(\alpha, \beta; \gamma; x) &= \sum_{j_n=0}^{\infty} \frac{(\alpha_n)_{j_n} (\beta_n)_{j_n}}{(\gamma)_{j_n} j_n!} x_n^{j_n} G_{n-1}(\alpha', \beta'; \gamma'; x') \qquad (16) \\ &= \frac{\Gamma(\gamma)}{\Gamma(\alpha_1) \Gamma(\gamma - \alpha_1)} \\ &\quad \times \int_0^1 dw\, w^{\alpha_1 - 1} (1 - w)^{\gamma - \alpha_1 - 1} (1 - x_1 w)^{-\sum_{j=1}^{n} \beta_j} G_{n-1}(\alpha', \beta'; \gamma'; x'). \qquad (17) \end{aligned}$$

Tensor integrals are obtained by differentiating I_{n+1} in terms of mass parameters. Eq.(15) shows that tensor integrals are also expressed by G_n.

Based on the above results, the problem of calculating one-loop integral is converted to one of establishing methods of expansion around 4 space-time dimension and numerical evaluations.

We show some samples of scalar case in the next section.

4. Calculations of n-point functions

In this section, we will discuss how to evaluate and expand G_n functions. Let us discuss in detail for 2-, 3- and 4-point functions separately.

4.1. Two-point function

Let's first consider the two-point function. This is a good example of the understanding of how to evaluate G_n. From the formula Eq.(10), the scalar two-point function is expressed by G_1 functions with $\alpha = 1$, $\beta = s + 3/2$ and $\gamma = s + 2$. Function G_1 is nothing but Gaussian hypergeometric function:

$$G_1(1, s+3/2; s+2; x) = \sum_{j=0}^{\infty} \frac{(1)_j \, (s+3/2)_j}{(s+2)_j} \frac{x^j}{j!} = F(1, s+3/2, s+2; x). \tag{18}$$

Combining the kinematical factor, we obtain

$$\begin{aligned} I_2(s) &= \frac{(p^2 + m_1^2 - m_2^2)(m_1^2)^{s+1}}{(s+1)E_1} F\left(1, s+\frac{3}{2}, s+2; -\frac{4p^2 m_1^2}{E_1}\right) \\ &\quad + \frac{(p^2 + m_2^2 - m_1^2)(m_2^2)^{s+1}}{(s+1)E_1} F\left(1, s+\frac{3}{2}, s+2; -\frac{4p^2 m_2^2}{E_1}\right), \end{aligned} \tag{19}$$

where $E_1 \equiv (p^2 - (m_1 + m_2)^2)(p^2 - (m_1 - m_2)^2)$. The case of $s = -\epsilon$ corresponds to the usual dimensional regularization $d = 4 - 2\epsilon$. It is more convenient for the expansion when half-integer parameter is converted to integer. By using identities of F, we obtain

$$I_2(s) = \frac{\xi_-(m_1^2)^s}{s+1} F\left(1, -s; s+2; \frac{\xi_-}{\xi_+}\right) + \frac{(1-\xi_-)(m_2^2)^s}{s+1} F\left(1, -s; s+2; \frac{1-\xi_-}{1-\xi_+}\right), \tag{20}$$

where $\xi_\pm = (p^2 - m_1^2 + m_2^2 \pm \sqrt{E_1})/(2p^2)$. After the conversion, G_1 can be expanded in arbitrary order of ϵ with multiple polylogarithmic functions[14, 15].

With the known analytic properties of F, it is found that two-point function may only be singular when $\xi_\pm = 0$, 1, ∞ and $\xi_1 = \xi_2$. These cases correspond to massless or on-shell limit. Let us investigate the case both of masses are taken massless limit as an example, where $\xi_- \to 0$ and $\xi_+ \to 1$. The first term in Eq.(20) goes to zero, but second term is not well-defined at this limit. However, with using identities of F, we can transform the expression into the well-defined form at this limit under the condition $\mathrm{Re}(s) > 0$:

$$\lim_{m_1, m_2 \to 0} I_2(s) = \lim_{m_1, m_2 \to 0} \frac{(-p^2\xi_+)^s}{s+1} F\left(s+1, -s; s+2; \frac{1}{\xi_+}\right) = (-p^2)^s B(s+1, s+1), \tag{21}$$

where B is beta-function. This means that we can select appropriate representations in terms of kinematical conditions.

4.2. Three-point function

Eq.(11) shows that scalar three-point function is obtained as a linear combination of G_2. It is expressed as

$$I_3(s) = \frac{1}{(s+1)(s+2)} \sum_{k_1=1}^{2} \sum_{k_2=1}^{3} h_{k_1,k_2} G_2\left(\{1,1\}, \{1/2, s+2\}; s+3; x_{1,(k_1,k_2)}, x_{2,(k_1,k_2)}\right). \tag{22}$$

Function G_2 is equivalent to Appell's function F_3:

$$G_2((\alpha_1, \alpha_2), (\beta_1, \beta_2), \gamma; x_1, x_2) = (1 - x_1)^{-\alpha_1} F_3\Big(\alpha_1, \alpha_2, \gamma - \beta_1 - \beta_2, \beta_2, \gamma; \frac{x_1}{x_1 - 1}, x_2\Big). \tag{23}$$

When $\alpha_1 = \alpha_2 = 1$, G_2 reduces to Appell's function F_1. It is also convenient when the half-integer parameter is transformed to integer as same as the case of two-point function. We apply nontrivial identity:

$$F_1(\alpha, \beta, \beta', \gamma; x, y) = (1+z)^{\alpha} F_D\Big(\alpha; \alpha-\gamma+1, 1-\alpha, 2\beta-1, \beta', \beta'; \gamma; z, \frac{z}{1+z}, \frac{2z}{1+z}, \frac{1}{v_+}, \frac{1}{v_-}\Big), \quad (24)$$

where $z = (1-\sqrt{1-x})/(1+\sqrt{1-x})$, $v_\pm = (1+\sqrt{1-x})/(y \mp \sqrt{y(y-x)})$ and F_D is on of Lauricella's functions[12]. For the case $s = -1-\epsilon$, Eq.(22) reduces to

$$G_2 \to F_1(\text{half-integer}) \to F_D(\text{integer}) \to (\text{expansion}) \to F_1(\text{integer}) \\ \to \text{multiple polylogarithmic functions in arbitrary order of } \epsilon.$$

Investigating the limit $\epsilon \to 0$, $1/\epsilon$ pole appears from $1/(s+1)$ of Eq.(22) for both massive and massless cases. However, this poles canceled out when all contributions are summed up for the massive case. So we can obtain the value of integration in this limit.

4.3. Four-point function

After performing δ-function integration, three Feynman parameters remain on the four-point function.

In this case, G_3 appears in Eq.(11).

$$G_3(\alpha, \beta; \gamma; x) = \frac{\Gamma(\gamma)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)\Gamma(\gamma-\alpha_1-\alpha_2-\alpha_3)} \\ \times \int_{\Delta_3} d^3u\, u_1^{\alpha_1-1} u_2^{\alpha_2-1} u_3^{\alpha_3-1} (1-u_1-u_2-u_3)^{\gamma-\alpha_1-\alpha_2-\alpha_3-1} \\ \times (1-x_1u_1)^{-\beta_1} (1-x_1u_1-x_2u_2)^{-\beta_2} (1-x_1u_1-x_2u_2-x_3u_3)^{-\beta_3}. \quad (25)$$

For scalar integral case, parameters take the values $\alpha = \{1,1,1\}$, $\beta = \{1/2, 1/2, s+5/2\}$, $\gamma = s+4$, $x = \{x_1, x_2, x_3\}$, and $s = -2-\epsilon$ for $d = 4-2\epsilon$. This function can be written in a linear combination of F_D up to $\mathcal{O}(\epsilon)$, which corresponds to finite order of I_4, since

$$G_3\left(\{1,1,1\}, \left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}-\epsilon\right\}; 2-\epsilon, x_1, x_2, x_3\right) \\ = C_1 F_1\left(1, 1, \frac{1}{2}; 2; \frac{x_2-x_1}{x_2-1}, \frac{x_3-x_1}{x_3-1},\right) + C_2 F_D\left(1, 2\epsilon, 1, 1, \frac{1}{2}, \frac{1}{2}; 2; \frac{1-\sqrt{1-x_3}}{2}, \frac{1}{\eta_1}, \frac{1}{\eta_2}, \frac{1}{\eta_3}, \frac{1}{\eta_4}\right) \\ + C_3 F_D\left(1, -\epsilon, 1, \frac{1}{2}; 2-\epsilon; x_1, \frac{x_1-x_2}{1-x_2}, \frac{x_1-x_3}{1-x_3}\right) + \mathcal{O}(\epsilon^2), \quad (26)$$

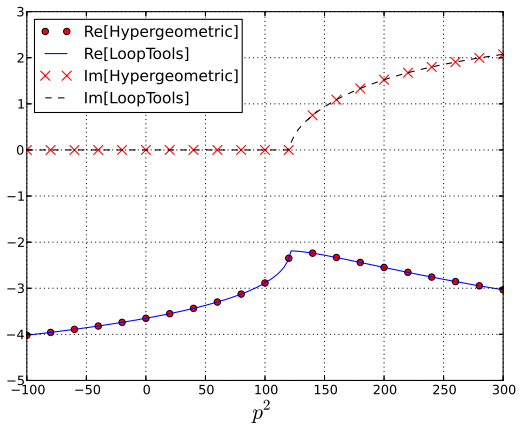
where the coefficients C_i's and η_i's are algebraic functions of x_1, x_2, and x_3. The half-integer parameters are converted to integers by using extended identities from Eq.(24).

4.4. Summary numerical calculation

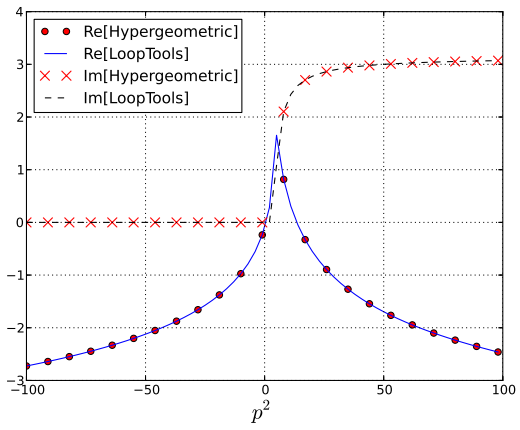
We have compared the numerical results between our method and `LoopTools`[13]. We show the compared results of two- and three-point function in Figs.1 and 2, respectively. The results are consistent in satisfactory accuracy.

5. Conclusion and discussion

In the discussion of Sec.2, it is necessary to select appropriate kinematical region in order to make Eq.(8) vanishes at the limit $m \to \infty$. However, we can show the following identity of

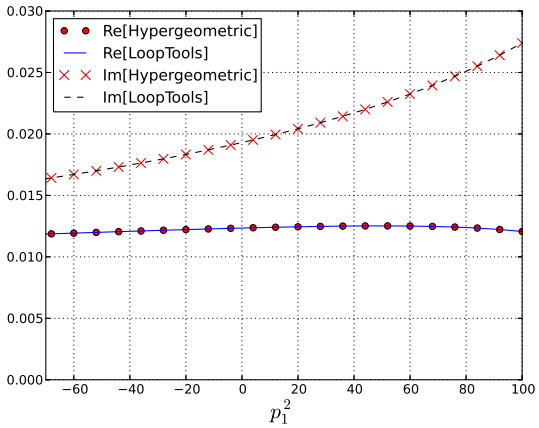


(a) $m_1^2 = 10$, $m_2^2 = 1$

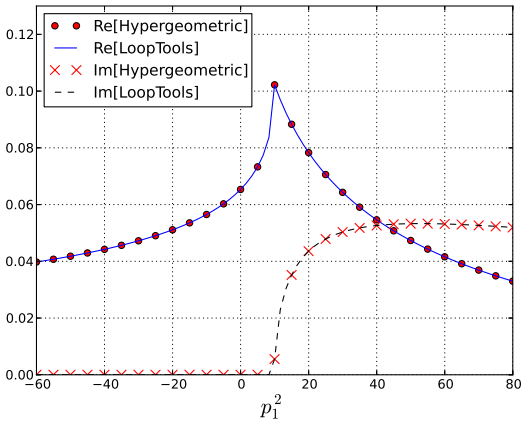


(b) $m_1^2 = 1$, $m_2^2 = 1.1$

Figure 1. Comparison of numerical results with `LoopTools` for two-point functions in two parameter sets (a) and (b). Circles and crosses are the numerical results of real and imaginary part of our calculation, respectively. Solid and dashed lines are the results of real and imaginary part which are obtained from `LoopTools`, respectively.



(a) $p_2^2 = -10$, $p_3^2 = 100$,
$m_1^2 = 1$, $m_2^2 = 10$, $m_3^2 = 100$



(b) $p_2^2 = -10$, $p_3^2 = -100$,
$m_1^2 = 1$, $m_2^2 = 2$, $m_3^2 = 3$

Figure 2. Comparison of numerical results of three-point functions for two parameter sets (a) and (b). Circles and crosses are the numerical results of real and imaginary part of our calculation, respectively. Solid and dashed lines are the results of real and imaginary part which are obtained from `LoopTools`, respectively.

Gauss hypergeometric function:

$$\mathcal{D}_n^s = \frac{1}{2(s+1)E_n} \sum_k \partial_k \left[\left(A^{-1} \partial \mathcal{D}_n \right)_k F\left(1, s + \frac{n}{2} + 1; s+2; -\frac{\mathcal{D}_n}{E_n} \right) \right]. \tag{27}$$

From this identity, one can derive recursion relation Eq.(9) and confirm it holds in all kinematical region[16].

We have shown that general one-loop integral is expressed by G_n, one of hypergeometric functions on complex Grassmannian. Especially, scalar two- and three-point functions are expressed in terms of Gaussian and Appell's functions, respectively for any kinematics variables and space-time dimension. Four-point function is expressed in Lauricella's functions up to finite order for arbitrary kinematical parameters. We have also shown the sample numerical calculation in terms of two-, and three-point functions and results are consistent with `LoopTools` package.

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