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Quantum oscillator models with a discrete position spectrum in the framework of Lie superalgebras

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Abstract. We present some algebraic models for the quantum oscillator based upon Lie superalgebras. The Hamiltonian, position and momentum operator are identified as elements of the Lie superalgebra, and then the emphasis is on the spectral analysis of these elements in Lie superalgebra representations. The first example is the Heisenberg-Weyl superalgebra $\mathfrak{sh}(2|2)$, which is considered as a "toy model". The representation considered is the Fock representation. The position operator has a discrete spectrum in this Fock representation, and the corresponding wavefunctions are in terms of Charlier polynomials. The second example is $\mathfrak{sl}(2|1)$, where we construct a class of discrete series representations explicitly. The spectral analysis of the position operator in these representations is an interesting problem, and gives rise to discrete position wavefunctions given in terms of Meixner polynomials. This model is more fundamental, since it contains the paraboson oscillator and the canonical oscillator as special cases.

1. Introduction

In all textbooks on quantum mechanics, it is described how the position wavefunctions of the one-dimensional canonical quantum oscillator are given in terms of Hermite polynomials. This is a first example of the relation between oscillator models and special functions. One of the alternative oscillator models is the so-called paraboson oscillator [1, 2], and here the position wavefunctions are given in terms of Laguerre polynomials [3, 4].

Both of these well-known oscillator models have an algebraic description as well: for the canonical oscillator this is in terms of the oscillator Lie algebra (and its Fock representation); for the paraboson oscillator this is in terms of the Lie superalgebra $\mathfrak{osp}(1|2)$ and its positive discrete series representations.

During the last years there has been increasing interest in other algebraic oscillator models, some of them inspired by applications in quantum optics or signal analysis. For these new oscillator models, the spectrum of the position operator can be continuous or discrete (with finite or infinite support). In the case of a discrete spectrum, one is faced with the interesting phenomenon of "discrete position wavefunctions", and there is a relation with discrete orthogonal polynomials.

The paper that triggered most of the work in this area is [5], where an oscillator model based on the Lie algebra $\mathfrak{su}(2)$ was introduced. In this model, the position wavefunctions are

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indeed discrete and expressed in terms of symmetric Krawtchouk polynomials. This model was extended by an extra parameter in [6, 7], with an underlying deformed $\mathfrak{su}(2)$ algebra, and position wavefunctions given by means of Hahn polynomials.

In the current contribution, we wish to present some examples of quantum oscillator models where the underlying algebra is a Lie superalgebra. For the first example, the underlying Lie superalgebra is the Heisenberg-Weyl superalgebra $\mathfrak{sh}(2|2)$, an algebra generated by one boson and one fermion pair. We study the spectral analysis of a position operator $\hat{q}$ in the Fock space representation of $\mathfrak{sh}(2|2)$. This operator has an infinite but discrete spectrum, and the corresponding position wavefunctions are given in terms of Charlier polynomials. We observe that also for an underlying orthosymplectic Lie superalgebra $\mathfrak{osp}(3|2)$, one would get the same position wavefunctions. This $\mathfrak{sh}(2|2)$ model is interesting because of its simplicity, but from the physical point of view it is maybe just a "toy model". The second example presented here has $\mathfrak{sl}(2|1)$ as underlying Lie superalgebra. In this model, the spectrum of a position operator can be discrete or continuous, depending on the value of a parameter γ. In the discrete case, the position wavefunctions are related to Meixner polynomials. In the continuous case, the position wavefunction reduce to the mentioned paraboson wavefunctions and thus, for a specific value of a representation parameter, to the canonical oscillator. Because of these special cases, the canonical oscillator is naturally embedded in the $\mathfrak{sl}(2|1)$ model, and it can be considered as more fundamental.

The examples presented here have been discussed in two previous papers. The $\mathfrak{sh}(2|2)$ toy model was analyzed in [8], and the $\mathfrak{sl}(2|1)$ in [9]. The purpose of the current contribution is to briefly present these examples again, compare them, give a relation to an $\mathfrak{osp}(3|2)$ model, and discuss some of the most significant properties.

To finalize this introduction, let us briefly recall the context of "algebraic oscillator models." For these models, one requires the same dynamics as for the classical or quantum oscillator, but the operators corresponding to position, momentum and Hamiltonian can be elements of some algebra different from the traditional Heisenberg (or oscillator) Lie algebra [5, 10, 11, 6, 12]. In the one-dimensional case, there are three (essentially self-adjoint) operators involved: the position operator $\hat{q}$, its corresponding momentum operator $\hat{p}$ and the Hamiltonian $\hat{H}$ which is the generator of time evolution. The main requirement is that these operators should satisfy the Hamilton-Lie equations (or the compatibility of Hamilton's equations with the Heisenberg equations):

$$[\hat{H}, \hat{q}] = -\mathrm{i}\hat{p}, \qquad [\hat{H}, \hat{p}] = \mathrm{i}\hat{q}, \tag{1}$$

in units with mass and frequency both equal to 1, and $\hbar = 1$. Contrary to the canonical case, the commutator $[\hat{q}, \hat{p}] = \mathrm{i}$ is not required. Apart from (1) and the self-adjointness, it is then common to require the following conditions [5]:

- all operators $\hat{q}$, $\hat{p}$, $\hat{H}$ belong to some Lie algebra or Lie superalgebra $\mathcal{A}$;
- the spectrum of $\hat{H}$ in (unitary) representations of $\mathcal{A}$ is equidistant.

2. The $\mathfrak{sh}(2|2)$ oscillator model

Consider a bosonic creation and annihilation operator $b^{\pm}$ with commutation relation

$$[b^-, b^+] = 1, \tag{2}$$

and a fermionic creation and annihilation operator $a^{\pm}$ with anticommutation relations

$$\{a^-, a^-\} = \{a^+, a^+\} = 0, \qquad \{a^-, a^+\} = 1. \tag{3}$$

Let us also assume that the two sets of operators commute with each other:

$$[b^{\xi}, a^{\eta}] = 0, \qquad \xi, \eta \in \{+, -\}. \tag{4}$$

Then the Lie superalgebra generated by the even elements $1, b^+, b^-$ and the odd elements a^+, a^- is known as the Heisenberg-Weyl superalgebra $\mathfrak{sh}(2|2)$ [13, 14]. It will be more convenient to work with a subalgebra of the enveloping algebra $\mathcal{U}(\mathfrak{sh}(2|2))$, namely the Lie superalgebra $\mathcal{S}$ with four odd basis elements

$$F^+ = a^+, \quad F^- = a^-, \quad Q^+ = b^+a^-, \quad Q^- = b^-a^+, \tag{5}$$

and even basis elements

$$E^+ = b^+, \quad E^- = b^-, \quad H = b^+b^- + a^+a^- \text{ and } 1. \tag{6}$$

The complete set of (anti)commutation relations among the elements of $\mathcal{S}$ is easy to determine [8]. The common Fock representation V of $\mathfrak{sh}(2|2)$ is generated by a vacuum vector $|0\rangle$ and by the relations $b^-|0\rangle = a^-|0\rangle = 0$. An orthonormal basis of V is given by the vectors

$$(a^+)^j \frac{(b^+)^m}{\sqrt{m!}}|0\rangle, \qquad m \in \{0, 1, 2, \ldots\},\ j \in \{0, 1\}.$$

It will be convenient to write these vectors as $|n\rangle$, with $n = 0, 1, 2, \ldots$, so V can be identified with the Hilbert space $\ell^2(\mathbb{Z}_+)$:

$$|2m\rangle = \frac{(b^+)^m}{\sqrt{m!}}|0\rangle, \qquad |2m+1\rangle = a^+\frac{(b^+)^m}{\sqrt{m!}}|0\rangle. \tag{7}$$

Then the action of all basis elements of $\mathcal{S}$ on these basis vectors is easy to determine. In particular, for the odd elements one finds:

$$\begin{aligned} F^+|n\rangle &= \frac{1}{2}(1+(-1)^n)|n+1\rangle, \qquad F^-|n\rangle = \frac{1}{2}(1-(-1)^n)|n-1\rangle, \\ Q^+|n\rangle &= \frac{1}{2}(1-(-1)^n)\sqrt{\frac{n+1}{2}}|n+1\rangle, \qquad Q^-|n\rangle = \frac{1}{2}(1+(-1)^n)\sqrt{\frac{n}{2}}|n-1\rangle. \end{aligned} \tag{8}$$

Note also that the usual conjugacy for the creation and annihilation operators leads to a natural $\star$-structure for $\mathcal{S}$, and the Fock representation is unitary with respect to this $\star$-structure.

In [8], we constructed an algebraic oscillator model with $\mathcal{S}$ as underlying Lie superalgebra. More precisely, using the reflection operator R with action $R|n\rangle = (-1)^n|n\rangle$, the Hamiltonian, position and momentum operators of the model are given by

$$\hat{H} = 2H + \frac{1}{2}R, \tag{9}$$

$$\hat{q} = \gamma F^+ + Q^+ + \gamma F^- + Q^-, \tag{10}$$

$$\hat{p} = \mathrm{i}\gamma F^+ + \mathrm{i}Q^+ - \mathrm{i}\gamma F^- - \mathrm{i}Q^-. \tag{11}$$

The spectrum of $\hat{H}$ in V coincides with that of the canonical oscillator, $\hat{H}|n\rangle = (n+\frac{1}{2})|n\rangle$, and the three self-adjoint operators satisfy (1). Herein, γ is a free parameter. Up to an arbitrary factor, (10) is the most general self-adjoint odd element in the Lie superalgebra $\mathcal{S}$. The main effort then goes to the spectral analysis of the position operator $\hat{q}$, which is represented by an infinite symmetric tridiagonal matrix in the ordered basis $\{|n\rangle,\ n = 0, 1, 2, \ldots\}$ of the Fock

space V:

$$\hat{q} = \begin{pmatrix} 0 & \gamma & & & & & & \\ \gamma & 0 & \sqrt{1} & & & & & \\ & \sqrt{1} & 0 & \gamma & & & & \\ & & \gamma & 0 & \sqrt{2} & & & \\ & & & \sqrt{2} & 0 & \gamma & & \\ & & & & \gamma & 0 & \sqrt{3} & \\ & & & & & \sqrt{3} & 0 & \ddots \\ & & & & & & \ddots & \ddots \end{pmatrix}. \tag{12}$$

For $\gamma > 0$, such a matrix is a Jacobi matrix, and its spectral theory is related to orthogonal polynomials [15, 16, 17]. One should construct polynomials $p_n(x)$ of degree n in x, with $p_{-1}(x) = 0$, $p_0(x) = 1$, and recurrence relation determined by the matrix (12),

$$\begin{aligned} x p_{2n}(x) &= \sqrt{n} p_{2n-1}(x) + \gamma p_{2n+1}(x), \\ x p_{2n+1}(x) &= \gamma p_{2n}(x) + \sqrt{n+1} p_{2n+2}(x), \qquad (n = 0, 1, 2, \ldots). \end{aligned} \tag{13}$$

If certain conditions are satisfied, the support of the weight function $w(x)$ for these polynomials determines the spectrum of the operator $\hat{q}$. Furthermore, for a real value x belonging to this support, the corresponding formal eigenvector of $\hat{q}$ is given by

$$v(x) = \sum_{n=0}^{\infty} p_n(x) \, |n\rangle. \tag{14}$$

In [8] we have shown that the solution of this recurrence relation leads to Charlier polynomials $C_n(x; a)$, defined by [18, 19, 20]:

$$C_n(x; a) = {}_2F_0\left(\begin{matrix} -n, -x \\ - \end{matrix} ; -\frac{1}{a} \right). \tag{15}$$

More particularly, for $\gamma \neq 0$, the solution of (13) is

$$p_{2n}(x) = \frac{(-\gamma)^n}{\sqrt{n!}} C_n(x^2; \gamma^2), \qquad p_{2n+1}(x) = -\frac{(-\gamma)^{n-1}}{\sqrt{n!}} \, x \, C_n(x^2 - 1; \gamma^2). \tag{16}$$

Then, using the orthogonality of Charlier polynomials [18], one deduces that the polynomials $p_n(x)$ satisfy a discrete orthogonality relation:

$$\sum_{x \in \mathbb{S}} w(x) p_n(x) p_m(x) = \mathrm{e}^{\gamma^2} \delta_{mn}, \tag{17}$$

where

$$\mathbb{S} = \{ \pm\sqrt{k} \mid k \in \mathbb{Z}_+ \} = \{ \ldots, -\sqrt{3}, -\sqrt{2}, -1, 0, 1, \sqrt{2}, \sqrt{3}, \ldots \}, \tag{18}$$

and where the weight function is given by

$$w(x) = \frac{1}{2} \frac{\gamma^{2k}}{k!} \qquad \text{for } x = \pm\sqrt{k} \quad (k = 1, 2, 3, \ldots) \tag{19}$$

and by $w(x) = 1$ for $x = 0$. So (18) gives the position spectrum.

It will be convenient to normalize the polynomials, and consider the corresponding orthonormal functions $\tilde{p}_n(x)$ satisfying $\sum_{x\in\mathbb{S}} \tilde{p}_m(x)\tilde{p}_n(x) = \delta_{mn}$. Then the normalized eigenvectors of $\hat{q}$, for an eigenvalue $x \in \mathbb{S}$, are given by

$$\tilde{v}(x) = \sum_{n=0}^{\infty} \tilde{p}_n(x)|n\rangle. \tag{20}$$

In such an expression, the overlap between the normalized $\hat{q}$-eigenvectors (20) and the $\hat{H}$-eigenvectors $|n\rangle$ can be interpreted as the position wavefunctions and will be denoted by $\varphi_n^{(\gamma)}(x)$. So these are discrete position wavefunctions, with support $\mathbb{S}$, given by

$$\varphi_n^{(\gamma)}(x) = \tilde{p}_n(x), \tag{21}$$

where x belongs to $\mathbb{S}$. The parameter γ originates from the freedom in choosing $\hat{q}$, in (10). In Figure 1 of [8], one can find example plots of $\varphi_n^{(\gamma)}(x)$, for certain values of n and of γ. So these are discrete plots, with the wavefunction consisting of "dots" positioned at $\pm\sqrt{k}$ ($k = \{0, 1, 2, 3, \ldots\}$). The shape of these wavefunctions depends on the value of γ. For $\gamma < 1$, the shape is close to a discrete version of the normalized Hermite wavefunctions of the canonical oscillator. As γ increases ($\gamma > 1$), the shape is closer to a discrete version of the paraboson wavefunctions. Note that these observation are just descriptive. For actual limits of the wavefunctions we refer to [8].

An additional remark relates the current structure also to the Lie superalgebra $\mathfrak{osp}(3|2)$. Indeed, consider the superalgebra generated by the same elements $a^{\pm}$ and $b^{\pm}$ satisfying (3) and (2), but with a different grading: $a^{\pm}$ are even elements, $b^{\pm}$ are odd elements, and they mutually anticommute:

$$\{b^{\xi}, a^{\eta}\} = 0, \qquad \xi, \eta \in \{+, -\}. \tag{22}$$

It is known that the superalgebra generated by these elements and relations is the enveloping algebra of $\mathfrak{osp}(3|2)$ [21]. So for $\mathfrak{osp}(3|2)$ one can construct a similar Fock space with the same basis vectors (7). In this representation space, the operator $\gamma a^+ + \gamma a^- + b^+a^- + a^+b^-$ has the same action as (10), so its spectrum and spectral analysis is the same. Note, however, that we are now working in a completely different superalgebra, namely $\mathfrak{osp}(3|2)$. In this context, the operator $\gamma a^+ + \gamma a^- + b^+a^- + a^+b^-$ does not have a direct meaning as a position operator: in fact, it is a linear combination of both even and odd elements of $\mathfrak{osp}(3|2)$.

To conclude this section, note that we are certainly not the first to relate Charlier polynomials to quantum oscillators. They have been related to the Weyl algebra in [22, 23], and appear as overlap coefficients between eigenvectors of a number operator and a "shifted" number operator [23]. In [24] some finite shift operators were defined closely related to $\hat{q}$ and $\hat{p}$, with a spectral analysis also in terms of Charlier polynomials. Still, it should be noticed that the operators (10)-(11) studied here are different from the ones investigated earlier in the literature, and the solution in terms of Charlier polynomials uses quite different properties of these polynomials.

3. The $\mathfrak{sl}(2|1)$ oscillator model

The example described in this section has been analyzed in detail in [9]. The idea is the same as in the previous section, but now we work with a different Lie superalgebra, which turns out to be more fundamental. This superalgebra is $\mathfrak{sl}(2|1)$, which has a basis consisting of four odd (or 'fermionic') elements F^+, F^-, G^+, G^- and four even (or 'bosonic') elements H, E^+, E^-, Z,

given by

$$F^+ = e_{32},\ G^+ = e_{13},\ F^- = e_{31},\ G^- = e_{23}, \tag{23}$$

$$H = \frac{1}{2}(e_{11} - e_{22}),\ E^+ = e_{12},\ E^- = e_{21},\ Z = \frac{1}{2}(e_{11} + e_{22}) + e_{33} \tag{24}$$

in terms (graded) 3×3 Weyl matrices e_{ij}. From this matrix form, the Lie superalgebra brackets (i.e. commutators and anti-commutators) can be deduced (see [25, p. 261], [26] or [12]).

$\mathfrak{sl}(2|1)$ has both finite and infinite-dimensional irreducible representations, and the unitarity of these representations depends on the choice of a $\star$-structure (or an adjoint operation) on the Lie superalgebra. Here this is:

$$Z^\dagger = Z, \quad H^\dagger = H, \quad (E^\pm)^\dagger = -E^\mp, \quad (F^\pm)^\dagger = \mp G^\mp, \quad (G^\pm)^\dagger = \pm F^\mp, \tag{25}$$

and for this $\star$-structure the finite-dimensional representations are not unitary. But one can construct a class of infinite-dimensional positive discrete series representations of $\mathfrak{sl}(2|1)$, Π_β, labeled by $\beta > 0$, which are unitary [9]. The representation space is $\ell^2(\mathbb{Z}_+)$ equipped with an orthonormal basis $|\beta, n\rangle$ ($n = 0, 1, 2, \ldots$), i.e. $\langle \beta, m|\beta, n\rangle = \delta_{m,n}$. For the actions of the $\mathfrak{sl}(2|1)$ basis elements on these vectors, it is handy to use the following "even" and "odd" functions, defined on integers n:

$$\mathcal{E}(n) = 1 \text{ if } n \text{ is even and 0 otherwise;} \qquad \mathcal{O}(n) = 1 \text{ if } n \text{ is odd and 0 otherwise.} \tag{26}$$

Let us just give the actions of the odd generators here:

$$\begin{aligned} F^+|\beta, n\rangle &= \mathcal{E}(n)\sqrt{\beta + \frac{n}{2}}\ |\beta, n+1\rangle, \qquad & F^-|\beta, n\rangle &= \mathcal{E}(n)\sqrt{\frac{n}{2}}\ |\beta, n-1\rangle, \\ G^+|\beta, n\rangle &= \mathcal{O}(n)\sqrt{\frac{n+1}{2}}\ |\beta, n+1\rangle, \qquad & G^-|\beta, n\rangle &= -\mathcal{O}(n)\sqrt{\beta + \frac{n-1}{2}}\ |\beta, n-1\rangle. \end{aligned} \tag{27}$$

The verification that this leads to a unitary irreducible representation is given in [9].

Just as before, we can construct a model by identifying three self-adjoint operators

$$\hat{H} = 2H + \frac{1}{2} - \beta, \tag{28}$$

$$\hat{q} = F^+ + \gamma G^+ - G^- + \gamma F^-, \tag{29}$$

$$\hat{p} = \mathrm{i}(F^+ + \gamma G^+ + G^- - \gamma F^-). \tag{30}$$

In the representation Π_β, the spectrum of $\hat{H}$ coincides with that of the canonical oscillator, and the three operators satisfy (1). Again, in (29), γ is a free parameter; the choice of the position operator $\hat{q}$ is such that it is (up to a factor) the most general self-adjoint odd element of $\mathfrak{sl}(2|1)$. Then the form of $\hat{p}$ follows from (1). In the (ordered) basis $\{|\beta, n\rangle,\ n = 0, 1, 2, \ldots\}$, the operator $\hat{q}$ is represented by an infinite symmetric tridiagonal matrix M_q:

$$M_q = \begin{pmatrix} 0 & R_0 & & & & \\ R_0 & 0 & S_1 & & & \\ & S_1 & 0 & R_1 & & \\ & & R_1 & 0 & S_2 & \\ & & & S_2 & 0 & \ddots \\ & & & & \ddots & \ddots \end{pmatrix}, \tag{31}$$

where

$$R_n = \sqrt{\beta + n}, \qquad S_n = \gamma\sqrt{n} \qquad (n = 0, 1, 2, \ldots). \tag{32}$$

For $\gamma > 0$, such a matrix is a Jacobi matrix, and its spectral theory has been discussed in [9]. Like in the previous section, one should first construct polynomials with a recurrence relation governed by the matrix (31), i.e. with $p_{-1}(x) = 0$, $p_0(x) = 1$, and

$$\begin{aligned} xp_{2n}(x) &= S_n p_{2n-1}(x) + R_n p_{2n+1}(x), \\ xp_{2n+1}(x) &= R_n p_{2n}(x) + S_{n+1} p_{2n+2}(x), \qquad (n = 0, 1, 2, \ldots). \end{aligned} \tag{33}$$

Such polynomials are orthogonal for some positive weight function $w(x)$, and the spectrum of M_q (or of $\hat{q}$) is the support of this weight function. Furthermore, for a real value x belonging to this support, the corresponding formal eigenvector of $\hat{q}$ is given by

$$v(x) = \sum_{n=0}^{\infty} p_n(x)\,|\beta, n\rangle. \tag{34}$$

In the current case, the solution of (33) depends on the value of γ. When $\gamma^2 \neq 1$, the solution of the recurrence relations (33) is given by [9]

$$\begin{aligned} p_{2n}(x) &= (-\gamma)^{-n}\sqrt{\frac{(\beta)_n}{n!}}\;{}_2F_1\left(\begin{matrix}-n, \frac{x^2}{1-\gamma^2} \\ \beta\end{matrix}; 1-\gamma^2\right), \\ p_{2n+1}(x) &= x(-\gamma)^{-n}\sqrt{\frac{(\beta+1)_n}{n!\beta}}\;{}_2F_1\left(\begin{matrix}-n, \frac{x^2}{1-\gamma^2}+1 \\ \beta+1\end{matrix}; 1-\gamma^2\right). \end{aligned} \tag{35}$$

When $\gamma^2 = 1$, the solution is of a different type and given by [9]

$$\begin{aligned} p_{2n}(x) &= (-\gamma)^{n}\sqrt{\frac{(\beta)_n}{n!}}\;{}_1F_1\left(\begin{matrix}-n \\ \beta\end{matrix}; x^2\right), \\ p_{2n+1}(x) &= x(-\gamma)^{n}\sqrt{\frac{(\beta+1)_n}{n!\beta}}\;{}_1F_1\left(\begin{matrix}-n \\ \beta+1\end{matrix}; x^2\right). \end{aligned} \tag{36}$$

In the first case, these polynomials can be related to the Meixner polynomial $M_n(k; \beta, c)$ of degree n in k, with parameters β and c [18, 19, 20]:

$$M_n(k; \beta, c) = \;{}_2F_1\left(\begin{matrix}-n, -k \\ \beta\end{matrix}; 1 - \frac{1}{c}\right), \tag{37}$$

and these satisfy a discrete orthogonality. In the second case ($\gamma^2 = 1$) these polynomials are related to Laguerre polynomials, and their orthogonality is continuous. For technical reasons, the case $\gamma^2 \neq 1$ should be split in two subcases ($|\gamma| < 1$ and $|\gamma| > 1$), and we shall consider only one of them. For $|\gamma| > 1$, the polynomials $p_n(x)$ satisfy a discrete orthogonality relation:

$$\sum_{x \in \mathbb{S}_1} w(x) p_n(x) p_m(x) = \left(\frac{\gamma^2}{\gamma^2 - 1}\right)^{\beta} \delta_{mn}, \tag{38}$$

where

$$\mathbb{S}_1 = \{\pm\sqrt{\gamma^2 - 1}\sqrt{k} \mid k \in \mathbb{Z}_+\}, \tag{39}$$

and where the weight function is given by

$$w(x) = \frac{1}{2}(1+\delta_{k,0})\frac{(\beta)_k}{k!}\gamma^{-2k} \qquad \text{for } x = \pm\sqrt{\gamma^2-1}\sqrt{k} \quad (k = 0, 1, 2, \ldots). \tag{40}$$

So in this case, the spectrum of the position operator $\hat{q}$ is discrete and given by (39). For $|\gamma| = 1$, the orthogonality relation of Laguerre polynomials implies that the polynomials $p_n(x)$ satisfy a continuous orthogonality relation:

$$\int_{-\infty}^{+\infty} w(x)p_n(x)p_m(x)dx = \Gamma(\beta)\delta_{mn}, \tag{41}$$

where

$$w(x) = \mathrm{e}^{-x^2}|x|^{2\beta-1}. \tag{42}$$

For $|\gamma| < 1$ the polynomials satisfy again a discrete orthogonality relation, deduced from the one for Meixner polynomials.

For an interpretation as position wavefunction, it will be useful to normalize the corresponding polynomials $p_n(x)$, in such a way that the normalized ones $\tilde{p}_n(x)$ satisfy $\sum_{x\in\mathbb{S}_1}\tilde{p}_m(x)\tilde{p}_n(x) = \delta_{mn}$ (or $\int \tilde{p}_m(x)\tilde{p}_n(x)dx = \delta_{mn}$ in case $\gamma^2 = 1$), and then

$$\tilde{v}(x) = \sum_{n=0}^{\infty} \tilde{p}_n(x)\,|\beta, n\rangle. \tag{43}$$

It will be convenient to write these coefficients as

$$\tilde{v}(x) = \sum_{n=0}^{\infty} \Phi_n^{(\beta,\gamma)}(x)\,|\beta, n\rangle. \tag{44}$$

Herein, x belongs to the spectrum of $\hat{q}$, and in the notation of the coefficients (or overlaps) $\Phi_n^{(\beta,\gamma)}(x)$ we have emphasized the dependence on the representation parameter β and the model parameter γ.

The overlaps $\Phi_n^{(\beta,\gamma)}(x)$ have again an interpretation as position wavefunctions. In this $\mathfrak{sl}(2|1)$ model, some interesting phenomena occur. For $\gamma^2 > 1$, these wavefunctions satisfy a discrete orthogonality, and their plots consist of discrete "dots" at positions determined by the support (39). Note that these dots get "closer" as γ gets closer to 1. When $\gamma^2 = 1$, the support becomes $\mathbb{R}$, and we get continuous wavefunctions. Plots of these wavefunctions can be found in Figures 1 and 2 of [9]. Note that the model parameter determines the type of the spectrum: discrete or continuous; and when discrete γ also determines how close the dots of the spectrum are to each other. The representation parameter β determines the shape of the wavefunctions: for $\beta = 1/2$ the shape (whether discrete or continuous) is close to that of the wavefunction of the canonical oscillator, for $\beta > 1/2$ the shape is close to that of the paraboson oscillator. More precisely, for $\gamma = 1$ and arbitrary $\beta > 0$, the wavefunctions $\Phi_n^{(\beta,1)}(x)$ *coincide* with the well known wavefunctions of the paraboson oscillator [9]. Since the canonical oscillator (boson oscillator) is a special case of the paraboson oscillator (namely with $\beta = 1/2$), one finds

$$\Phi_n^{(1/2,1)}(x) = \frac{1}{2^{n/2}\pi^{1/4}\sqrt{n!}}\mathrm{e}^{-x^2/2}H_n(x), \tag{45}$$

with $H_n(x)$ the common Hermite polynomial.

It should be mentioned that this relation to the paraboson oscillator (and hence to the canonical oscillator) also follows from the underlying algebra. Indeed, $\mathfrak{sl}(2|1)$ contains the paraboson algebra $\mathfrak{osp}(1|2)$ as a subalgebra, and the representations Π_β correspond to irreducible representations of $\mathfrak{osp}(1|2)$. In this context, the position operator for the paraboson oscillator equals the expression (29) with $\gamma = 1$. This, by the way, explains why we have chosen – in the $\mathfrak{sl}(2|1)$ case – as position operator the most general self-adjoint odd element of the superalgebra.

4. Concluding remarks

We have presented here two algebraic models for the quantum oscillator based on Lie superalgebras. The first model comes from the Lie superalgebra $\mathfrak{sh}(2|2)$, and is rather a toy model. It is easy to analyze, and serves mainly as supporting example to tackle more complicated models. For the $\mathfrak{sh}(2|2)$ case, there is one model parameter γ which arises by allowing the "most general" expression for a possible position operator. There is only one natural unitary representation (the Fock space), so this makes the spectral analysis not too difficult.

The second model comes from the Lie superalgebra $\mathfrak{sl}(2|1)$, and is more fundamental. For the $\mathfrak{sl}(2|1)$ case, there is again one model parameter γ by allowing the "most general" expression for a possible position operator. But now there is a natural class of unitary representations, each representation Π_β characterized by a positive real number β. The spectral analysis of the position operator and its corresponding position wavefunctions is more subtle. The model parameter γ plays an important role in the support of the position wavefunctions (which can be discrete or continuous), and the representation parameter β plays a role in the "shape" of the wavefunctions (closer to the canonical case or to the paraboson case). The $\mathfrak{sl}(2|1)$ model includes, as special cases (i.e. for special values of the parameters γ and β) the paraboson oscillator and the canonical oscillator.

As a final remark, let us mention that the current examples are both sufficiently simple to compute other relevant physical quantities, such as the action of $[\hat{q}, \hat{p}]$, in the relevant representations. We refer to [9, 8] for these computations.

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