

# A relation between massive scalar field in $AdS_{d+1}$ and diffusion in channels

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**Abstract.** It is shown that, when the diffusion coefficient is a constant and is taken a particular family of channels, the Fick-Jacobs equation is invariant under conformal symmetry. In addition, using the diffusion coefficient and the geometric parameters of the channels, a representation for the conformal algebra is obtained. Furthermore, it is found that for these systems the Fick-Jacobs equation is equivalent to the Schrödinger equation for the 1-dimensional conformal quantum mechanics. Moreover, using this equivalence, it is found a relation between a massive scalar field equation in  $AdS_{d+1}$  background and Fick-Jacobs equation, where the geometric parameter of the channels and the geometric parameters of  $AdS_{d+1}$  are identified.

## 1. Introduction

Anti-de Sitter space  $AdS_{d+1}$  is the maximally symmetric solution of Einstein's equations with a negative cosmological constant. This space can be represented as the hyperboloid

$$X_0^2 + X_{d+1}^2 - \sum_{i=1}^{d-1} X_i^2 = R^2 \quad (1)$$

in the flat  $d + 2$ -dimensional space with metric

$$ds^2 = -dX_0^2 - dX_{d+1}^2 + \sum_{i=1}^{d-1} X_i^2. \quad (2)$$

In Poincaré coordinates, the  $AdS_{d+1}$  metric takes the form [1, 2]

$$ds^2 = \frac{R^2}{z^2} \left( \eta_{\mu\nu} dx^\mu dx^\nu + dz^2 \right), \quad \mu = 0, 1, \dots, d \quad (3)$$

where  $\eta_{\mu\nu}$  is the Minkowski metric and  $z$  is the so call holographic coordinate. In addition the cosmological constant is given by

$$\Lambda = -\frac{d(d-1)}{2R^2}. \quad (4)$$

In this space, the equation of motion for a massive scalar field,  $\phi(x, z)$ , is

$$\frac{1}{R^2} \frac{\partial^2 \phi(x, z)}{\partial z^2} + \frac{1}{R^2} \partial^\mu \partial_\mu \phi(x, z) - \frac{(d-1)}{R^2 z} \frac{\partial \phi(x, z)}{\partial z} - \frac{m^2}{z^2} \phi(x, z) = 0. \quad (5)$$



An amazing result in physics is given by the  $AdS_{d+1}/CFT_d$  correspondence, which allows a relation between  $(d+1)$ -dimensional gravitational theory and certain classes of  $d$ -dimensional Yang-Mills theories [1, 2]. Recently this correspondence has been extended, for example there is a correspondence between a gravitational theory and condensed matter  $AdS/CMT$  [5] and also there is a duality between a gravitational theory and fluid dynamics *gravity/fluid* [6, 7]. Furthermore, Brownian motion has been studied in this frame-work [8, 9]. In the  $AdS_{d+1}/CFT_d$  correspondence, in order to get a conformal field theory the limit  $z \rightarrow 0$  is taken [1, 2], then the holographic coordinate disappears in the  $CFT_d$ . However, the holographic coordinate is important too, for example if we take

$$\phi(x, z) = e^{-ip \cdot x} z^{\frac{(d-1)}{2}} \psi(z), \quad p^2 = -M_d^2, \quad (6)$$

the equation (5) becomes

$$\left( -\frac{1}{R^2} \frac{\partial^2}{\partial z^2} + \frac{g}{z^2} \right) \psi(z) = \frac{M_d^2}{R^2} \psi(z), \quad g = \frac{1}{R^2} \left( \left( \frac{d-1}{2} \right) \left( \frac{d+1}{2} \right) + m^2 R^2 \right). \quad (7)$$

Using this last equation, it is possible to get the spectrum of the light hadrons, where  $z$  is related with the invariant separation of the quark constituents [3, 4]. Now, the equation (7) is the static Schrödinger equation for the conformal quantum mechanics [34], which appears in different contexts, from black-holes to atomic physics [35, 36, 37]. Notice that, if a system is related with the conformal quantum mechanics, then it is also related with a massive scalar field in  $AdS_{d+1}$ .

The conformal group is the symmetry group for both Anti-de Sitter space and conformal field theory, for this reason the conformal symmetry is important in the  $AdS_{d+1}/CFT_d$  correspondence [2]. This symmetry appears in other systems, for example the free Schrödinger equation is invariant under a non-relativistic conformal transformations, which is known as the Schrödinger group [10, 11]. In fact this group has been important to study the non-relativistic  $AdS_{d+1}/CFT_d$  correspondence [12, 13]. Some work about the Schrödinger group can be seen in [14, 15, 16, 17, 18, 19, 20, 21]. Now, the simplest model of diffusion is described by the Fick equation and Sophus Lie showed that this equation is invariant under the Schrödinger group [22]. Another study about diffusion phenomena and the Schrödinger group can be seen in [23].

When the diffusion is in a channel, which has the shape of surface of revolution with cross sectional area  $A(x)$ , the Fick equation has to be changed to the Fick-Jacobs equation [24]

$$\frac{\partial C(x, t)}{\partial t} = \frac{\partial}{\partial x} \left[ D(x) A(x) \frac{\partial}{\partial x} \left( \frac{C(x, t)}{A(x)} \right) \right], \quad (8)$$

where  $C(x, t)$  is the particle concentration and  $D(x)$  is the diffusion coefficient. This last equation is important to study diffusion in biological channels, zeolites and nano-channels [25, 26, 27, 28, 29, 30, 31]. In this paper we will show that, when the diffusion coefficient is a constant and the cross sectional area is  $A(x) = (b + \lambda x)^{2\nu}$  (where  $\nu, b$ , and  $\lambda$  are real constants), the Fick-Jacobs equation is equivalent to the conformal quantum mechanics. In addition, a relation between a massive scalar field equation in  $AdS_{d+1}$  background and Fick-Jacobs equation is found, in which the diffusion coefficient and cosmological constant for  $AdS_{d+1}$  space are associated. Furthermore, it is found that the geometric parameter of the channels and the geometric parameters of  $AdS_{d+1}$  space are related too. In this case the axial coordinate of the channel  $x$  and the holographic coordinate  $z$  of  $AdS_{d+1}$  are identified.

This paper is organized in the following way: in section 2 a brief review about the conformal quantum mechanics is given; in section 3 it is shown that the Fick-Jacobs equation is invariant under conformal symmetry for a family of channels; in section 4 it is shown that the Fick-Jacobs equation is equivalent to the conformal quantum mechanics for a set of particular channels and an exact solution for this equation is given. In section 5 a relation between Fick-Jacobs equation and a massive scalar field in  $AdS_{d+1}$  is found. Finally, in section 6 a summary is given.

## 2. Conformal Quantum Mechanics

The Schrödinger equation for the 1-dimensional conformal quantum mechanics is given by

$$i\hbar \frac{\partial \psi(x, t)}{\partial t} = H \psi(x, t) \quad H = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{g}{x^2}, \quad (9)$$

which is invariant under the non-relativistic conformal symmetry

$$t' = \frac{\alpha t + \beta}{\gamma t + \delta}, \quad x' = \frac{ax}{\gamma t + \delta}, \quad a^2 = \alpha\delta - \beta\gamma \neq 0, \quad (10)$$

in order to keep the wave equation invariant under the last transformation the wave function must transform like

$$\psi'(x', t') = \sqrt{\gamma t + \delta} e^{i \frac{m}{2\hbar} \Phi(x, t)} \psi(x, t) \quad (11)$$

where

$$\Phi(x, t) = -\frac{\gamma x^2}{\gamma t + \delta}. \quad (12)$$

In this case the group generators are given by  $H$  and

$$K_1 = tH - \frac{1}{2} \left( xp - \frac{i}{2} \right), \quad (13)$$

$$K_2 = t^2 H - t \left( xp - \frac{i}{2} \right) + \frac{mx^2}{2} \quad (14)$$

and the algebra

$$[H, K_1] = iH, \quad [H, K_2] = 2iK_1, \quad [K_1, K_2] = iK_2 \quad (15)$$

is satisfied. Using this algebra, it is possible to show that the operator  $H, K_1, K_2$  are conserved.

The classical system with the general potential  $V(r) = gr^z$  was first studied by Jacobi [32] and the quantum system with  $z = -2$  was originally proposed by Jackiw [33]. The spectrum of the Hamiltonian (9) was found by de Alfaro, Fubini and Furlan [34]. It is worth to mention that the conformal quantum mechanics appears in different contexts, from black-holes to atomic physics [35, 36, 37] and has been proposed as the  $CFT_1$  dual to  $AdS_2$  [38, 39]. We will show that this Hamiltonian appears in diffusion phenomena too.

### 3. Conformal symmetry and Fick-Jacobs equation

Now, we will study the family of channels with cross sectional area

$$A(x) = (b + \lambda x)^{2\nu}, \quad (\nu, b, \lambda = \text{constant}). \quad (16)$$

In this case, taking  $D(x) = D_0$  as a constant and using the change of variable

$$y = b + \lambda x, \quad (17)$$

the Fick-Jacobs equation (8) becomes

$$\frac{\partial C(y, t)}{\partial t} = \lambda^2 D_0 \left( \frac{\partial^2}{\partial y^2} - \frac{2\nu}{y} \frac{\partial}{\partial y} + \frac{2\nu}{y^2} \right) C(y, t). \quad (18)$$

This equation is invariant under conformal transformation (10), where the concentration transforms as

$$C'(y', t') = (\gamma t + \delta)^{\frac{1-2\nu}{2}} e^{-\frac{1}{4D_0} \Phi(y, t)} C(y, t). \quad (19)$$

and  $\Phi$  is given by (12). If we take  $p = -i \frac{\partial}{\partial y}$  the equation (18) can be written like

$$\frac{\partial C(y, t)}{\partial t} = \mathbf{H} C(y, t), \quad (20)$$

where

$$\mathbf{H} = \lambda^2 D_0 \left( -p^2 - i \frac{2\nu}{y} p + \frac{2\nu}{y^2} \right). \quad (21)$$

Now, using

$$\mathbf{K}_1 = t\mathbf{H} - \frac{1}{2} \left( yp - \frac{i}{2} + i\nu \right), \quad (22)$$

$$\mathbf{K}_2 = t^2\mathbf{H} - t \left( yp - \frac{i}{2} + i\nu \right) - \frac{y^2}{4\lambda^2 D_0}. \quad (23)$$

we obtained the conformal algebra

$$[\mathbf{H}, \mathbf{K}_1] = i\mathbf{H}, \quad [\mathbf{H}, \mathbf{K}_2] = 2i\mathbf{K}_1, \quad [\mathbf{K}_1, \mathbf{K}_2] = i\mathbf{K}_2.$$

However, these operators are not conserved.

### 4. Conformal quantum mechanics and Fick-Jacobs equation

In the last section we obtained a kind of not hermitian quantum mechanics. However, the equation (18) is related with the usual conformal quantum mechanics. In fact, using the no unitary transformation

$$C(y, t) = (b + \lambda x)^\nu \psi(y, t) \quad (24)$$

the equation (18) becomes

$$\frac{\partial \psi(y, t)}{\partial t} = -H \psi(y, t), \quad (25)$$

here

$$H = -\lambda^2 D_0 \frac{\partial^2}{\partial y^2} + \frac{g}{y^2}, \quad g = \lambda^2 D_0 \nu (\nu - 1). \quad (26)$$

Notice that  $H$  is hermitian and is the Hamiltonian for the conformal quantum mechanics (9). In fact, proposing  $\psi(y, t) = e^{-Et} \Psi(y)$ , we get the static Schrödinger equation for the conformal quantum mechanics

$$E\Psi(y) = H\Psi(y). \quad (27)$$

Then, the family of channels (16) is associated with the family of conformal Hamiltonians (26). Observe that for each Hamiltonian (26) we have two channels, namely each Hamiltonian is associated with two  $\nu$  values. For example,  $\nu = 0$  represents a cylindrical channel while  $\nu = 1$  represents a conical channel, but both gave the Hamiltonian

$$H = -\lambda^2 D_0 \frac{\partial^2}{\partial y^2}. \quad (28)$$

Now, using the change of variable (17), we found that the exact solution for the Fick-Jacobs equation is given by

$$C_\nu(x, t) = B e^{-Et} (b + \lambda x)^{\frac{2\nu+1}{2}} J_{\pm(\frac{2\nu-1}{2})} \left( \pm \sqrt{\frac{E}{\lambda^2 D_0}} (b + \lambda x) \right), \quad (29)$$

here  $B$  is a constant. The parameter  $E$  can be obtained from the boundary condition. For example, if  $C_\nu(L, t) = 0$ , the values

$$E_n = \frac{\rho_n^2 \lambda^2 D_0}{b + \lambda L} \quad (30)$$

are obtained, where  $\rho_n$  is the  $n$ th root for the Bessel function of the order  $\left(\frac{2\nu-1}{2}\right)$ . In this case the initial condition  $C_\nu(x, 0) = 0$  can be written as a Fourier-Bessel series.

## 5. AdS/Fick-Jacobs

Now, if the holographic coordinate  $z \neq 0$ , then the massive scalar field in  $AdS_{d+1}$  can be expressed as the equation (6), where the field  $\psi(z)$  satisfies the static Schrödinger equation for the conformal quantum mechanics. Then, if the cross sectional area is given by  $A(x) = (b + \lambda x)^{2\nu}$ , with the mapping

$$b + \lambda x \longleftrightarrow z \quad (31)$$

$$\lambda^2 D_0 \longleftrightarrow R^{-2} = -\frac{2\Lambda}{d(d+1)}, \quad (32)$$

$$\nu(\nu-1) \longleftrightarrow \left(\frac{d-1}{2}\right)\left(\frac{d+1}{2}\right) + m^2 R^2, \quad (33)$$

$$E \longleftrightarrow \frac{M_d^2}{R^2}, \quad (34)$$

a relation between the Fick-Jacobs equation (26) and a massive scalar field in  $AdS_{d+1}$  is obtained. Notice that the cosmological constant  $\Lambda$  leads the expansion of  $AdS_{d+1}$  universe, while

the diffusion coefficient  $D_0$  leads the diffusion of particles in the channel and these both parameters are related. In addition, it can be seen that  $\nu$  and  $d, R$  are related too, it is worth to mention that  $\nu$  drives the geometric properties of the channel, while  $d$  and  $R$  drive the geometric properties of the  $AdS_{d+1}$  space.

In the usual  $AdS_{d+1}/CFT_d$  duality, the limit  $z \rightarrow 0$  is taken [1]. However there is a correspondence with  $z$  no constant, like  $AdS/QCD$  correspondence [3]. In this paper a correspondence between the wave equation for a massive scalar field in  $AdS_{d+1}$  and Fick-Jacobs equation was gotten. Even more the axial coordinate  $x$  of the channel and the holographic coordinate  $z$  of  $AdS_{d+1}$  are identified.

## 6. Summary

Recently mathematical techniques developed in quantum physics have been employed to study biological systems [40]. Now, symmetries are important in quantum systems, then symmetries might be important in biological systems too. In this paper was studied the Fick-Jacobs equation which models diffusion in biological channels, zeolites and nano-channels. It was shown that, when the diffusion coefficient is a constant and is taken a particular family of channels, this equation is invariant under conformal symmetry. In addition, using the diffusion coefficient and the geometric parameters of the channels, a representation for the conformal algebra was obtained. Furthermore, it was found that for these systems the Fick-Jacobs equation is equivalent to the Schrödinger equation for the 1-dimensional conformal quantum mechanics. Moreover, using this equivalence, it was found a relation between a massive scalar field equation in  $AdS_{d+1}$  background and Fick-Jacobs equation, where the geometric parameter of the channels and the geometric parameters of  $AdS_{d+1}$  are identified.

It is well known that symmetry methods are very useful in quantum mechanics. However, in biological systems these methods are few employed. The Fick-Jacobs equation is an example where symmetries are important and useful. In a future work we will study other equations that describe biological systems using these methods.

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