

# Hamiltonians with two-ladder spectra and solutions to the Painlevé IV equation

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**Abstract.** Some supersymmetric partners of the harmonic oscillator Hamiltonian possess third-order differential ladder operators and, thus, they realize the second-order polynomial Heisenberg algebras. The corresponding spectra consist of an infinite ladder of energy levels plus a finite one that can be placed at any position with respect to the infinite ladder. Both of them have equally spaced energy levels. Departing from these Hamiltonians several families of real and complex solutions to the Painlevé IV equation with real parameters will be generated.

## 1. Introduction

The interest in analyzing systematically the links which can be established between supersymmetric quantum mechanics (SUSY QM) and nonlinear differential equations is increasing over time [1–3]. In fact, there is a well known connection between the SUSY partners of the free particle and solutions to the KdV equation [4]. One of the aims in this paper is to discuss the existence of an additional link, now between SUSY QM and Painlevé IV (PIV) equation. By exploiting this fact, a procedure for generating solutions of the PIV equation can be easily designed [2]. Up to our knowledge, the first people who established the link between first-order SUSY QM and PIV equation were Veselov and Shabat [5], Dubov, Eleonsky and Kulagin [6], Adler [7]. This was further explored for higher-order SUSY by Andrianov, Cannata, Ioffe and Nishnianidze [8], Fernández, Negro and Nieto [9], Carballo, Fernández, Negro and Nieto [10], Mateo and Negro [11], Bermudez and Fernández [1–3, 12] (see also [13]).

In order to perform this task, we have organized the paper as follows. In sections 2 and 3 we will introduce the SUSY QM and the second-order polynomial Heisenberg algebras respectively. In section 4 we will apply the SUSY QM to the harmonic oscillator while in section 5 we will establish the corresponding connection with Painlevé IV equation. In addition, several solutions to this equation, real and complex, will be derived. Our conclusions are contained in section 6.

## 2. SUSY QM

Let us consider the following intertwining relations [14–21]:

$$H_{i+1}A_{i+1}^+ = A_{i+1}^+H_i, \quad (1)$$

$$A_{i+1}^+ = \frac{1}{\sqrt{2}} \left[ -\frac{d}{dx} + \alpha_{i+1}(x, \epsilon_{i+1}) \right], \quad (2)$$



$$H_i = -\frac{1}{2} \frac{d^2}{dx^2} + V_i(x), \quad i = 0, \dots, k. \quad (3)$$

Thus, the following equations are satisfied:

$$\alpha'_{i+1}(x, \epsilon_{i+1}) + \alpha_{i+1}^2(x, \epsilon_{i+1}) = 2[V_i(x) - \epsilon_{i+1}], \quad (4)$$

$$V_{i+1}(x) = V_i(x) - \alpha'_{i+1}(x, \epsilon_{i+1}). \quad (5)$$

The key point in this treatment is to realize that the solutions to the Riccati equation (4) can be obtained by algebraic means in terms of those of the Riccati equation for the previous system [18, 19], namely,

$$\alpha_{i+1}(x, \epsilon_{i+1}) = -\alpha_i(x, \epsilon_i) - \frac{2(\epsilon_i - \epsilon_{i+1})}{\alpha_i(x, \epsilon_i) - \alpha_i(x, \epsilon_{i+1})}, \quad (6)$$

which implies that  $\alpha_{i+1}(x, \epsilon_{i+1})$  is determined either from  $i + 1$  solutions  $\alpha_1(x, \epsilon_j)$  of the initial Riccati equation

$$\alpha'_1(x, \epsilon_j) + \alpha_1^2(x, \epsilon_j) = 2[V_0(x) - \epsilon_j], \quad j = 1, \dots, i + 1, \quad (7)$$

or from  $i + 1$  solutions  $u_j$  of the associated Schrödinger equation

$$-\frac{1}{2}u_j'' + V_0(x)u_j = \epsilon_j u_j, \quad j = 1, \dots, i + 1. \quad (8)$$

The initial and final Hamiltonians,  $H_0$  and  $H_k$ , satisfy the following intertwining relations

$$H_k B_k^+ = B_k^+ H_0, \quad H_0 B_k = B_k H_k, \quad (9)$$

$$B_k = A_1 \dots A_k, \quad B_k^+ = A_k^+ \dots A_1^+. \quad (10)$$

In the same way, the initial and final potentials  $V_0, V_k$  are linked through

$$V_k(x) = V_0(x) - \{\ln[W(u_1, \dots, u_k)]\}'', \quad (11)$$

where  $W(u_1, \dots, u_k)$  denotes the Wronskian of the  $k$  solutions  $\{u_1(x), \dots, u_k(x)\}$ . In addition, the eigenfunctions of  $H_k$  are given by

$$\psi_n^{(k)} = [(E_n - \epsilon_1) \dots (E_n - \epsilon_k)]^{-\frac{1}{2}} B_k^+ \psi_n, \quad E_n, \quad (12)$$

$$\psi_{\epsilon_j}^{(k)} \propto \frac{W(u_1, \dots, u_{j-1}, u_{j+1}, \dots, u_k)}{W(u_1, \dots, u_k)}, \quad \epsilon_j. \quad (13)$$

Thus, given the potential  $V_0(x)$  and  $k$  solutions  $\{u_1(x), \dots, u_k(x)\}$  of the initial stationary Schrödinger equation associated with the factorization energies  $\{\epsilon_1, \dots, \epsilon_k\}$ , the new potential  $V_k(x)$  of equation (11) is essentially determined, with eigenfunctions and eigenvalues given in equations (12-13).

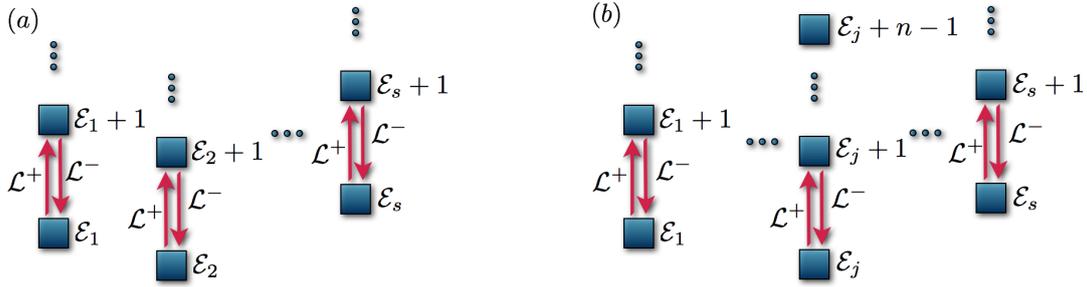
### 3. Second-order polynomial Heisenberg algebras

The  $m$ -th order polynomial Heisenberg algebras (PHA) are deformations of the Heisenberg-Weyl algebra of kind [10] (see also [22, 23]):

$$[H, \mathcal{L}^\pm] = \pm \mathcal{L}^\pm, \quad (14)$$

$$[\mathcal{L}^-, \mathcal{L}^+] \equiv Q_{m+1}(H + 1) - Q_{m+1}(H) = P_m(H), \quad (15)$$

$$Q_{m+1}(H) = \mathcal{L}^+ \mathcal{L}^- = \prod_{i=1}^{m+1} (H - \mathcal{E}_i). \quad (16)$$



**Figure 1.** The spectrum of systems ruled by polynomial Heisenberg algebras can have either  $s$  infinite ladders (a) or  $s - 1$  infinite plus a finite one if equation (19) is satisfied (b).

One of the simplest realizations is a differential one, in which  $H$  is a one-dimensional Schrödinger Hamiltonian

$$H = -\frac{1}{2} \frac{d^2}{dx^2} + V(x), \quad (17)$$

and  $\mathcal{L}^\pm$  are differential ladder operators of order  $(m + 1)$ -th.

The spectral properties of systems ruled by PHA depend on the number  $s$  of physical eigenstates of  $H$  which also belong to the kernel of  $\mathcal{L}^-$ :

$$\mathcal{L}^- \psi_{\mathcal{E}_i} = 0, \quad H \psi_{\mathcal{E}_i} = \mathcal{E}_i \psi_{\mathcal{E}_i}, \quad i = 1, \dots, s. \quad (18)$$

Departing from them,  $s$  physical ladders of  $H$  can be constructed through the action of  $\mathcal{L}^+$ , see figure 1(a). If, however, it happens that

$$(\mathcal{L}^+)^{n-1} \psi_{\mathcal{E}_j} \neq 0, \quad (\mathcal{L}^+)^n \psi_{\mathcal{E}_j} = 0 \quad \Rightarrow \quad \mathcal{E}_k = \mathcal{E}_j + n, \quad (19)$$

then the  $j$ -th ladder starts from  $\mathcal{E}_j$  but ends at  $\mathcal{E}_j + n - 1$ , as shown in figure 1(b).

In this work we are particularly interested in second-order PHA, with  $m = 2$ , for which:

$$Q_3(H) = (H - \mathcal{E}_1)(H - \mathcal{E}_2)(H - \mathcal{E}_3), \quad (20)$$

$$P_2(H) = 3H^2 + [3 - 2(\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3)]H \quad (21)$$

$$+ 1 - (\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3) + \mathcal{E}_1\mathcal{E}_2 + \mathcal{E}_1\mathcal{E}_3 + \mathcal{E}_2\mathcal{E}_3. \quad (22)$$

In this case our differential ladder operators  $\mathcal{L}^\pm$  are of third order, and the corresponding Hamiltonian could have up to three independent physical ladders starting from  $\mathcal{E}_1$ ,  $\mathcal{E}_2$  and  $\mathcal{E}_3$ .

In order to identify the corresponding systems, let us take  $\mathcal{L}^\pm$  such that [8–10]:

$$\mathcal{L}^+ = I_1^+ I_2^+, \quad (23)$$

$$I_1^+ = \frac{1}{\sqrt{2}} \left[ -\frac{d}{dx} + f(x) \right], \quad I_2^+ = \frac{1}{2} \left[ \frac{d^2}{dx^2} + g(x) \frac{d}{dx} + h(x) \right], \quad (24)$$

$$H I_1^+ = I_1^+ (H_a + 1), \quad H_a I_2^+ = I_2^+ H. \quad (25)$$

By using the standard expressions for first and second-order SUSY QM we obtain

$$-f' + f^2 = 2(V - \mathcal{E}_3), \quad (26)$$

$$V_a = V + f' - 1 = V + g', \quad (27)$$

$$\frac{g''}{2g} - \left( \frac{g'}{2g} \right)^2 - g' + \frac{g^2}{4} + \frac{(\mathcal{E}_1 - \mathcal{E}_2)^2}{g^2} + \mathcal{E}_1 + \mathcal{E}_2 - 2 = 2V, \quad (28)$$

$$h = -\frac{g'}{2} + \frac{g^2}{2} - 2V + \mathcal{E}_1 + \mathcal{E}_2 - 2. \quad (29)$$

By decoupling this system it turns out that

$$f(x) = x + g(x), \quad (30)$$

$$h(x) = -x^2 + \frac{g'}{2} - \frac{g^2}{2} - 2xg + a, \quad (31)$$

$$V(x) = \frac{x^2}{2} - \frac{g'}{2} + \frac{g^2}{2} + xg + \mathcal{E}_3 - \frac{1}{2}, \quad (32)$$

where  $g$  satisfies

$$g'' = \frac{g'^2}{2g} + \frac{3}{2}g^3 + 4xg^2 + 2(x^2 - a)g + \frac{b}{g}, \quad (33)$$

which is the Painlevé IV equation with parameters ( $\Delta = \mathcal{E}_1 - \mathcal{E}_2$ ):

$$a = \mathcal{E}_1 + \mathcal{E}_2 - 2\mathcal{E}_3 - 1, \quad b = -2\Delta^2. \quad (34)$$

The extremal states of our system turn out to be

$$\psi_{\mathcal{E}_1} \propto \left( \frac{g'}{2g} - \frac{g}{2} - \frac{\Delta}{g} - x \right) \exp \left[ \int \left( \frac{g'}{2g} + \frac{g}{2} - \frac{\Delta}{g} \right) dx \right], \quad (35)$$

$$\psi_{\mathcal{E}_2} \propto \left( \frac{g'}{2g} - \frac{g}{2} + \frac{\Delta}{g} - x \right) \exp \left[ \int \left( \frac{g'}{2g} + \frac{g}{2} + \frac{\Delta}{g} \right) dx \right], \quad (36)$$

$$\psi_{\mathcal{E}_3} \propto \exp \left( -\frac{x^2}{2} - \int g dx \right). \quad (37)$$

In particular, it is important to note that

$$g(x) = -x - \{\ln[\psi_{\mathcal{E}_3}(x)]\}'. \quad (38)$$

This means that, if we would know systems ruled by second-order PHA, particularly its extremal states, we could find solutions to the PIV equation. This idea will be explored further in section 5.

#### 4. Harmonic oscillator SUSY partners

In order to apply the SUSY techniques, we need the solution of the Schrödinger equation for  $V_0(x) = x^2/2$  and arbitrary  $\epsilon$ , which is given by [24]:

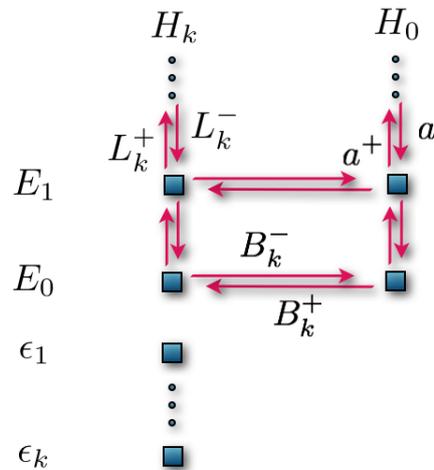
$$u = e^{-\frac{x^2}{2}} \left[ {}_1F_1 \left( \frac{1-2\epsilon}{4}, \frac{1}{2}; x^2 \right) + 2x\nu \frac{\Gamma(\frac{3-2\epsilon}{4})}{\Gamma(\frac{1-2\epsilon}{4})} {}_1F_1 \left( \frac{3-2\epsilon}{4}, \frac{3}{2}; x^2 \right) \right]. \quad (39)$$

Let us perform now a  $k$ -th order SUSY transformation by using  $\{u_1, \dots, u_k\}$  such that  $\epsilon_k < \dots < \epsilon_1 < 1/2$ ,  $|\nu_j| < 1$  for odd  $j$ ,  $|\nu_j| > 1$  for even  $j$ . Thus,  $k$  new levels are created so that

$$\text{Sp}(H_k) = \{\epsilon_j, E_n, j = k, \dots, 1, n = 0, 1, \dots\}. \quad (40)$$

Let us note that there are differential ladder operators for  $H_k$  of order  $2k + 1$ :

$$L_k^- = B_k^+ a B_k, \quad L_k^+ = B_k^+ a^+ B_k. \quad (41)$$



**Figure 2.** Diagram representing the two SUSY partner Hamiltonians  $H_0$  and  $H_k$  and their associated ladder operators  $a^+, a$  and  $L_k^\pm$ , respectively.

Since

$$Q_{2k+1}(H_k) = \left(H_k - \frac{1}{2}\right) \prod_{i=1}^k (H_k - \epsilon_i - 1)(H_k - \epsilon_i), \quad (42)$$

it is seen that they generate a PHA of order  $2k$ , namely:

$$[L_k^-, L_k^+] = P_{2k}(H_k). \quad (43)$$

Moreover, since to every root  $\epsilon_j$  of  $Q_{2k+1}(H_k)$  there is associated other one  $\epsilon_j + 1$ , we can say that  $\text{Sp}(H_k)$  contains  $k$  one-step ladders, each one of them starting and ending at  $\epsilon_j, j = 1, \dots, k$ . In addition, there is an infinite ladder starting from  $E_0 = 1/2$  (see diagram in figure 2).

### 5. Solutions of PIV equation through SUSY QM

As we saw in the previous section,  $L_1^\pm$  are third-order differential ladder operators, but  $L_k^\pm$  are of order greater than three for  $k > 1$ . Thus the question arises: is it possible to reduce to three the order of these ladder operators for  $k > 1$ ? If so, we could obtain systems which perhaps would supply us with solutions to the PIV equation [25]. The answer turns out to be positive, and it is contained in the following theorem [1, 2].

**Theorem.** Suppose that  $H_k$  is generated by  $k$  connected seed solutions

$$u_j = a^{j-1}u_1, \quad \epsilon_j = \epsilon_1 - (j - 1), \quad j = 1, \dots, k, \quad (44)$$

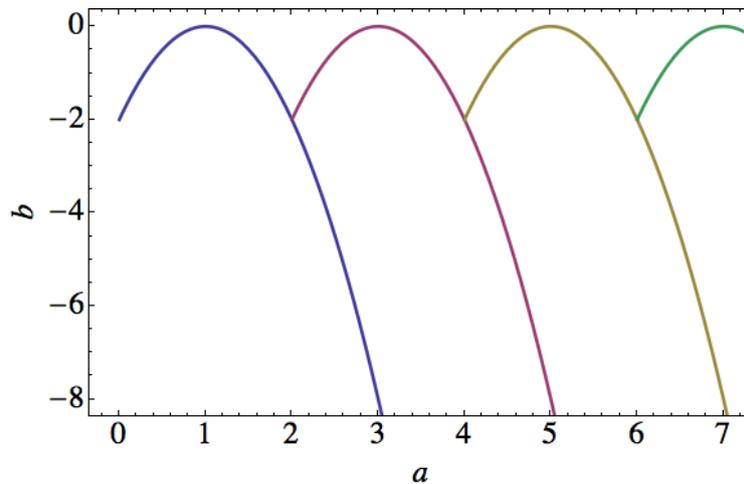
$u_1(x)$  being an independent seed solution in the form given in equation (39) such that  $\epsilon_1 < 1/2$  and  $|\nu_1| < 1$ . Thus

$$L_k^+ = P_{k-1}(H_k)l_k^+, \quad (45)$$

where  $P_{k-1}(H_k) = (H_k - \epsilon_1) \dots (H_k - \epsilon_{k-1})$  and  $l_k^+$  turns out to be a third-order differential ladder operator such that

$$[H_k, l_k^+] = l_k^+, \quad (46)$$

$$l_k^+ l_k^- = (H_k - \epsilon_k) \left(H_k - \frac{1}{2}\right) (H_k - \epsilon_1 - 1). \quad (47)$$



**Figure 3.** Parameter space where we can obtain real non-singular solutions to the PIV equation.

Let us note that the operators  $l_k^\pm$  already connect the levels  $\{\epsilon_j, j = 1, \dots, k\}$ , which form a finite ladder of length  $k$  starting from  $\epsilon_k = \epsilon_1 - (k - 1)$  and finishing at  $\epsilon_1$ . Moreover, the operator  $l_k^+$  annihilates only the eigenstate associated to  $\epsilon_1$  while the operator  $l_k^-$  annihilates the following three extremal states:

$$\psi_{\mathcal{E}_1} \propto B_k^+ e^{-x^2/2}, \quad \mathcal{E}_1 = 1/2, \quad (48)$$

$$\psi_{\mathcal{E}_2} \propto B_k^+ a^+ u_1, \quad \mathcal{E}_2 = \epsilon_1 + 1, \quad (49)$$

$$\psi_{\mathcal{E}_3} \propto \frac{W(u_1, \dots, u_{k-1})}{W(u_1, \dots, u_k)}, \quad \mathcal{E}_3 = \epsilon_k = \epsilon_1 - (k - 1). \quad (50)$$

From equation (38), it is clear now that departing from the third extremal state we can generate non-singular real solutions to the PIV equation. Indeed, the potential and the corresponding solution to the PIV equation are given by:

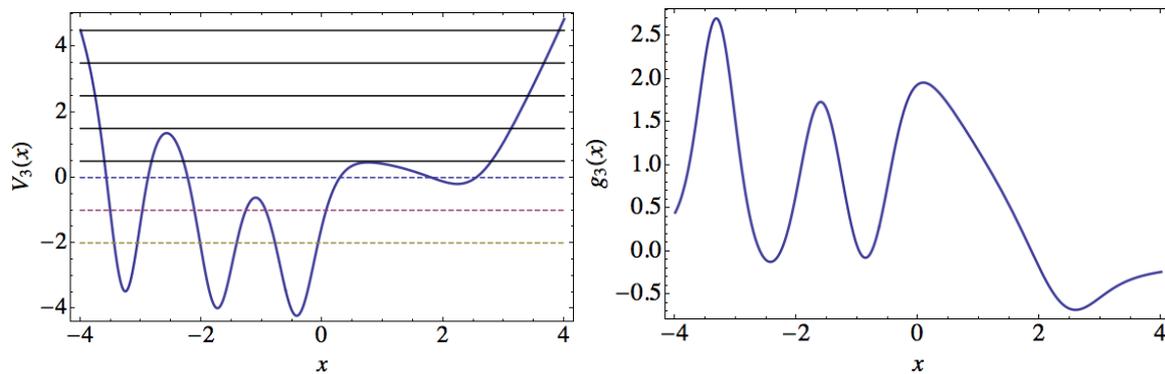
$$V(x) = \frac{x^2}{2} - [\ln W(u_1, \dots, u_k)]'', \quad (51)$$

$$g(x) = \begin{cases} -x + [\ln u_1]' & \text{for } k = 1 \\ -x - \left[ \ln \frac{u_1}{W(u_1, u_2)} \right]' & \text{for } k = 2 \\ -x - \left[ \ln \frac{W(u_1, \dots, u_{k-1})}{W(u_1, \dots, u_k)} \right]' & \text{for } k > 2 \end{cases} \quad (52)$$

and the connection between the parameters  $\epsilon_1, k$  of the SUSY transformation and  $a, b$  of the PIV equation is as follows

$$a = -\epsilon_1 + 2k - \frac{3}{2}, \quad b = -2 \left( \epsilon_1 + \frac{1}{2} \right)^2. \quad (53)$$

The curves on the parameter space  $a$ - $b$  where we have real non-singular solutions to the PIV equation are shown in figure 3. An example of the SUSY partner potential of the oscillator as well as the corresponding PIV solution is given in figure 4.



**Figure 4.** Third-order SUSY partner potential of the oscillator (left) and PIV solution (right) for  $\epsilon_1 = 0, \nu_1 = 0.99$ .

Let us stress that the restrictions  $\epsilon_1 < E_0$ ,  $|\nu_1| < 1$  were needed to produce a non-singular potential  $V_k(x)$  and its PIV solution, i.e., the full finite ladder of  $H_k$  is placed below  $E_0$ . From the viewpoint of spectral design it would be important to surpass this restriction so that the finite ladder of  $H_k$  could be placed above  $E_0$  (at least partially). This can be done by using complex transformation functions  $u_1$  but associated to  $\epsilon_1 \in \mathbb{R}$  [12, 26]. The price to pay is that the generated potentials will now be complex, but with real energy spectra. The main change with respect to our previous treatment is that the employed complex seed solution reads now

$$u_1(x) = e^{-x^2/2} \left[ {}_1F_1 \left( \frac{1-2\epsilon_1}{4}, \frac{1}{2}; x^2 \right) + \Lambda x {}_1F_1 \left( \frac{3-2\epsilon_1}{4}, \frac{3}{2}; x^2 \right) \right], \quad (54)$$

where  $\Lambda = \lambda + i\kappa$  ( $\lambda, \kappa \in \mathbb{R}$ ) and  $\epsilon_1 \in \mathbb{R}$ , which leads once again to a Hamiltonian having third-order differential ladder operators.

The extremal states become once again those given in equations (48-50). The PIV solution and its associated parameters are

$$g_k(x) = -x - \left\{ \ln [\psi_{\mathcal{E}_3}(x)] \right\}' = -x - \left\{ \ln \left[ \frac{W(u_1, \dots, u_{k-1})}{W(u_1, \dots, u_k)} \right] \right\}', \quad (55)$$

$$a_i = -\epsilon_1 + 2k - \frac{3}{2}, \quad b_i = -2 \left( \epsilon_1 + \frac{1}{2} \right)^2. \quad (56)$$

Note that the cyclic permutations of the indexes in the extremal states and associated energies leads now to two additional non-singular solutions of the PIV equation:

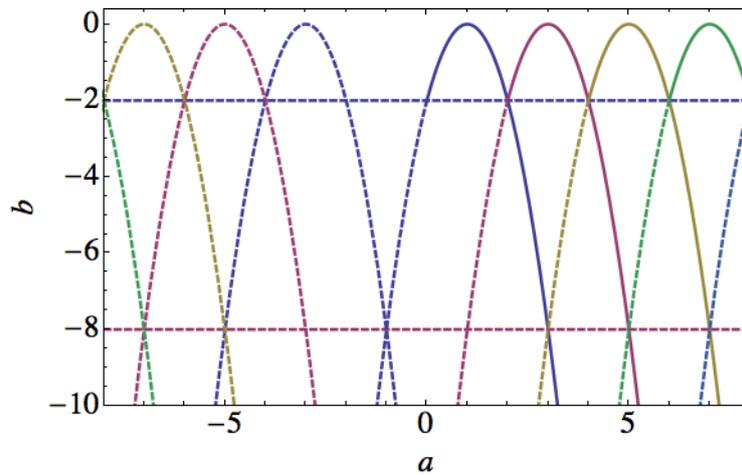
$$g_k(x) = -x - \left\{ \ln \left[ B_k^+ e^{-x^2/2} \right] \right\}', \quad a_{ii} = 2\epsilon_1 - k, \quad b_{ii} = -2k^2, \quad (57)$$

$$g_k(x) = -x - \left\{ \ln \left[ B_k^+ a^+ u_1 \right] \right\}', \quad a_{iii} = -\epsilon_1 - k - \frac{3}{2}, \quad b_{iii} = -2 \left( \epsilon_1 - k + \frac{1}{2} \right)^2. \quad (58)$$

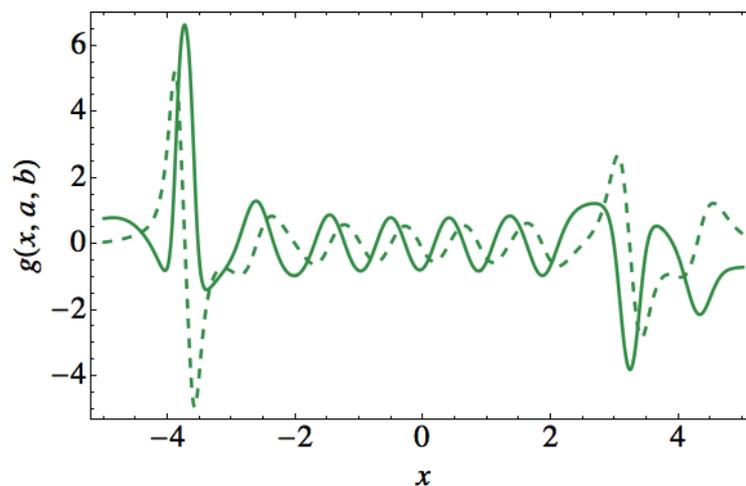
In the parameter space of the PIV equation now we get additional curves on which we have non-singular (complex) solutions (see figure 5). An example of this kind of complex solutions can be seen in figure 6.

## 6. Conclusions

In this work we have shown that the general Schrödinger Hamiltonians ruled by second-order polynomial Heisenberg algebras, i.e., having third-order differential ladder operators, are linked to the PIV equation. Moreover, a prescription for generating solutions of the PIV equation



**Figure 5.** Parameter space where we can obtain complex non-singular solutions to the PIV equation.



**Figure 6.** Real and imaginary parts (solid and dashed lines) of a complex solution to PIV for  $a_{ii} = 12$ ,  $b_{ii} = -8$  ( $k = 2$ ,  $\epsilon_1 = 7$ ,  $\lambda = \kappa = 1$ )

departing from the SUSY partners of the harmonic oscillator has been proposed. The associated potentials have two physical ladders of equally spaced energy levels: an infinite one starting from  $1/2$  and a finite one, placed either completely below  $1/2$  for real transformation functions or at any arbitrary position on the real energy axis for complex ones, both for  $\epsilon_1 \in \mathbb{R}$ .

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