Cotangent bundle over all the compact Hermitian symmetric spaces and projective superspace

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Abstract. We construct the $\mathcal{N}=2$ supersymmetric nonlinear sigma model on the cotangent bundle over the compact Hermitian symmetric space $E_7/E_6 \times U(1)$ by using the projective superspace formalism which is an off-shell superfield formulation in four-dimensional space-time. We also give a simple formula giving the hyper-Kähler potential of the cotangent bundle over all the compact Hermitian symmetric spaces.

1. Introduction

Supersymmetry (SUSY) has intimate relations to the complex geometry [1]. In particular, the target space of SUSY nonlinear sigma models (NLSMs) possessing 8 supercharges must be hyper-Käbler manifold [2]. It means that SUSY is a powerful tool to construct a new hyper-Käbler manifold. In other words, a new hyper-K̈haler manifold can be obtained through a new SUSY NLSM with 8 supercharges. One of the most convenient formulations to construct SUSY NLSMs with 8 supercharges is the projective superspace formalism [3, 4, 5, 6] being an off-shell superfield formulation in theories with 8 supercharges [1].

The projective superspace formalism is a powerful devise to construct $\mathcal{N}=2$ SUSY NLSMs. With the use of this formalism, first the SUSY NLSMs on the tangent bundles over the compact and non-compact classical Hermitian symmetric as well as, using the generalized Legendre transform [4], the cotangent bundles corresponding to the hyperkähler metrics have been constructed [9, 10, 11, 12]. Developed the results in [9, 10, 11, 12], the cotangent bundle over the compact type of $E_6/SO(10)\times U(1)$ have been also constructed [13]. After these works, the closed form expression of the cotangent bundle over any compact Hermitian symmetric space was obtained [14], which gives a form of the cotangent bundle presented in a matrix form. We elaborated on this result to provide a new closed formula for cotangent bundle action, and applied it to construction of the $\mathcal{N}=2$ SUSY NLSM on the cotangent bundle over the compact $E_7/E_6\times U(1)$ [15]. In this proceeding, we review our work based on [15].

[1] There is another $\mathcal{N}=2$ off-shell superfield formulation called the harmonic superspace formalism [7, 8].

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2. $\mathcal{N}=2$ sigma models in the projective superspace

We start with a family of four-dimensional $\mathcal{N}=2$ off-shell supersymmetric nonlinear sigma models that are described in ordinary $\mathcal{N}=1$ superspace by the action:

$$S[\Upsilon,\breve{\Upsilon}] = \frac{1}{2\pi\mathrm{i}} \oint \frac{\mathrm{d}\zeta}{\zeta} \int \mathrm{d}^8 z\, K\big(\Upsilon^I(\zeta), \breve{\Upsilon}^{\bar{J}}(\zeta)\big)\,, \quad z^M = (x^\mu, \theta_\alpha, \bar{\theta}^{\dot{\alpha}}), \tag{2.1}$$

where $\mu = 0,1,2,3$ and $\alpha, \dot{\alpha} = 1,2$. The complex coordinate ζ which is called the projective coordinate parameterizes $SU(2)_R/U(1)$ where $SU(2)_R$ is the internal symmetry. Action in the projective superspace is written by using contour integral over ζ, and reality conditions are imposed using complex conjugation of ζ composed with the antipodal map. The action is formulated in terms of the so-called polar multiplet [4, 5, 6], one of $\mathcal{N}=2$ multiplets living in the projective superspace. The polar multiplet is described by the so-called arctic superfield $\Upsilon(\zeta)$ and antarctic superfield $\breve{\Upsilon}(\zeta)$ that are generated by an infinite set of ordinary $\mathcal{N}=1$ superfields:

$$\Upsilon(\zeta) = \sum_{n=0}^{\infty} \Upsilon_n \zeta^n = \Phi + \Sigma\,\zeta + O(\zeta^2)\ , \qquad \breve{\Upsilon}(\zeta) = \sum_{n=0}^{\infty} \bar{\Upsilon}_n (-\zeta)^{-n}\,, \tag{2.2}$$

where $\breve{\Upsilon}$ is the conjugation of Υ under the composition of complex conjugation with the antipodal map on the Riemann sphere, $\bar{\zeta} \to -1/\zeta$. Here Φ is chiral, Σ complex linear,

$$\bar{D}_{\dot{\alpha}} \Phi = 0\ , \qquad \bar{D}^2 \Sigma = 0\,, \tag{2.3}$$

and the remaining component superfields are unconstrained complex superfields. The above theory is obtained as a minimal $\mathcal{N}=2$ extension of the general four-dimensional $\mathcal{N}=1$ supersymmetric nonlinear sigma model [1]

$$S[\Phi,\bar{\Phi}] = \int \mathrm{d}^8 z\, K(\Phi^I, \bar{\Phi}^{\bar{J}})\,, \tag{2.4}$$

where K is the Kähler potential of a Kähler manifold $\mathcal{M}$.

To describe the theory in terms of the physical superfields Φ and Σ only, all the auxiliary superfields have to be eliminated with the aid of the corresponding algebraic equations of motion

$$\oint \frac{\mathrm{d}\zeta}{\zeta}\, \zeta^n\, \frac{\partial K(\Upsilon,\breve{\Upsilon})}{\partial \Upsilon^I} \ = \ \oint \frac{\mathrm{d}\zeta}{\zeta}\, \zeta^{-n}\, \frac{\partial K(\Upsilon,\breve{\Upsilon})}{\partial \breve{\Upsilon}^{\bar{I}}} \ = \ 0\ , \qquad n \geq 2\,. \tag{2.5}$$

Let $\Upsilon_*(\zeta) \equiv \Upsilon_*(\zeta; \Phi, \bar{\Phi}, \Sigma, \bar{\Sigma})$ denote a unique solution subject to the initial conditions

$$\Upsilon_*(0) = \Phi\ , \qquad \dot{\Upsilon}_*(0) = \Sigma\,. \tag{2.6}$$

For a general Kähler manifold $\mathcal{M}$, the auxiliary superfields $\Upsilon_2, \Upsilon_3, \dots$, and their conjugates, can be eliminated only perturbatively. Their elimination can be carried out using the ansatz [16]

$$\Upsilon^I_n = \sum_{p=0}^{\infty} G^I{}_{J_1 \dots J_{n+p}\, \bar{L}_1 \dots \bar{L}_p}(\Phi,\bar{\Phi})\, \Sigma^{J_1} \dots \Sigma^{J_{n+p}}\, \bar{\Sigma}^{\bar{L}_1} \dots \bar{\Sigma}^{\bar{L}_p}\,, \qquad n \geq 2\,. \tag{2.7}$$

Upon elimination of the auxiliary superfields, the action (2.1) should take the form [9, 10]

$$S_{\rm tb}[\Phi,\Sigma] \ = \ \int \mathrm{d}^8 z\, \Big\{ K\big(\Phi,\bar{\Phi}\big) + \mathcal{L}\big(\Phi,\bar{\Phi},\Sigma,\bar{\Sigma}\big) \Big\}\,, \tag{2.8}$$

where

$$\mathcal{L}\big(\Phi,\bar{\Phi},\Sigma,\bar{\Sigma}\big)=\sum_{n=1}^{\infty}\mathcal{L}_{I_1\cdots I_n\bar{J}_1\cdots\bar{J}_n}\big(\Phi,\bar{\Phi}\big)\Sigma^{I_1}\ldots\Sigma^{I_n}\bar{\Sigma}^{\bar{J}_1}\ldots\bar{\Sigma}^{\bar{J}_n}. \tag{2.9}$$

The first term in the expansion (2.9) is given as $\mathcal{L}_{I\bar{J}}=-g_{I\bar{J}}\big(\Phi,\bar{\Phi}\big)$ and the tensors $\mathcal{L}_{I_1\cdots I_n\bar{J}_1\cdots\bar{J}_n}$ for $n>1$ are functions of the Riemann curvature $R_{I\bar{J}K\bar{L}}\big(\Phi,\bar{\Phi}\big)$ and its covariant derivatives. Eq. (2.8) is written by the base manifold coordinate Φ and the tangent space coordinate Σ. Therefore this is the tangent bundle action.

The complex linear tangent variables Σ's in (2.8) can be dualized into chiral one-forms, in accordance with the generalized Legendre transform [4]. To construct a dual formulation, we consider the first-order action

$$S=\int \mathrm{d}^8z\,\Big\{\,K\big(\Phi,\bar{\Phi}\big)+\mathcal{L}\big(\Phi,\bar{\Phi},\Sigma,\bar{\Sigma}\big)+\Psi_I\,\Sigma^I+\bar{\Psi}_{\bar{I}}\bar{\Sigma}^{\bar{I}}\Big\}\,, \tag{2.10}$$

where the tangent vector Σ^I is now a complex unconstrained superfield, while the one-form Ψ is chiral, $\bar{D}_{\dot{\alpha}}\Psi_I=0$. Integrating out Σ's and their conjugate should give the cotangent-bundle action

$$S_{\rm ctb}[\Phi,\Psi]\;\;=\;\;\int \mathrm{d}^8z\,\Big\{\,K\big(\Phi,\bar{\Phi}\big)+\mathcal{H}\big(\Phi,\bar{\Phi},\Psi,\bar{\Psi}\big)\Big\}\,, \tag{2.11}$$

where

$$\mathcal{H}\big(\Phi,\bar{\Phi},\Psi,\bar{\Psi}\big)=\sum_{n=1}^{\infty}\mathcal{H}^{I_1\cdots I_n\bar{J}_1\cdots\bar{J}_n}\big(\Phi,\bar{\Phi}\big)\Psi_{I_1}\ldots\Psi_{I_n}\bar{\Psi}_{\bar{J}_1}\ldots\bar{\Psi}_{\bar{J}_n}\,, \tag{2.12}$$

with $\mathcal{H}^{I\bar{J}}\big(\Phi,\bar{\Phi}\big)=g^{I\bar{J}}\big(\Phi,\bar{\Phi}\big)$. The variables (Φ^I,Ψ_J) parametrize the cotangent bundle $T^*\mathcal{M}$ of the Kähler manifold $\mathcal{M}$. The action (2.11) is written by chiral superfields only, so the action gives hyperkähler potential.

3. The tangent bundle

In what follows, we consider the case when $\mathcal{M}$ is a Hermitian symmetric space:

$$\nabla_L R_{I_1\bar{J}_1I_2\bar{J}_2}=\bar{\nabla}_{\bar{L}}R_{I_1\bar{J}_1I_2\bar{J}_2}=0\,. \tag{3.1}$$

Then, the algebraic equations of motion (2.5) are known to be equivalent to the holomorphic geodesic equation (with complex evolution parameter) [9, 10]

$$\frac{\mathrm{d}^2\Upsilon^I_*(\zeta)}{\mathrm{d}\zeta^2}+\Gamma^I{}_{JK}\Big(\Upsilon_*(\zeta),\bar{\Phi}\Big)\,\frac{\mathrm{d}\Upsilon^J_*(\zeta)}{\mathrm{d}\zeta}\,\frac{\mathrm{d}\Upsilon^K_*(\zeta)}{\mathrm{d}\zeta}=0\,, \tag{3.2}$$

under the same initial conditions (2.6). Here $\Gamma^I{}_{JK}(\Phi,\bar{\Phi})$ are the Christoffel symbols for the Kähler metric $g_{I\bar{J}}(\Phi,\bar{\Phi})=\partial_I\partial_{\bar{J}}K(\Phi,\bar{\Phi})$. In particular, we have

$$\Upsilon^I_2=-\frac{1}{2}\Gamma^I{}_{JK}\big(\Phi,\bar{\Phi}\big)\,\Sigma^J\Sigma^K\,. \tag{3.3}$$

According to the principles of projective superspace [4, 5], the action (2.1) is invariant under the $\mathcal{N}=2$ supersymmetry transformations

$$\delta\Upsilon^I(\zeta)=\mathrm{i}\,\big(\varepsilon^\alpha_i Q^i_\alpha+\bar{\varepsilon}^i_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}_i\big)\,\Upsilon^I(\zeta), \tag{3.4}$$

when $\Upsilon^I(\zeta)$ is viewed as a $\mathcal{N}=2$ superfield. Here $i=1,2$ is the $SU(2)_R$ index. However, since the action is given in $\mathcal{N}=1$ superspace, it is only the $\mathcal{N}=1$ supersymmetry which is manifestly realized. The second hidden supersymmetry can be shown to act on the physical superfields Φ and Σ as follows (see, e.g. [6])

$$\delta\Phi^I = \bar{\varepsilon}_{\dot\alpha}\bar{D}^{\dot\alpha}\Sigma^I\,, \qquad \delta\Sigma^I = -\varepsilon^\alpha D_\alpha\Phi^I + \bar{\varepsilon}_{\dot\alpha}\bar{D}^{\dot\alpha}\Upsilon^I_2\,. \tag{3.5}$$

Upon elimination of the auxiliary superfields, the action (2.8), which is associated with the Hermitian symmetric space $\mathcal{M}$, is invariant under

$$\delta\Phi^I = \bar{\varepsilon}_{\dot\alpha}\bar{D}^{\dot\alpha}\Sigma^I\,, \qquad \delta\Sigma^I = -\varepsilon^\alpha D_\alpha\Phi^I - \frac{1}{2}\bar{\varepsilon}_{\dot\alpha}\bar{D}^{\dot\alpha}\Big\{\Gamma^I{}_{JK}\big(\Phi,\bar\Phi\big)\,\Sigma^J\Sigma^K\Big\}\,. \tag{3.6}$$

The invariance of the action (2.8) under (3.6) requires to satisfy the the following first-order differential equation

$$\frac{1}{2}R_{K\bar{J}L}{}^I\,\frac{\partial\mathcal{L}}{\partial\Sigma^I}\,\Sigma^K\Sigma^L + \frac{\partial\mathcal{L}}{\partial\bar\Sigma^{\bar J}} + g_{I\bar J}\,\Sigma^I = 0\,. \tag{3.7}$$

The solution to the equation (3.7) was first presented in [13] in the form

$$\mathcal{L} = -\frac{\mathrm{e}^{R_{\Sigma,\bar\Sigma}}-1}{R_{\Sigma,\bar\Sigma}}\,|\Sigma|^2\,, \qquad |\Sigma|^2 := g_{I\bar J}\Sigma^I\bar\Sigma^{\bar J}\,. \tag{3.8}$$

An alternative form of the solution was obtained in [14]:

$$\mathcal{L}(\Phi,\bar\Phi,\Sigma,\bar\Sigma) = -\frac{1}{2}\mathbf{\Sigma}^T\mathbf{g}\frac{\ln(\mathbf{1}+\mathbf{R}_{\Sigma,\bar\Sigma})}{\mathbf{R}_{\Sigma,\bar\Sigma}}\mathbf{\Sigma}\,, \tag{3.9}$$

where

$$\mathbf{\Sigma} = \begin{pmatrix}\Sigma^I\\ \bar\Sigma^{\bar I}\end{pmatrix}\,, \qquad \mathbf{g} = \begin{pmatrix}0 & g_{I\bar J}\\ g_{\bar I J} & 0\end{pmatrix}\,, \tag{3.10}$$

$$\mathbf{R}_{\Sigma,\bar\Sigma} = \begin{pmatrix}0 & (R_\Sigma)^I{}_{\bar J}\\ (R_{\bar\Sigma})^{\bar I}{}_J & \end{pmatrix} = \begin{pmatrix}0 & \frac{1}{2}R_K{}^I{}_{L\bar J}\Sigma^K\Sigma^L\\ \frac{1}{2}R_{\bar K}{}^{\bar I}{}_{\bar L J}\bar\Sigma^{\bar K}\bar\Sigma^{\bar L} & \end{pmatrix}\,. \tag{3.11}$$

We shall rewrite (3.9) using the following differential operators

$$R_{\Sigma,\bar\Sigma} = -(R_\Sigma)^I{}_{\bar J}\bar\Sigma^{\bar J}\frac{\partial}{\partial\Sigma^I}, \qquad \bar{R}_{\Sigma,\bar\Sigma} = -(R_{\bar\Sigma})^{\bar I}{}_J\Sigma^J\frac{\partial}{\partial\bar\Sigma^{\bar I}}\,. \tag{3.12}$$

These operators were originally introduced in [13]. They satisfy the relation

$$R_{\Sigma,\bar\Sigma}\mathcal{L}^{(n)} = \bar{R}_{\Sigma,\bar\Sigma}\mathcal{L}^{(n)}\,. \tag{3.13}$$

Performing the Taylor expansion for (3.9), we have

$$\mathcal{L} = \sum_{n=1}^{\infty} c_n F^{(n)}\,, \qquad c_n = \frac{(-1)^n}{n}\,, \tag{3.14}$$

where the functions $F^{(n)}$ given by

$$F^{(2k+2)} = \Sigma^I g_{I\bar J}\Big(\big(R_{\bar\Sigma}R_\Sigma\big)^k\Big)^{\bar J}{}_{\bar K}(R_{\bar\Sigma})^{\bar K}{}_L\Sigma^L\,, \quad k=0,1,2\ldots \tag{3.15}$$

$$F^{(2k+1)} = \Sigma^I g_{I\bar J}\Big(\big(R_{\bar\Sigma}R_\Sigma\big)^k\Big)^{\bar J}{}_{\bar K}\bar\Sigma^{\bar K}\,, \quad k=0,1,2\ldots \tag{3.16}$$

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satisfy relations

$$\Sigma^I \frac{\partial}{\partial \Sigma^I} F^{(n)} = \bar{\Sigma}^{\bar{I}} \frac{\partial}{\partial \bar{\Sigma}^{\bar{I}}} F^{(n)} = n F^{(n)} \,, \tag{3.17}$$

$$\frac{\partial}{\partial \Sigma^I} F^{(2k+2)} = (2k+2) g_{I\bar{J}} \left((R_{\bar{\Sigma}} R_{\Sigma})^k R_{\bar{\Sigma}} \right)^{\bar{J}}{}_{K} \Sigma^K , \tag{3.18}$$

$$\frac{\partial}{\partial \Sigma^I} F^{(2k+1)} = (2k+1) g_{I\bar{J}} \left((R_{\bar{\Sigma}} R_{\Sigma})^k \right)^{\bar{J}}{}_{\bar{K}} \bar{\Sigma}^{\bar{K}} . \tag{3.19}$$

Using these identities, one can easily prove

$$F^{(n+1)} = \frac{(-R_{\Sigma,\bar{\Sigma}})^n}{n!} \, |\Sigma|^2 \ . \tag{3.20}$$

Putting this into (3.14), we arrive at

$$\mathcal{L} = - \sum_{n=1}^{\infty} \frac{(R_{\Sigma,\bar{\Sigma}})^{n-1}}{n!} \, |\Sigma|^2 = - \int_0^1 \mathrm{d}t \, \mathrm{e}^{t R_{\Sigma,\bar{\Sigma}}} \, |\Sigma|^2 = - \frac{\mathrm{e}^{R_{\Sigma,\bar{\Sigma}}} - 1}{R_{\Sigma,\bar{\Sigma}}} \, |\Sigma|^2 \ . \tag{3.21}$$

This form coincides with (3.8).

4. The cotangent bundle

We start with (2.10) to construct a dual formulation [13]. The action can be shown to be invariant under the following supersymmetry transformations

$$\begin{aligned}
\delta \Phi^I &= \frac{1}{2} \bar{D}^2 \big\{ \overline{\varepsilon \theta} \, \Sigma^I \big\} \,, \\
\delta \Sigma^I &= - \varepsilon^\alpha D_\alpha \Phi^I - \frac{1}{2} \bar{\varepsilon}_{\dot{\alpha}} \bar{D}^{\dot{\alpha}} \Big\{ \Gamma^I{}_{JK} \big(\Phi, \bar{\Phi} \big) \, \Sigma^J \Sigma^K \Big\} - \frac{1}{2} \overline{\varepsilon \theta} \, \Gamma^I{}_{JK} \big(\Phi, \bar{\Phi} \big) \, \Sigma^J \bar{D}^2 \Sigma^K \,, \\
\delta \Psi_I &= - \frac{1}{2} \bar{D}^2 \Big\{ \overline{\varepsilon \theta} \, K_I \big(\Phi, \bar{\Phi} \big) \Big\} + \frac{1}{2} \bar{D}^2 \Big\{ \overline{\varepsilon \theta} \, \Gamma^K_{IJ} \big(\Phi, \bar{\Phi} \big) \, \Sigma^J \Big\} \, \Psi_K \,.
\end{aligned} \tag{4.22}$$

These transformations induce the supersymmetry transformations making the action (2.11) invariant

$$\begin{aligned}
\delta \Phi^I &= \frac{1}{2} \bar{D}^2 \big\{ \overline{\varepsilon \theta} \, \Sigma^I \big(\Phi, \bar{\Phi}, \Psi, \bar{\Psi} \big) \big\} \,, \\
\delta \Psi_I &= - \frac{1}{2} \bar{D}^2 \Big\{ \overline{\varepsilon \theta} \, K_I \big(\Phi, \bar{\Phi} \big) \Big\} + \frac{1}{2} \bar{D}^2 \Big\{ \overline{\varepsilon \theta} \, \Gamma^K_{IJ} \big(\Phi, \bar{\Phi} \big) \, \Sigma^J \big(\Phi, \bar{\Phi}, \Psi, \bar{\Psi} \big) \Big\} \, \Psi_K \,,
\end{aligned} \tag{4.23}$$

with

$$\Sigma^I \big(\Phi, \bar{\Phi}, \Psi, \bar{\Psi} \big) = \frac{\partial}{\partial \Psi_I} \, \mathcal{H} \big(\Phi, \bar{\Phi}, \Psi, \bar{\Psi} \big) \equiv \mathcal{H}^I \,. \tag{4.24}$$

The requirement of invariance under such transformations can be shown to be equivalent to the following nonlinear equation on $\mathcal{H}$ [13]:

$$\mathcal{H}^I \, g_{I\bar{J}} - \frac{1}{2} \, \mathcal{H}^K \mathcal{H}^L \, R_{K\bar{J}L}{}^I \, \Psi_I = \bar{\Psi}_{\bar{J}} \,. \tag{4.25}$$

This equation also follows directly from (3.7) using the definition of the Ψ's, or, as a consequence of the superspace Legendre transform.

The closed form expression of $\mathcal{H}$ for any Hermitian symmetric space was obtained as a solution of (4.25), which is given by [14]

$$\mathcal{H}(\Phi,\bar{\Phi},\Psi,\bar{\Psi}) = \frac{1}{2}\mathbf{\Psi}^T\mathbf{g}^{-1}\mathcal{F}(-\mathbf{R}_{\Psi,\bar{\Psi}})\mathbf{\Psi}\,, \tag{4.26}$$

where

$$\mathbf{\Psi} = \begin{pmatrix}\Psi_I\\ \bar{\Psi}_{\bar{I}}\end{pmatrix}, \qquad \mathbf{g}^{-1} = \begin{pmatrix}0 & g^{I\bar{J}}\\ g^{\bar{I}J} & 0\end{pmatrix}, \tag{4.27}$$

$$\mathbf{R}_{\Psi,\bar{\Psi}} = \begin{pmatrix}0 & (R_\Psi)_I{}^{\bar{J}}\\ (R_{\bar{\Psi}})_{\bar{I}}{}^J & 0\end{pmatrix} = \begin{pmatrix}0 & \frac{1}{2}R_I{}^{K\bar{J}L}\Psi_K\Psi_L\\ \frac{1}{2}R_{\bar{I}}{}^{\bar{K}J\bar{L}}\bar{\Psi}_{\bar{K}}\bar{\Psi}_{\bar{L}} & 0\end{pmatrix}, \tag{4.28}$$

and

$$\mathcal{F}(x) = \frac{1}{x}\left[\sqrt{1+4x} - 1 - \ln\left(\frac{1+\sqrt{1+4x}}{2}\right)\right]. \tag{4.29}$$

We rewrite (4.26) to a convenient form for our purpose. In a similar way to the tangent bundle case, we introduce the following differential operators

$$R_{\Psi,\bar{\Psi}} = -(R_\Psi)_I{}^{\bar{J}}\bar{\Psi}_{\bar{J}}\frac{\partial}{\partial\Psi_I}, \quad \bar{R}_{\Psi,\bar{\Psi}} = -(R_{\bar{\Psi}})_{\bar{I}}{}^J\Psi_J\frac{\partial}{\partial\bar{\Psi}_{\bar{I}}}. \tag{4.30}$$

They satisfy

$$R_{\Psi,\bar{\Psi}}\mathcal{H}^{(n)} = \bar{R}_{\Psi,\bar{\Psi}}\mathcal{H}^{(n)}. \tag{4.31}$$

Performing the Taylor expansion for (4.26), we have

$$\mathcal{H} = \sum_{n=1}^{\infty}c_nG^{(n)}, \qquad c_n = \frac{(-1)^{n-1}\mathcal{F}^{(n-1)}(0)}{(n-1)!}, \tag{4.32}$$

where the terms $G^{(n)}$ are given by

$$G^{(2k+2)} = \Psi_Ig^{I\bar{J}}\Big(\big(R_{\bar{\Psi}}R_\Psi\big)^k\Big)_{\bar{J}}{}^{\bar{K}}(R_{\bar{\Psi}})_{\bar{K}}{}^L\Psi_L\,, \quad k=0,1,2\ldots \tag{4.33}$$

$$G^{(2k+1)} = \Psi_Ig^{I\bar{J}}\Big(\big(R_{\bar{\Psi}}R_\Psi\big)^k\Big)_{\bar{J}}{}^{\bar{K}}\bar{\Psi}_{\bar{K}}\,, \quad k=0,1,2\ldots \tag{4.34}$$

and satisfy following relations

$$\Psi_I\frac{\partial}{\partial\Psi_I}G^{(n)} = \bar{\Psi}_{\bar{I}}\frac{\partial}{\partial\bar{\Psi}_{\bar{I}}}G^{(n)} = nG^{(n)}\,, \tag{4.35}$$

$$\frac{\partial}{\partial\Psi_I}G^{(2k+2)} = (2k+2)g^{I\bar{J}}\Big(\big(R_{\bar{\Psi}}R_\Psi\big)^k\Big)_{\bar{J}}{}^{\bar{K}}(R_{\bar{\Psi}})_{\bar{K}}{}^L\Psi_L\,, \tag{4.36}$$

$$\frac{\partial}{\partial\Psi_I}G^{(2k+1)} = (2k+1)g^{I\bar{J}}\Big(\big(R_{\bar{\Psi}}R_\Psi\big)^k\Big)_{\bar{J}}{}^{\bar{K}}\bar{\Psi}_{\bar{K}}\,. \tag{4.37}$$

By using the above identities, one can show

$$G^{(n+1)} = \frac{(-R_{\Psi,\bar{\Psi}})^n}{n!}\,|\Psi|^2\,, \qquad |\Psi|^2 := g^{I\bar{J}}\Psi_I\bar{\Psi}_{\bar{J}}\,. \tag{4.38}$$

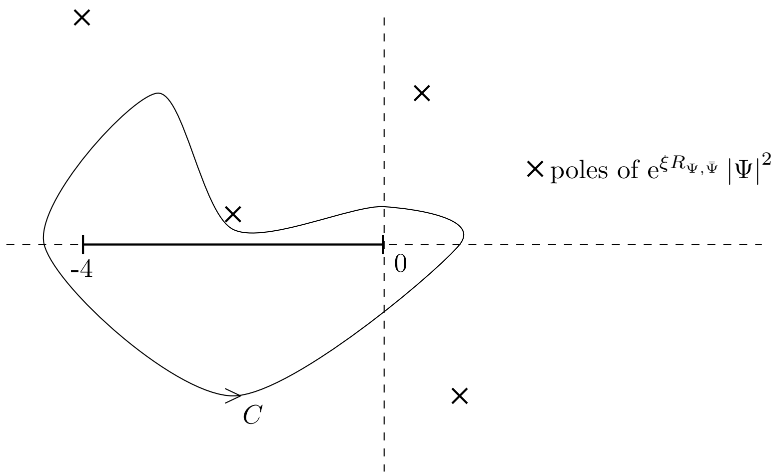

Figure 1. The contour of integration in eq. (4.42).

Putting together (4.32) and (4.38) we find

$$\mathcal{H} = \sum_{n=0}^{\infty} \frac{\mathcal{F}^{(n)}(0)}{n!} \frac{R^n_{\Psi,\bar{\Psi}}}{n!} |\Psi|^2 \,. \tag{4.39}$$

Now we use of the formula,

$$\frac{x^n}{n!} = \oint_C \frac{\mathrm{d}\xi}{2\pi \mathrm{i}} \frac{\mathrm{e}^{\xi x}}{\xi^{n+1}} \,, \tag{4.40}$$

where, at this point, contour C can be any closed loop containing the origin with positive (counterclockwise) orientation. Using this identity we obtain

$$\mathcal{H} = \sum_{n=0}^{\infty} \frac{\mathcal{F}^{(n)}(0)}{n!} \oint_C \frac{\mathrm{d}\xi}{2\pi \mathrm{i}} \frac{e^{\xi R_{\Psi,\bar{\Psi}}}}{\xi^{n+1}} |\Psi|^2 \,, \tag{4.41}$$

where now the contour C must be chosen such that it lies in the region where the factor $e^{\xi R_{\Psi,\bar{\Psi}}}|\Psi|^2$ is analytic. With this in mind we can rewrite eq. (4.41) as

$$\mathcal{H} = \oint_C \frac{\mathrm{d}\xi}{2\pi \mathrm{i}} \frac{\mathcal{F}(1/\xi)}{\xi} \mathrm{e}^{\xi R_{\Psi,\bar{\Psi}}} |\Psi|^2 \,. \tag{4.42}$$

The function $\mathcal{F}(1/\xi)/\xi$ gives a branch cut between -4 and 0. Therefore, the contour has to be chosen so that it does not cross the branch cut because the $\mathcal{H}$ is real and regular. Combining both requirements we see that the contour C in addition to avoiding the branch cut must be also bounded by the poles of the factor $e^{\xi R_{\Psi,\bar{\Psi}}}|\Psi|^2$ (see fig. 1). In the subsequent section, we derive the tangent bundle as well as the cotangent bundle over $E_7/[E_6 \times U(1)]$ by using (3.21) and (4.42).

5. The compact Hermitian symmetric space $E_7/(E_6 \times U(1))$

Locally, the compact Hermitian symmetric space $E_7/(E_6 \times U(1))$ can be described by complex variables Φ^i transforming in the **27** representation of the E_6 and their conjugates [17, 18].

$$\Phi^i \,, \qquad \bar{\Phi}_i := (\Phi^i)^* \,, \qquad i = 1, \ldots, 27 \,. \tag{5.1}$$

The Kähler potential is

$$K(\Phi,\bar\Phi)=\ln\left(1+\Phi^i\bar\Phi_i+\frac{1}{4}|\Gamma_{ijk}\Phi^j\Phi^k|^2+\frac{1}{36}|\Gamma_{ijk}\Phi^i\Phi^j\Phi^k|^2\right). \tag{5.2}$$

Here we have used the notation, $|\Gamma_{ijk}\Phi^j\Phi^k|^2=(\Gamma_{ijk}\Phi^j\Phi^k)(\Gamma^{ilm}\bar\Phi_l\bar\Phi_m)$ and $|\Gamma_{ijk}\Phi^i\Phi^j\Phi^k|^2=(\Gamma_{ijk}\Phi^i\Phi^j\Phi^k)(\Gamma^{lmn}\bar\Phi_l\bar\Phi_m\bar\Phi_n)$, where Γ_{ijk} is a rank-3 symmetric tensor covariant under the E_6. Its complex conjugate is denoted by Γ^{ijk}.

Products of these tensors satisfy an identity [19]

$$\Gamma_{ijk}\Gamma^{ijl}=10\delta_k^l\,. \tag{5.3}$$

And

$$\Gamma_{ijk}\Big(\Gamma^{ilm}\Gamma^{jnp}+\Gamma^{ilp}\Gamma^{jnm}+\Gamma^{iln}\Gamma^{jmp}\Big)=\delta_k^l\Gamma^{mnp}+\delta_k^p\Gamma^{mnl}+\delta_k^n\Gamma^{mlp}+\delta_k^m\Gamma^{lnp}\,, \tag{5.4}$$

which is called the Springer relation [20].

Let us calculate the tangent bundle Lagrangian by using (3.21). In our notation, the first-order differential operator defined in (3.12) is given by

$$R_{\Sigma,\bar\Sigma}=-\frac{1}{2}\Sigma^i\bar\Sigma_j\Sigma^k R_{i\ k}^{\ j\ l}(g^{-1})_l^{\ m}\frac{\partial}{\partial\Sigma^m}\,, \tag{5.5}$$

where $(g^{-1})_i^{\ j}=(g_j^{\ i})^{-1}$ is the inverse metric of $g_i^{\ j}$, that is $g_i^{\ k}(g^{-1})_k^{\ j}=\delta_i^{\ j}$. Since we are considering a symmetric space, it is actually sufficient to carry out the calculations at a particular point, say at $\Phi=0$. The Riemann tensor at $\Phi=0$ is given as

$$\begin{aligned}R_{i\ k}^{\ j\ l}\Big|_{\Phi=0} &= \partial_k\partial^l g_i^{\ j}-(g^{-1})_m^{\ n}\partial_n g_i^{\ j}\partial^m g_k^{\ l}\Big|_{\Phi=0}\\ &= -\delta_i^{\ j}\delta_k^{\ l}+\Gamma_{mik}\Gamma^{mlj}-\delta_k^{\ j}\delta_i^{\ l}\,.\end{aligned} \tag{5.6}$$

Substituting this into (5.5), we have

$$R_{\Sigma,\bar\Sigma}=xD-\frac{1}{2}y\partial_x-\frac{1}{3}z\partial_y\,,\qquad D:=x\partial_x+y\partial_y+z\partial_z\,, \tag{5.7}$$

where we have used (5.4) and invariant quantities under the E_6 action

$$x:=\Sigma^i\bar\Sigma_i\,, \tag{5.8}$$
$$y:=(\Gamma_{ijk}\Sigma^j\Sigma^k)(\Gamma^{ilm}\bar\Sigma_l\bar\Sigma_m)\,, \tag{5.9}$$
$$z:=(\Gamma_{ijk}\Sigma^i\Sigma^j\Sigma^k)(\Gamma^{lmn}\bar\Sigma_l\bar\Sigma_m\bar\Sigma_n)\,. \tag{5.10}$$

Using the Baker-Campbell-Hausdorff expansion formula, one can show, that

$$\mathrm{e}^{tR_{\Sigma,\bar\Sigma}}=\mathrm{e}^{txD}\mathrm{e}^{-\frac{t}{2}y\partial_x}\mathrm{e}^{-\frac{t}{3}z\partial_y}\mathrm{e}^{\frac{t^2}{12}z\partial_x}\mathrm{e}^{-\frac{t^2}{4}yD}\mathrm{e}^{-\frac{t^3}{18}zD}\,. \tag{5.11}$$

By straight-forward application of each exponential we obtain

$$\mathrm{e}^{tR_{\Sigma,\bar\Sigma}}x=-\partial_t\ln\Omega(t;x,y,z)\,,\qquad \Omega(t;x,y,z):=1-tx+\frac{t^2}{4}y-\frac{t^3}{36}z\,. \tag{5.12}$$

Plugging (5.12) into (3.21) we get the tangent bundle action at $\Phi=0$.

$$\mathcal{L}=\ln\Big(1-x+\frac{1}{4}y-\frac{1}{36}z\Big)\,. \tag{5.13}$$

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This result can be extended to an arbitrary point Φ of the base manifold by replacing

$$x \to \; g_i{}^j \Sigma^i \bar{\Sigma}_j \,, \tag{5.14}$$

$$\frac{1}{4} y \to \; \frac{1}{2} (g_i{}^j \Sigma^i \bar{\Sigma}_j)^2 + \frac{1}{4} R_i{}^j{}_k{}^l \Sigma^i \bar{\Sigma}_j \Sigma^k \bar{\Sigma}_l \,, \tag{5.15}$$

$$-\frac{1}{36} z \to -\frac{1}{6} (g_i{}^j \Sigma^i \bar{\Sigma}_j)^3 - \frac{1}{4} (g_i{}^j \Sigma^i \bar{\Sigma}_j)(R_k{}^m{}_l{}^n \Sigma^k \bar{\Sigma}_m \Sigma^l \bar{\Sigma}_n) - \frac{1}{12} |g_i{}^j R_j{}^k{}_l{}^m \bar{\Sigma}_k \Sigma^l \bar{\Sigma}_m|^2 \,. \tag{5.16}$$

Let us turn to the cotangent bundle action. Again we restrict the calculations to the origin of the base manifold $\Phi = 0$. Defining E_6 invariant quantities in terms of the cotangent vector

$$\tilde{x} := \Psi_i \bar{\Psi}^i \,, \tag{5.17}$$

$$\tilde{y} := (\Gamma^{ijk} \Psi_j \Psi_k)(\Gamma_{ilm} \bar{\Psi}^l \bar{\Psi}^m) \,, \tag{5.18}$$

$$\tilde{z} := (\Gamma^{ijk} \Psi_i \Psi_j \Psi_k)(\Gamma_{lmn} \bar{\Psi}^l \bar{\Psi}^m \bar{\Psi}^n) \,, \tag{5.19}$$

we obtain the differential operator (4.30) of the form

$$R_{\Psi,\bar{\Psi}} = \tilde{x} \tilde{D} - \frac{1}{2} \tilde{y} \partial_{\tilde{x}} - \frac{1}{3} \tilde{z} \partial_{\tilde{y}} \,, \quad \tilde{D} := \tilde{x} \partial_{\tilde{x}} + \tilde{y} \partial_{\tilde{y}} + \tilde{z} \partial_{\tilde{z}} \,. \tag{5.20}$$

Repeating the same calculation as below (5.11), we find

$$e^{\xi R_{\Psi,\bar{\Psi}}} \tilde{x} = -\partial_\xi \ln \Omega(\xi; \tilde{x}, \tilde{y}, \tilde{z}) \,, \tag{5.21}$$

where the function Ω is given in (5.12). Eq. (4.42) with the above leads to

$$\mathcal{H} = -\oint_C \frac{\mathrm{d}\xi}{2\pi \mathrm{i}} \frac{\mathcal{F}(1/\xi)}{\xi} \frac{-\tilde{x} + \frac{\xi}{2} \tilde{y} - \frac{\xi^2}{12} \tilde{z}}{1 - \xi \tilde{x} + \frac{\xi^2}{4} \tilde{y} - \frac{\xi^3}{36} \tilde{z}} \,. \tag{5.22}$$

Note that the factor in (4.41) $e^{\xi R_{\Psi,\bar{\Psi}}} |\Psi|^2$ produces poles, given by the roots of the cubic equation:

$$1 - \xi x + \frac{\xi^2}{4} y - \frac{\xi^3}{36} z = \frac{z}{36} (\xi_1 - \xi)(\xi_2 - \xi)(\xi_3 - \xi) = 0 \,. \tag{5.23}$$

By construction, these poles are not inside the contour which only encircles the branch cut between the origin $\xi = 0$ and $\xi = -4$. Since the function $\mathcal{F}(1/\xi)/\xi$ has no singularity at infinity in the ξ-plane, the contour can be equivalently respected as encircling the poles ξ_1, ξ_2 and ξ_3 in the opposite direction. Thus, we can apply the Residue theorem, picking additional minus sign from the orientation of contour (which kills another minus given by eq. (5.21)) to obtain

$$\mathcal{H} = \frac{\mathcal{F}(1/\xi_1)}{\xi_1} + \frac{\mathcal{F}(1/\xi_2)}{\xi_2} + \frac{\mathcal{F}(1/\xi_3)}{\xi_3} \,. \tag{5.24}$$

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The explicit form of ξ_1, ξ_2 and ξ_3 are given by

$$\begin{aligned}
\xi_1 &= \frac{3\tilde{y}}{\tilde{z}} + \frac{2^{1/3}(-81\tilde{y}^2 + 108\tilde{x}\tilde{z})}{3\tilde{z}(-1458\tilde{y}^3 + 2916\tilde{x}\tilde{y}\tilde{z} - 972\tilde{z}^2 + A)^{1/3}} \\
&\quad - \frac{(-1458\tilde{y}^3 + 2916\tilde{x}\tilde{y}\tilde{z} - 972\tilde{z}^2 + A)^{1/3}}{3 \cdot 2^{1/3}\tilde{z}}, \qquad (5.25) \\
\xi_2 &= \frac{3\tilde{y}}{\tilde{z}} - \frac{(1 + i\sqrt{3})(-81\tilde{y}^2 + 108\tilde{x}\tilde{z})}{3 \cdot 2^{2/3}\tilde{z}(-1458\tilde{y}^3 + 2916\tilde{x}\tilde{y}\tilde{z} - 972\tilde{z}^2 + A^{1/3})} \\
&\quad + \frac{1 - i\sqrt{3}}{6 \cdot 2^{1/3}\tilde{z}}(-1458\tilde{y}^3 + 2916\tilde{x}\tilde{y}\tilde{z} - 972\tilde{z}^2 + A^{1/3}), \qquad (5.26) \\
\xi_3 &= \frac{3\tilde{y}}{\tilde{z}} - \frac{(1 - i\sqrt{3})(-81\tilde{y}^2 + 108\tilde{x}\tilde{z})}{3 \cdot 2^{2/3}\tilde{z}(-1458\tilde{y}^3 + 2916\tilde{x}\tilde{y}\tilde{z} - 972\tilde{z}^2 + A^{1/3})} \\
&\quad + \frac{1 + i\sqrt{3}}{6 \cdot 2^{1/3}\tilde{z}}(-1458\tilde{y}^3 + 2916\tilde{x}\tilde{y}\tilde{z} - 972\tilde{z}^2 + A^{1/3}), \qquad (5.27)
\end{aligned}$$

where $A = \sqrt{4(-81\tilde{y}^2 + 108\tilde{x}\tilde{z})^3 + (-1458\tilde{y}^3 + 2916\tilde{x}\tilde{y}\tilde{z} - 972\tilde{z}^2)^2}$. The result at an arbitrary point of Φ can be obtained by the following replacements

$$\begin{aligned}
\tilde{x} &\to (g^{-1})_i{}^j \Psi_j \bar{\Psi}^i\,, \qquad (5.28) \\
\frac{1}{4}\tilde{y} &\to \frac{1}{2}((g^{-1})_i{}^j \Psi_j \bar{\Psi}^i)^2 + \frac{1}{4}\tilde{R}_i{}^j{}_k{}^l \bar{\Psi}^i \Psi_j \bar{\Psi}^k \Psi_l\,, \qquad (5.29) \\
-\frac{1}{36}\tilde{z} &\to -\frac{1}{6}((g^{-1})_i{}^j \Psi_j \bar{\Psi}^j)^3 - \frac{1}{4}((g^{-1})_i{}^j \Psi_j \bar{\Psi}^i)(\tilde{R}_k{}^l{}_m{}^n \bar{\Psi}^k \Psi_l \bar{\Psi}^m \Psi_n) \\
&\quad - \frac{1}{12}|(g^{-1})_i{}^j \tilde{R}_j{}^k{}_l{}^m \Psi_k \bar{\Psi}^l \Psi_m|^2\,, \qquad (5.30)
\end{aligned}$$

where $\tilde{R}_i{}^j{}_k{}^l = (g^{-1})_i{}^m (g^{-1})_n{}^j (g^{-1})_k{}^p (g^{-1})_q{}^l R_m{}^n{}_p{}^q$.

6. Generalization

Our reformulation suggests an alternative expression for the cotangent bundle over any compact Hermitian symmetric space. In order to explain it, let us look at the equation (5.22). We can rewrite it as follows

$$\mathcal{H} = -\oint_C \frac{\mathrm{d}\xi}{2\pi \mathrm{i}} \frac{\mathcal{F}(1/\xi)}{\xi} \partial_\xi \ln \Omega(\xi; \tilde{x}, \tilde{y}, \tilde{z})\,. \qquad (6.31)$$

It is easy to see that Ω is related to the Kähler potential $K(\Phi, \bar{\Phi})$ by uniform rescaling of coordinates

$$\ln \Omega(\xi; \tilde{x}, \tilde{y}, \tilde{z}) = K(\Phi \to -\sqrt{\xi}\Psi, \bar{\Phi} \to \sqrt{\xi}\bar{\Psi}) := K_\xi(\Psi, \bar{\Psi})\,. \qquad (6.32)$$

Thus, we see that the cotangent bundle action for any Hermitian symmetric space is given by

$$\mathcal{H} = -\oint_C \frac{\mathrm{d}\xi}{2\pi \mathrm{i}} \frac{\mathcal{F}(1/\xi)}{\xi} \partial_\xi K_\xi(\Psi, \bar{\Psi})\,, \qquad (6.33)$$

where, as in the $E_7/(E_6 \times U(1))$ case, the contour C must encircle the branch cut of $\mathcal{F}(1/\xi)/\xi$ and it must be bounded by the poles of $\partial_\xi K_\xi$, in the same manner as depicted in fig. 1. Since for all Hermitian symmetric spaces the Ω is a finite-order polynomial in ξ, we may write

$$\Omega \sim \Pi_i\big(\xi - \xi_i(\Psi, \bar{\Psi})\big)\,, \qquad (6.34)$$

where ξ_i are the solutions of characteristic equation $\Omega = 0$. Since we are interested in (derivative of) logarithm of the above quantity, the constant of proportionality is actually not important. Thus, we are led to

$$\partial_\xi K_\xi = \sum_\xi \frac{1}{\xi - \xi_i} . \tag{6.35}$$

Substituting this back into (6.33) and using the Residue theorem, we arrive at

$$\mathcal{H} = -\sum_i \oint_C \frac{\mathrm{d}\xi}{2\pi \mathrm{i}} \frac{\mathcal{F}(1/\xi)}{\xi} \frac{1}{\xi - \xi_i} = \sum_i \frac{\mathcal{F}(1/\xi_i)}{\xi_i} , \tag{6.36}$$

where minus sing is absorbed because the contour C encircles poles in the clockwise direction. We checked that the results obtained by this formula for all the compact Hermitian symmetric spaces perfectly coincide with the previous results in [9, 10, 12, 13].

7. Conclusions

In this proceeding we have reviewed how to construct the closed formulas for tangent and cotangent bundle action over the compact Hermitian symmetric space $E_7/(E_6 \times U(1))$, starting from the result in [14]. This particular case allowed us to establish the correspondence between a cotangent bundle action and poles of derivatives of the Kähler potential under uniform rescaling of coordinates. Based on this correspondence we have present a general formula for a cotangent bundle action for any compact Hermitian symmetric space in Eq. (6.36). In the former methods in [9, 10, 12, 13], the Legendre transform is used to obtain a cotangent bundle action. This procedure, although in principle applicable to the $E_7/(E_6 \times U(1))$ case, is not straightforward and one often must find a suitable ansatz to solve involved algebraic equations. Our simple formulae of the cotangent bundle action obtained from the result in [14] avoids such a complication and clearly demonstrates the reduction of work in constructing of a cotangent bundle action.

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