Representations of ℓ-conformal Galilei algebra and hierarchy of invariant equation

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Abstract. The ℓ-conformal Galilei algebra, denoted by $\mathfrak{g}_\ell(d)$, is a particular non-semisimple Lie algebra specified by a positive integer d and a spin value ℓ. The algebra $\mathfrak{g}_\ell(d)$ admits central extensions. We study lowest weight representations, in particular Verma modules, of $\mathfrak{g}_\ell(d)$ with the central extensions for $d = 1, 2$. We give a classification of irreducible modules over $d = 1$ algebras and a condition of the Verma modules over $d = 2$ algebras being reducible. As an application of the representation theory, hierarchies of differential equations are derived. The Lie group generated by $\mathfrak{g}_\ell(d)$ with the central extension is a kinematical symmetry of the differential equations.

1. Introduction

By ℓ-conformal Galilei algebra we mean a particular class of non-semisimple Lie algebras. Each Lie algebra in the class is specified by a pair of numbers (d, ℓ). The possible values of d and ℓ are given by

$$d = 1,\ 2,\ 3,\ \ldots, \qquad \ell = \frac{1}{2},\ 1,\ \frac{3}{2},\ 2,\ \ldots$$

The Lie algebra specified by a given pair (d, ℓ) has the generators

$$D,\ H,\ C,\ M_{ij} = -M_{ji},\ P_i^{(n)}, \qquad i, j = 1, 2, \ldots, d, \quad n = 0, 1, \cdots, 2\ell$$

and the generators satisfy the following nonvanishing commutation relations:

$$\begin{aligned}
&[D, H] = 2H, \qquad [D, C] = -2C, \qquad [C, H] = D, \\
&[M_{ij}, M_{k\ell}] = -\delta_{ik} M_{j\ell} - \delta_{j\ell} M_{ik} + \delta_{i\ell} M_{jk} + \delta_{jk} M_{i\ell}, \\
&[H, P_i^{(n)}] = -n P_i^{(n-1)}, \qquad [D, P_i^{(n)}] = 2(\ell - n) P_i^{(n)}, \\
&[C, P_i^{(n)}] = (2\ell - n) P_i^{(n+1)}, \qquad [M_{ij}, P_k^{(n)}] = -\delta_{ik} P_j^{(n)} + \delta_{jk} P_i^{(n)}.
\end{aligned} \tag{1}$$

This Lie algebra was introduced as a nonrelativistic conformal algebra in [1, 2]. Indeed one may see that it has the subalgebra $\langle D, H, C \rangle \simeq sl(2, \mathbb{R}) \simeq so(2, 1)$ which is the conformal algebra in $(1 + 1)$ dimensional spacetime. The commutative subalgebra $\langle P_i^{(n)} \rangle$ carries the spin ℓ

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representation of $sl(2,\mathbb{R})$ subalgebra and the vector representation of $so(d)$ subalgebra generated by M_{ij}. The parameter d indicates the dimension of space on which the generators act as infinitesimal transformations of the spacetime when a particular realization is considered. We denote the algebra specified by (d,ℓ) by $\mathfrak{g}_\ell(d)$.

It is known that $\mathfrak{g}_\ell(d)$ has two distinct type of central extensions according to the values of d and ℓ :

(i) mass extension existing for any d and $\ell \in \mathbb{Z}_{>0}+\frac{1}{2}$

$$[P_i^{(m)}, P_j^{(n)}] = \delta_{ij}\,\delta_{m+n,2\ell}\,I_m M, \qquad I_m = (-1)^{m+\ell+\frac{1}{2}}(2\ell-m)!\,m!. \tag{2}$$

(ii) exotic extension existing only for $d=2$ and $\ell \in \mathbb{Z}_{>0}$

$$[P_i^{(m)}, P_j^{(n)}] = \epsilon_{ij}\,\delta_{m+n,2\ell}\,I_m \Theta, \qquad I_m = (-1)^{m}(2\ell-m)!\,m!, \tag{3}$$

where ϵ_{ij} is the antisymmetric tensor with $\epsilon_{12}=1$. Simple explanation of the existence of two distinct central extensions is found in [9]. The Schrödinger algebra considered by Niederer corresponds to $\mathfrak{g}_{1/2}(d)$ with the mass central extension [3]. The exotic extension was first found for $\ell=1$ in the study of classical mechanics having higher order time derivatives [10, 11, 12].

The purpose of the present work is to develop a representation theory of $\mathfrak{g}_\ell(d)$ with the central extensions for $d=1,2$. Namely, we shall investigate representations of $\mathfrak{g}_\ell(2)$ for all possible values of ℓ but those of $\mathfrak{g}_\ell(1)$ for half-integer values of ℓ. As an application of the representation theory we derive partial differential equations which have kinematical invariance of the group generated by $\mathfrak{g}_\ell(d)$. The choice of particular values of d is motivated by the observation that $\mathfrak{g}_\ell(d)$ has more complicated structure for $d>2$ as the spatial rotation becomes non-abelian. There are a couple of reasons of focusing on the central extensions: as shown in [5], in the case of $\mathfrak{g}_{1/2}(1)$ without the mass central extension, the representation theory is more involved and the resulting differential equations are not of physical importance. In many physical applications of $\mathfrak{g}_\ell(d)$ the central extensions play an important roles (see for example [11, 13, 14]).

The present work is an extension of the previous works [5, 6, 7, 8, 16]. In [5, 6] a classification of the irreducible lowest weight representations of $\mathfrak{g}_{1/2}(d)$ $(d=1,2,3)$ and differential equations having kinematical symmetries generated by the algebras are given. Such differential equations for $\mathfrak{g}_{1/2}(d)$ with any d have been obtained in [7, 8]. On the other hand [16] classifies the irreducible highest weight representations of $\mathfrak{g}_1(2)$ with the exotic central extension. All those works show that one may apply the techniques such as Verma modules, singular vectors to investigate irreducible representations of $\mathfrak{g}_\ell(d)$. They also showed that differential equations having the group generated by $\mathfrak{g}_\ell(d)$ as kinematical symmetry may be obtained by the method developed for semisimple Lie groups [4].

We organize this paper as follows. In the next section we provide an explicit formula of singular vectors in the Verma modules over $\mathfrak{g}_\ell(1)$ with the mass central extension. This is used to give a classification of irreducible lowest weight modules of the algebras. We then employ the method of [4] to derive differential equations, such that kinematical symmetries of the equations are given by the Lie group generated by $\mathfrak{g}_\ell(1)$ with the mass central extension. In §3 we study reducibility of Verma modules over $\mathfrak{g}_\ell(2)$ with the mass or the exotic central extensions by explicit construction of singular vectors. By the result we derive differential equations having kinematical symmetries generated by $\mathfrak{g}_\ell(2)$ with the central extensions. We close the paper with some remarks in §4.

2. Representation theory of $d=1$ conformal Galilei algebra and invariant equations

2.1. Classification of irreducible representations

We start our investigations with $\mathfrak{g}_\ell(1)$ with the mass central extension, i.e., ℓ being a positive half-integer. The algebra, denoted simply by $\mathfrak{g}$, has no generators of spatial rotations and does not need an index for space coordinates. Namely, one may write the generators of $\mathfrak{g}$ as $D, H, C, P^{(n)}, M$ with $n = 0, 1, \ldots, 2\ell$. Their nonvanishing commutators are given by (1) and (2). Our aim in this subsection is to give a classification of the lowest weight modules of $\mathfrak{g}$. This will be done by a way analogous to the case of semisimple Lie algebras. That is, we introduce the Verma module over $\mathfrak{g}$ which is in a sense the largest lowest weight module for a given set of lowest weights. Then use the theorem that any lowest weight module over $\mathfrak{g}$ with the same set of lowest weight is isomorphic to a quotient of the Verma module [15].

To define the Verma modules we make the following vector space decomposition of $\mathfrak{g}$:

$$\begin{aligned} \mathfrak{g}^+ &= \langle\, H, P^{(0)}, P^{(1)}, \cdots, P^{(\ell-\frac{1}{2})}\,\rangle, \\ \mathfrak{g}^0 &= \langle\, D, M\,\rangle, \\ \mathfrak{g}^- &= \langle\, C, P^{(\ell+\frac{1}{2})}, P^{(\ell+\frac{3}{2})}, \cdots, P^{(2\ell)}\,\rangle. \end{aligned} \tag{4}$$

Then one may see that $[\mathfrak{g}^0, \mathfrak{g}^\pm] \subset \mathfrak{g}^\pm$, that is, this is analogues to the triangular decomposition of semisimple Lie algebras. Suppose that there exists a lowest weight vector defined by

$$\begin{aligned} &D\,|\delta,\mu\rangle = -\delta\,|\delta,\mu\rangle\,, \quad M\,|\delta,\mu\rangle = -\mu\,|\delta,\mu\rangle\,, \\ &X\,|\delta,\mu\rangle = 0, \quad {}^\forall X \in \mathfrak{g}^- \end{aligned} \tag{5}$$

A Verma module $V^{\delta,\mu}$ over $\mathfrak{g}$ with the lowest weights $(-\delta, -\mu)$ is defined by $V^{\delta,\mu} = U(\mathfrak{g}^+)\,|\delta,\mu\rangle$ where $U(\mathfrak{g}^+)$ is the universal enveloping algebra of $\mathfrak{g}^+$. More precisely, one may defined $V^{\delta,\mu}$ as the vector space with the following basis:

$$V^{\delta,\mu} = \left\{ H^h \prod_{j=0}^{\ell-\frac{1}{2}} P^{k_j}_{\ell-\frac{1}{2}-j}\,|\delta,\mu\rangle \;\middle|\; h, k_0, k_1, \cdots, k_{\ell-\frac{1}{2}} \in \mathbb{Z}_{\geq 0} \right\}. \tag{6}$$

The vector space $V^{\delta,\mu}$ carries a representation of $\mathfrak{g}$ specified by the lowest weights $(-\delta, -\mu)$. In general, the representation is not irreducible (reducible). Reducibility of $V^{\delta,\mu}$ is detected by singular vectors. A singular vector $|v_q\rangle$ is a vector having the same property as $|\delta,\mu\rangle$ but different eigenvalues. If the Verma module has a singular vector, then the representation is reducible, since the space $I^{\delta,\mu} = U(\mathfrak{g}^+)\,|v_q\rangle$ is obviously an invariant subspace of $V^{\delta,\mu}$. To obtain an irreducible module we consider a quotient module $V^{\delta,\mu}/I^{\delta,\mu}$. If the quotient module has no singular vectors then the quotient module is irreducible. If the module has singular vectors we take a quotient again and repeat the same procedure until we have an irreducible module. In this way one may arrive at a complete classification of irreducible lowest weight modules over $\mathfrak{g}$. Here we give only the results of this procedure. One may find the proofs in [17] (In [17] highest weight Verma modules are considered. It is easy to convert it to the lowest weight modules).

Proposition 1 *If* $2\delta - 2(q-1) + (\ell + \frac{1}{2})^2 = 0$ $q \in \mathbb{Z}_{>0}$ *and* $\mu \neq 0$, *then* $V^{\delta,\mu}$ *has precisely one singular vector:*

$$|v_q\rangle = \mathcal{S}^q\,|0\rangle\,, \quad \mathcal{S} = 2((\ell - \tfrac{1}{2}))^2\mu H + (P^{(\ell-\frac{1}{2})})^2 \tag{7}$$

Namely, $|v_q\rangle$ *satisfies the relations:*

$$\begin{aligned} &D\,|v_q\rangle = (2q-\delta)\,|v_q\rangle\,, \quad M\,|v_q\rangle = -\mu\,|v_q\rangle\,, \\ &X\,|v_q\rangle = 0, \quad {}^\forall X \in \mathfrak{g}^- \end{aligned} \tag{8}$$

Theorem 2 *The irreducible lowest weight modules of* $d = 1$ *ℓ-CGA with the mass central extension are listed as follows* $(\mu \neq 0)$:

- $V^{\delta,\mu}$ *if* $2\delta - 2(q-1) + (\ell + \frac{1}{2})^2 \neq 0$,
- $V^{\delta,\mu}/I^{\delta,\mu}$ *if* $2\delta - 2(q-1) + (\ell + \frac{1}{2})^2 = 0$,

where $I^{\delta,\mu} = U(\mathfrak{g}^+)\,|v_q\rangle$ *and* $q \in \mathbb{Z}_{>0}$. *All modules are infinite dimensional.*

2.2. Kinematically invariant differential equations

In this subsection we derive differential equations invariant under the group generated by $\mathfrak{g}$. Here the *invariance* means the kinematical symmetry in the sense of [3]. This is done by the method developed for real connected semisimple Lie groups [4]. We first summarize the recipe for deriving such differential equations.

Let g be a Lie algebra and $\mathsf{G} = \exp(\mathsf{g})$ be the group generated by g. Suppose that g admits a triangular-like decomposition such as (4): $\mathsf{g} = \mathsf{g}^+ \oplus \mathsf{g}^0 \oplus \mathsf{g}^-$. Then the Lie group G also has the corresponding decomposition: $\mathsf{G} = \mathsf{G}^+\mathsf{G}^0\mathsf{G}^-$ where $\mathsf{G}^\pm = \exp(\mathsf{g}^\pm)$ and $\mathsf{G}^0 = \exp(\mathsf{g}^0)$. Consider a Verma module V^Λ over g with the lowest weight vector $|0\rangle$ and the lowest weight Λ :

$$\begin{aligned} & V^\Lambda = U(\mathsf{g}^+)\,|0\rangle\,, \\ & X\,|0\rangle = 0, \qquad X \in \mathsf{g}^-, \\ & X\,|0\rangle = \Lambda(X)\,|0\rangle\,, \qquad X \in \mathsf{g}^0 \end{aligned} \tag{9}$$

We assume that the Verma module has a singular vector $|v_s\rangle = \mathcal{P}\,|0\rangle\,, \mathcal{P} \in U(\mathsf{g}^+)$.

Consider a C^∞ function on G having the property of *right covariance*:

$$f(gg^0g^-) = e^{\Lambda(X)} f(g), \quad {}^\forall g \in \mathsf{G},\ {}^\forall g^0 = e^X \in \mathsf{G}^0,\ {}^\forall g^- \in \mathsf{G}^- \tag{10}$$

Thus the function $f(g)$ is actually a function on the coset $\mathsf{G}/\mathsf{G}^0\mathsf{G}^-$. Now consider the space C^Λ of right covariant functions on G. We introduce the right action of g on C^Λ by the standard formula:

$$\pi_R(X)f(g) = \frac{d}{d\tau} f(ge^{\tau X})\bigg|_{\tau=0}, \quad X \in \mathsf{g},\ g \in \mathsf{G} \tag{11}$$

Then it is immediate to verify by the right covariance (10) that $f(g)$ has the properties of lowest weight vector:

$$\begin{aligned} & \pi_R(X)f(g) = 0, \qquad X \in \mathsf{g}^- \\ & \pi_R(X)f(g) = \Lambda(X)f(g), \qquad X \in \mathsf{g}^0 \end{aligned} \tag{12}$$

Thus one may have a realization of the Verma module V^Λ in terms of C^Λ.

One may consider the left regular representation of G on C^Λ defined by

$$(T^\Lambda(g)f)(g') = f(g^{-1}g'). \tag{13}$$

We remark that we take the same lowest weights for g and G. Recalling that a singular vector induces an another lowest weight representation with the different lowest weight Λ', it is not difficult to see that the singular vector in the representation π_R gives an intertwining operator of two representations of G :

$$\pi_R(\mathcal{P})T^\Lambda = T^{\Lambda'}\pi_R(\mathcal{P}). \tag{14}$$

As will be seen later $\pi_R(\mathcal{P})$ is, in general, a differential operator. If $\pi_R(\mathcal{P})$ has a kernel, i.e.,

$$\pi_R(\mathcal{P})\psi = 0, \tag{15}$$

then we have a differential equation. Furthermore the solution ψ to the differential equation (15) is transformed to another solution by the left action of G because of the relation (14). Thus equation (15) is the desired equation.

Now we apply the recipe to $\mathfrak{g}$, i.e., $\mathfrak{g}_\ell(1)$ with the mass central extension. Parameterizing an element of $G^+ = \exp(\mathfrak{g}^+)$ as $g^+ = \exp(tH + \sum_{j=0}^{\ell-\frac{1}{2}} x_j P^{(j)})$ the right action of H and $P^{(k)}$ becomes differential operators:

$$\pi_R(H) = \frac{\partial}{\partial t} + \sum_{j=1}^{\ell-\frac{1}{2}} jx_j \frac{\partial}{\partial x_{j-1}}, \qquad \pi_R(P^{(k)}) = \frac{\partial}{\partial x_k}. \tag{16}$$

By (7) and (15) we have proved the following:

Proposition 3 *The Lie group generated by* $\mathfrak{g}$ *is a kinematical symmetry of the following hierarchy of partial differential equations:*

$$\left(a_\ell \mu \Big(\frac{\partial}{\partial t} + \sum_{j=1}^{\ell-\frac{1}{2}} j x_j \frac{\partial}{\partial x_{j-1}} \Big) + \frac{\partial^2}{\partial^2 x_{\ell-\frac{1}{2}}} \right)^q \psi(t, x_i) = 0, \qquad q \in \mathbb{Z}_{>0} \tag{17}$$

where $a_\ell = 2((\ell - \frac{1}{2})!)^2$.

For $\ell = 1/2$ (17) recovers a hierarchy of heat/Schrödinger equations in one space dimension obtained in [5, 7, 8]:

$$\left(2\mu \frac{\partial}{\partial t} + \frac{\partial^2}{\partial x^2} \right)^q \psi(t,x) = 0.$$

The central charge μ is interpreted as (imaginary) mass. For higher values of ℓ we observe an interesting deviation from the heat/Schrödinger equation and the obtained equations are highly nontrivial. As an illustration we show the hierarchy of equations for $\ell = 3/2$ and $5/2$:

$$\left(2\mu \Big(\frac{\partial}{\partial t} + x_1 \frac{\partial}{\partial x_0} \Big) + \frac{\partial^2}{\partial x_1^2} \right)^q \psi(t, x_0, x_1) = 0, \quad \ell = 3/2$$
$$\left(8\mu \Big(\frac{\partial}{\partial t} + x_1 \frac{\partial}{\partial x_0} + 2x_2 \frac{\partial}{\partial x_1} \Big) + \frac{\partial^2}{\partial x_2^2} \right)^q \psi(t, x_0, x_1, x_2) = 0, \quad \ell = 5/2$$

3. Representation theory of $d = 2$ conformal Galilei algebra and invariant equations

In this section we study $\mathfrak{g}_\ell(2)$ with the mass or the exotic central extensions. We give a condition for reducibility of Verma modules and differential equations having kinematical invariance. Here we present our main results without proofs. The proofs and other details are found in the separate publication [18].

3.1. Mass central extension

The algebra $\mathfrak{g}_\ell(2)$ with the mass central extension, we denote it simply by $\hat{\mathfrak{g}}$, has the following generators:

$$D,\ H,\ C,\ M_{12},\ P_i^{(n)}, M, \qquad i,j = 1,2, \quad n = 0,1,\cdots,2\ell$$

where ℓ is a positive half-integer. We make the following changes of generators:

$$P_\pm^{(n)} = P_1^{(n)} \pm i P_2^{(n)}, \qquad J = -iM_{12}, \qquad M = 2M, \tag{18}$$

then their nonvanishing commutators are given by

$$\begin{aligned}
&[D,H] = 2H, \qquad [D,C] = -2C, \qquad [C,H] = D, \\
&[H, P_\pm^{(n)}] = -nP_\pm^{(n-1)}, \qquad [D, P_\pm^{(n)}] = 2(\ell - n) P_\pm^{(n)}, \\
&[C, P_\pm^{(n)}] = (2\ell - n) P_\pm^{(n+1)}, \qquad [J, P_\pm^{(n)}] = \pm P_\pm^{(n)}, \\
&[P_\pm^{(m)}, P_\mp^{(n)}] = \delta_{m+n,2\ell}\, I_m M,
\end{aligned} \tag{19}$$

where I_m is given by (2). We introduce a vector space decomposition of $\hat{\mathfrak{g}}$:

$$\begin{aligned}
\hat{\mathfrak{g}}^+ &= \langle\ H,\ P_\pm^{(n)}\ \rangle, \quad n = 0,1,\cdots,\ell - \frac{1}{2} \\
\hat{\mathfrak{g}}^0 &= \langle\ D,\ J,\ M\ \rangle, \\
\hat{\mathfrak{g}}^- &= \langle\ C,\ P_\pm^{(n)}\ \rangle, \quad n = \ell + \frac{1}{2}, \cdots, 2\ell
\end{aligned} \tag{20}$$

This corresponds to the triangular decomposition of semisimple Lie algebras.

Suppose the existence of the lowest weight vector $|\delta, r, \mu\rangle$ defined by

$$\begin{aligned} D\,|\delta,r,\mu\rangle &= -\delta\,|\delta,r,\mu\rangle\,, \qquad J\,|\delta,r,\mu\rangle = -r\,|\delta,r,\mu\rangle\,, \\ M\,|\delta,r,\mu\rangle &= -\mu\,|\delta,r,\mu\rangle\,, \qquad X\,|\delta,r,\mu\rangle = 0, \ {}^\forall X \in \hat{\mathfrak{g}}^- \end{aligned} \tag{21}$$

One may build a Verma module $V^{\delta,r,\mu}$ on $|\delta,r,\mu\rangle$ whose basis is given by

$$H^k \prod_{n=0}^{\ell-\frac{1}{2}} (P_+^{(n)})^{a_n} (P_-^{(n)})^{b_n}\,|\delta,r,\mu\rangle\,, \tag{22}$$

where k, a_n, b_n $(n = 0, 1, \ldots, \ell - \frac{1}{2})$ are non-negative integers. The Verma module $V^{\delta,r,\mu}$ is not always irreducible as indicated in the following proposition.

Proposition 4 *The Verma module $V^{\delta,r,\mu}$ has a singular vector if $\delta - q + (\ell + \frac{1}{2})^2 + 1 = 0$ for a positive integer q. It is given by*

$$|v_q\rangle = (a_\ell \mu H + P_+^{(\ell-\frac{1}{2})} P_-^{(\ell-\frac{1}{2})})^q\,|\delta,r,\mu\rangle\,, \quad \alpha_\ell = \left(\left(\ell - \frac{1}{2}\right)!\right)^2 \tag{23}$$

and satisfies the relations

$$\begin{aligned} &D\,|v_q\rangle = (2q-\delta)\,|v_q\rangle\,, \quad J\,|v_q\rangle = -r\,|v_q\rangle\,, \quad M\,|v_q\rangle = -\mu\,|v_q\rangle\,, \\ &X\,|v_q\rangle = 0, \quad {}^\forall X \in \hat{\mathfrak{g}}^- \end{aligned} \tag{24}$$

The existence of the singular vectors (24) allows us to derive differential equations such that their kinematical symmetry is given by the Lie group $\exp(\hat{\mathfrak{g}})$ as was done in §2.2. Let us parametrize an element $g \in \exp(\hat{\mathfrak{g}}^+)$ as

$$g = \exp(tH)\exp\left(\sum_{n=0}^{\ell-\frac{1}{2}} (x_n P_+^{(n)} + y_n P_-^{(n)})\right), \tag{25}$$

then the right action of elements in $\hat{\mathfrak{g}}^+$ on a right covariant function $f(g)$ with $g \in \exp(\hat{\mathfrak{g}})$ becomes differential operators:

$$\begin{aligned} \pi_R(H) &= \frac{\partial}{\partial t} + \sum_{n=1}^{\ell-\frac{1}{2}} n\left(x_n \frac{\partial}{\partial x_{n-1}} + y_n \frac{\partial}{\partial y_{n-1}}\right), \\ \pi_R(P_+^{(n)}) &= \frac{\partial}{\partial x_n}, \qquad \pi_R(P_-^{(n)}) = \frac{\partial}{\partial y_n}. \end{aligned} \tag{26}$$

By this and Proposition 4 we have prove the following:

Proposition 5 *The Lie group generated by $\hat{\mathfrak{g}}$ is a kinematical symmetry of the following hierarchy of partial differential equations:*

$$\left[a_\ell \mu \left(\frac{\partial}{\partial t} + \sum_{n=1}^{\ell-\frac{1}{2}} n\left(x_n \frac{\partial}{\partial x_{n-1}} + y_n \frac{\partial}{\partial y_{n-1}}\right)\right) + \frac{\partial^2}{\partial x_{\ell-\frac{1}{2}} \partial y_{\ell-\frac{1}{2}}}\right]^q \psi(t, x_i, y_i) = 0, \tag{27}$$

where $q \in \mathbb{Z}_{>0}$.

For $\ell = 1/2$ (27) yields the following form:

$$\left(\mu \frac{\partial}{\partial t} + \frac{\partial^2}{\partial x \partial y}\right)^q \psi(t,x,y) = 0.$$

By an appropriate change of variables, this recovers a hierarchy of heat/Schrödinger equations in two space dimension obtained in [6, 7, 8].

3.2. Exotic central extension

We denote $\mathfrak{g}_\ell(2)$ with the exotic central extension by $\check{\mathfrak{g}}$. It has the generators

$$D,\ H,\ C,\ M_{12},\ P_i^{(n)}, \Theta, \qquad i,j=1,2, \quad n=0,1,\cdots,2\ell$$

where ℓ is a positive integer. We denote the central element of $\check{\mathfrak{g}}$ by Θ to make clear the difference from the case of $\hat{\mathfrak{g}}$. We redefined the generators as

$$P_\pm^{(n)} = P_1^{(n)} \pm iP_2^{(n)}, \qquad J=-iM_{12}, \qquad \Theta=-2i\Theta, \tag{28}$$

then the nonvanishing commutators of $\check{\mathfrak{g}}$ are identical to (19) except the central extension. The central extension for $\check{\mathfrak{g}}$ in terms of new generators is given by

$$[P_\pm^{(m)}, P_\mp^{(n)}] = \pm\delta_{m+n,2\ell} I_m \Theta \tag{29}$$

with I_m defined in (3).

We make a triangular like decomposition of $\check{\mathfrak{g}}$:

$$\begin{aligned}
&\check{\mathfrak{g}}^+ = \langle\ H,\ P_+^{(\ell)},\ P_\pm^{(n)}\ \rangle, \quad n=0,1,\cdots,\ell-1\\
&\check{\mathfrak{g}}^0 = \langle\ D,\ J,\ \Theta\ \rangle,\\
&\check{\mathfrak{g}}^- = \langle\ C,\ P_-^{(\ell)},\ P_\pm^{(n)}\ \rangle, \quad n=\ell+1,\ell+2,\cdots,2\ell
\end{aligned} \tag{30}$$

The lowest weight vector $|\delta,r,\theta\rangle$ is defined as usual:

$$\begin{aligned}
&D\,|\delta,r,\theta\rangle = -\delta\,|\delta,r,\theta\rangle\,, \qquad J\,|\delta,r,\theta\rangle = -r\,|\delta,r,\theta\rangle\,,\\
&\Theta\,|\delta,r,\theta\rangle = \theta\,|\delta,r,\theta\rangle\,, \qquad X\,|\delta,r,\theta\rangle = 0 \text{ for } {}^\forall X\in\check{\mathfrak{g}}^-.
\end{aligned} \tag{31}$$

We construct a Verma module $V^{\delta,r,\theta}$ over $\check{\mathfrak{g}}$ by repeated applications of an element of $\check{\mathfrak{g}}^+$ on $|\delta,r,\theta\rangle$. A basis of $V^{\delta,r,\theta}$ is given by

$$H^h (P_+^{(\ell)})^{a_\ell} \prod_{n=0}^{\ell-1} (P_+^{(n)})^{a_n} (P_-^{(n)})^{b_n}\,|\delta,r,\mu\rangle\,, \tag{32}$$

where h, a_n $(n=0,1,\ldots,\ell), b_n$ $(n=0,1,\ldots,\ell-1)$ are non-negative integers. The Verma module $V^{\delta,r,\theta}$ is not always irreducible as $\mathfrak{g}$ or $\hat{\mathfrak{g}}$.

Proposition 6 *The Verma module $V^{\delta,r,\theta}$ has a singular vector if $\delta-q+\ell(\ell+1)+1=0$ for a positive integer q. It is given by*

$$|v_q\rangle = (\alpha_\ell\theta H + (-1)^\ell P_-^{(\ell-1)}P_+^{(\ell)})^q\,|\delta,r,\theta\rangle\,, \quad \alpha_\ell = \ell!(\ell-1)! \tag{33}$$

and satisfies the relations

$$\begin{aligned}
&D\,|v_q\rangle = (2q-\delta)\,|v_q\rangle\,, \quad J\,|v_q\rangle = -r\,|v_q\rangle\,, \quad \Theta\,|v_q\rangle = \theta\,|v_q\rangle\,,\\
&X\,|v_q\rangle = 0, \quad {}^\forall X\in\check{\mathfrak{g}}^-
\end{aligned} \tag{34}$$

This generalize the result for $\ell=1$ investigated in [16].

Now let us derive differential equations having kinematical symmetries generated by $\check{\mathfrak{g}}$. We parametrize an element $g\in\exp(\check{\mathfrak{g}}^+)$ as

$$g = e^{tH}\exp\left(\sum_{n=0}^{\ell-1}(x_nP_+^{(n)} + y_nP_-^{(n)}) + x_\ell P_+^{(\ell)}\right). \tag{35}$$

Then the right action of elements in $\check{\mathfrak{g}}^+$ on a right covariant function $f(g)$ with $g \in \exp(\check{\mathfrak{g}})$ becomes differential operators:

$$\begin{aligned}
\pi_R(H) &= \frac{\partial}{\partial t} + \sum_{n=0}^{\ell-1}(n+1)x_{n+1}\frac{\partial}{\partial x_n} + \sum_{n=0}^{\ell-2}(n+1)y_{n+1}\frac{\partial}{\partial y_n}, \\
\pi_R(P_+^{(n)}) &= \frac{\partial}{\partial x_n}, \qquad \pi_R(P_-^{(n)}) = \frac{\partial}{\partial y_n}.
\end{aligned} \tag{36}$$

By this and Proposition 6 we have prove the following:

Proposition 7 *The Lie group generated by $\hat{\mathfrak{g}}$ is a kinematical symmetry of the following hierarchy of partial differential equations:*

$$\left[\alpha_\ell \theta \left(\frac{\partial}{\partial t} + \sum_{n=1}^{\ell} n x_n \frac{\partial}{\partial x_{n-1}} + \sum_{n=1}^{\ell-1} n y_n \frac{\partial}{\partial y_{n-1}} \right) + (-1)^\ell \frac{\partial^2}{\partial y_{\ell-1}\partial x_\ell}\right]^q \psi = 0, \tag{37}$$

where $q \in \mathbb{Z}_{>0}$.

4. Concluding remarks

We have studied the lowest weight representations of ℓ-conformal Galilei algebras with central extensions for $d = 1, 2$. For $d = 1$ algebras we gave a classification of all irreducible lowest weight modules, however, the result for $d = 2$ algebras is partial in the sense that a full classification of irreducible modules has not been done yet. In fact, our knowledge on irreducible modules of $\mathfrak{g}_\ell(d)$ is very limited. We summarize in Table 1 the works on lowest (highest) weight representations of $\mathfrak{g}_\ell(d)$ have been done so far. In Table 1 'full classification' lists the algebras for which a classification of irreducible lowest (highest) weight modules has been given. On the other hand 'partial result' means that it has been shown that some Verma modules are reducible because of the existence of singular vectors, but reducibility of other Verma modules are still open. The algebras are indicated by a pair (d, ℓ). Recalling that d takes any positive integers and ℓ takes any spin values, one can see that, what has been done so far is very little compared with the whole family of $\mathfrak{g}_\ell(d)$. There are many things to be done in order to achieve a complete classification of irreducible modules of $\mathfrak{g}_\ell(d)$. However the method of the classification employed in [5, 6, 16, 17], which is useful for lower values of d, will not be efficient for higher values of d. We need to develop an alternate method.

Table 1. Works on irreducible modules of $\mathfrak{g}_\ell(d)$.

full classification	partial result
(1,any half-integer)	$(2,\ell \geq \frac{3}{2})$
$(2,\frac{1}{2})$	$(d \geq 4, \frac{1}{2})$
(2,1)	
$(3,\frac{1}{2})$	

In the present work we also discuss differential equations having kinematical symmetries generated by $\mathfrak{g}_\ell(d)$. The differential equations for $\ell = 1/2$ are a hierarchy of free Schrödinger equations. However physical implication of other equations is still an open problem. Because of the method of construction, we obtained linear differential equations having kinematical symmetries. It may be interesting to consider differential equations having full Lie symmetries generated by $\mathfrak{g}_\ell(d)$ along the line of [19, 20, 21, 22, 23].

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