

Single and collective regimes in three-level systems interacting with a one-mode electromagnetic field

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Abstract. A semiclassical analysis is presented to determine the quantum phase transition from single to collective regimes in three-level atoms in the presence of a radiation field. The energy surfaces of the Ξ , V , and Λ configurations are constructed by taking the expectation value with respect to $U(3)$ coherent states that carry the totally symmetric representation, determined by the total number of atoms. The corresponding stability and equilibrium properties are calculated by means of the catastrophe theory, discovering the bifurcation and Maxwell sets. We establish that the atoms with Λ and Ξ configurations have the presence of double points, i.e., there are two independent quantum states with the same energy that can be obtained depending on the values of the dipolar strengths of the interaction between the atoms and the radiation field. Additionally, the Ξ configuration exhibits a fixed triple point.

1. Introduction

The theory of quantum electrodynamics (QED) is used to study the interaction of atomic or molecular matter with a radiation field [1]. It requires a quantum treatment of both the particles and the field. QED is one of the most successful theories of physics, as is evident from the examples of the Lamb shift, the anomalous magnetic moment of the electron, and the electrodynamic relative displacement of the ground state of the positronium. Traditionally, the Hamiltonian related to a system of non-interacting atoms in an electromagnetic field is formed by three parts:

$$H = H_a + H_f + H_{af}, \quad (1)$$

where the interaction term H_{af} is responsible for the exchange of energy between the radiation field and the atomic sector. When the radiation field is considered classically and fixed externally, a semiclassical method has been developed in which only the atomic and coupling parts are described within the quantum mechanics formalism [2]. In this contribution, we take into account the complete Hamiltonian within the quantum formalism.

The radiation field is usually described in terms of an infinite set of harmonic oscillators, one for each mode of radiation. The amplitude of the electric field associated to a mode of frequency ω is given by $E_0 = \sqrt{\hbar\omega/(2\epsilon_0 V)}$ with ϵ_0 the permittivity of free space, and V the quantization volume. Each atom is considered as a two-level system, and its coupling to the radiation is described by the Rabi frequency $\Omega_{af} = d_{ge}E_0/\hbar$. Here d_{ge} denotes the matrix element of the electric dipole operator of the atom between the ground and excited states. In free space the



probability of photon emission per unit time is proportional to the Rabi frequency and to the number of modes per unit of frequency interval [3, 4], that is,

$$\Gamma_0 = \frac{\omega^3}{3\pi\hbar c^3} \frac{|d_{ge}|^2}{\epsilon_0}. \quad (2)$$

The probability of survival of an atom in its excited state is given by the Fermi rule $P_e(t) = e^{-\Gamma_0 t}$. This exponential law describes an irreversible decay process of the atom to its ground state, g . However, this can be changed dramatically in a high-Q cavity regime, like that of emission of radiation between mirrors. The radiation rate in a cavity of volume V is given by [3]

$$\Gamma_{cav} = \Gamma_0 \frac{Q\lambda^3}{V}, \quad (3)$$

where Q denotes the quality factor of the cavity (which measures the lifetime of a photon in the cavity), and λ is the wavelength of the radiation field. For some high-Q cavities, one can achieve storage times of a photon running from a few milliseconds to a fraction of a second. Therefore, we see that the rate of emission in free space can be increased by the ratio $\frac{Q\lambda^3}{V}$, an advantage which in [5] was exploited to enhance Γ_0 by a factor of approximately 500. This type of cavity leads to a reversible atom-field evolution [6].

A review of the dynamics of two- and three-level systems interacting with quantized cavity electromagnetic fields is given in [7]. The Jaynes-Cummings model is the coupling of a two-level system with a harmonic oscillator and it corresponds to the typical situation of QED in a cavity. This model can also describe an ion in a trap when two ionic states are coupled by laser beams with those states of the oscillator in the trap. The model was introduced in 1963 by Jaynes and Cummings as an idealization of the matter-field coupling in free space [8]. Many theoretical predictions of this model, such as the existence of collapse and revivals in the Rabi oscillations [9], the formation of macroscopic quantum states, the many experimental studies of Rydberg atoms with very large principal quantum number within single-mode cavities, and measures of entanglement associated with spin-squeezed states, have been confirmed [10].

A system of N_a non-interacting two-level atoms coupled to a quantized electromagnetic field, using the dipolar and the rotating wave approximations, was described by the Tavis-Cummings Model [11, 12], having since then an extensive use in quantum optics [10]. The investigation of the phase transitions of the system has been carried out in the thermodynamic limit [13, 14], and at zero temperature [15, 16, 17]. The quantum description of a system of N_a non-interacting atoms under the action of a one-mode electromagnetic field allowed Dicke to predict the superradiance phenomenon [18]. The Dicke model does not consider the rotating wave approximation and it has been physically realized using a QED cavity with Bose-Einstein condensates [19, 20]. The localization of the quantum phase transitions in the dipolar strength parameters was studied in [21, 22, 23], and its relation with the position of extremal values for the Rényi-Wehrl entropies in [24]. The system of three-level atoms in the Ξ and Λ configurations interacting with a one mode radiation field together with a dipole-dipole interaction between the atoms has been used to define a new concept about atomic squeezing [25, 26]. Spin variances for the V and Λ configurations of an ensemble of atoms interacting with two light fields, a coherent pump state and a squeezed vacuum as a probe, have been calculated by means of Langevin equations derived from the Bloch equations [27]. By means of the Holstein-Primakoff transformation, normal and superradiant stable states for the Λ configuration have been identified in the thermodynamic limit [28].

More recently, we obtained in the rotating wave approximation (RWA), analytical expressions of the quantum phase transitions and their order from the single to collective regimes for three-level atoms interacting with a one-mode electromagnetic field for the Ξ , Λ , and V

configurations [29]. These transitions appear reflected in the ground state energy surface \mathcal{E}^c and in their corresponding total number of excitations \mathcal{M}^c , when plotted as functions of the dipolar coupling constants (control parameters).

In this contribution we consider three-level atoms in a one-mode electromagnetic cavity in the three possible atomic configurations Ξ , Λ , and V . We review thoroughly the procedure to define the separatrix of the system in terms of the bifurcation and Maxwell sets. This includes the use of good variational states associated to the coherent states of $U(3)$ to describe the matter part, together with the Heisenberg-Weyl coherent states for the description of the field. The separatrix divides the single and collective regimes of the system, which can be shown through the analytic expression for the vector states together with the corresponding statistical properties of light and matter. We establish the presence of double points for the Ξ , and Λ atomic configurations; for the Ξ case we additionally find the existence of a fixed triple point. This is independent of the number of atoms of the system, and depends only on the values of the dipole coupling strengths between the three-level states.

2. Model Hamiltonian

The one atom Hamiltonian can be written in terms of a three dimensional complete set of states as follows

$$H_a = \sum_{k,j=1}^3 (H_a)_{kj} |k\rangle\langle j| = \sum_{j=1}^3 E_j |j\rangle\langle j|, \quad (4)$$

where in the last expression, we use the fact that the matrix element $(H_a)_{kj}$ of the Hamiltonian satisfies $(H_a)_{kj} = E_j \delta_{jk}$. This means that the set of states $\{|k\rangle\}$ with $k = 1, 2, 3$, are eigenstates of the atom.

The one-mode electromagnetic field of frequency Ω is denoted by the Hamiltonian of a harmonic oscillator $H_f = \hbar\Omega a^\dagger a$, and the coupling interaction is given by $H_{af} = -\vec{d} \cdot \vec{E}$ with $\vec{d} = e\vec{r}$ denoting the dipole moment operator and \vec{E} the external quantum electric field at the position of the atom. In the long wavelength approximation the coupling interaction term is given by

$$H_{af} = -\frac{1}{2} \sum_{j \neq k=1}^3 \mu_{jk} (|j\rangle\langle k| + |k\rangle\langle j|)(a + a^\dagger), \quad (5)$$

where we have defined $\mu_{jk} \equiv \sqrt{\frac{2\pi\hbar\Omega}{V}} \sum_{\beta} (\vec{E}_0 \cdot \vec{e}_{\beta} (x_{\beta})_{jk})$. Besides, we have used the fact that the parameters μ_{jk} , which are proportional to the dipolar strengths, are only different from zero when $j \neq k$. We take, without loss of generality, $\mu_{jk} = \mu_{kj}$.

Therefore the one three-level atom Hamiltonian interacting with a one-mode radiation field can be written in the form

$$H = \Omega a^\dagger a + \sum_{j=1}^3 \omega_j A_{jj}^{(1)} - \frac{1}{2} \sum_{j \neq k=1}^3 \mu_{jk} (A_{jk}^{(1)} + A_{kj}^{(1)})(a + a^\dagger), \quad (6)$$

where we have defined $A_{ij}^{(1)} = |i\rangle\langle j|$ and it is easy to prove that these operators satisfy the commutation relations of a unitary group in 3 dimensions, $U(3)$. We use from here onwards $E_j = \hbar\omega_j$ and units where $\hbar = 1$. This Hamiltonian presents the phenomenon of collapses and revivals in the probabilities of finding the atom in a determined energy level in a dynamic evolution of an arbitrary state, for a sufficiently large average number of photons in the cavity ($\nu \approx 10$).

The extension of the problem to consider N_a three-level atoms is straightforward through the replacement of the matter operators as follows: $A_{jk} = \sum_s A_{jk}^{(s)}$, where $A_{jk}^{(s)}$ denotes the

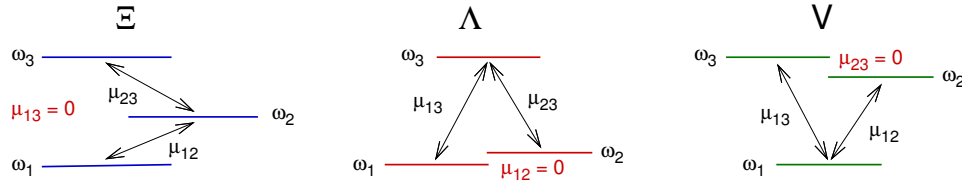


Figure 1. Atomic configurations and dipolar coupling parameters.

atomic operator of the s -th atom and satisfies again the commutation relations of a $U(3)$ group. By means of the rotating wave approximation (RWA), that is, neglecting the terms in the Hamiltonian that do not conserve the total number of excitations, we arrive to the expression of the Hamiltonian model that is going to be used to determine the region in parameter space in which there is a sudden change in the properties of the ground state of the system,

$$H = \Omega a^\dagger a + \omega_1 A_{11} + \omega_2 A_{22} + \omega_3 A_{33} - \frac{\mu_{12}}{\sqrt{N_a}} (A_{12} a^\dagger + A_{21} a) - \frac{\mu_{13}}{\sqrt{N_a}} (A_{13} a^\dagger + A_{31} a) - \frac{\mu_{23}}{\sqrt{N_a}} (A_{23} a^\dagger + A_{32} a). \quad (7)$$

We have included the factor $\sqrt{N_a}$ in the expression of the dipolar strengths to replace the quantization volume by its corresponding density of atoms. Additionally we are considering that the energy levels obey the frequency inequalities $\omega_1 \leq \omega_2 \leq \omega_3$, in all of the atomic configurations mentioned above. It can be easily proved that the total number of excitations operator given by

$$M_\Xi = a^\dagger a + A_{22} + 2A_{33}, \quad M_\Lambda = a^\dagger a + A_{33}, \quad M_V = a^\dagger a + A_{22} + A_{33}, \quad (8)$$

commutes with respect to the corresponding Hamiltonian. In Fig.(1) we show that: for the Ξ atomic configuration one takes $\mu_{13} = 0$; for Λ atoms we consider $\mu_{12} = 0$; and for the V case, $\mu_{23} = 0$. The condition $\mu_{ij} = 0$ means that the transition from the atomic state j to the state i is forbidden.

3. Energy surface

The energy surface is defined by means of the expectation value of the Hamiltonian with respect to a variational test function. The structure of the Hamiltonian (7) leads us to choose the Heisenberg-Weyl coherent states as a good starting point for the field part and the $U(3)$ coherent states for the matter. Major reasons for using these variational states are the facts that one can obtain analytical expressions for the expectation values of all the matter and field observables, and that they form a basis for the complete Hilbert space of the system in terms of four complex parameters: one to describe the field, α , and three associated to the matter part, γ_k , with $k = 1, 2, 3$. In this work we are going to consider only totally symmetric representations of the $U(3)$ group; then the corresponding coherent states depend only on two complex parameters γ_2 and γ_3 . Therefore, the variational state constructed from the tensor product of matter and field components can be written in the form [29]

$$\begin{aligned} |\alpha; N_a \gamma_2, \gamma_3\rangle &= e^{-|\alpha|^2/2} e^{\alpha a^\dagger} |0\rangle_F \otimes \frac{(b_1^\dagger + \gamma_3 b_2^\dagger + \gamma_2 b_3^\dagger)^{N_a}}{\sqrt{N_a!} (1 + |\gamma_2|^2 + |\gamma_3|^2)^{N_a/2}} |0, 0, 0\rangle_F, \\ &= \frac{\sqrt{N_a!} e^{-|\alpha|^2/2}}{(1 + |\gamma_2|^2 + |\gamma_3|^2)^{N_a/2}} \sum_{\nu=0}^{\infty} \sum_{q=0}^{N_a} \sum_{r=0}^q \frac{\alpha^\nu \gamma_2^{N_a-q} \gamma_3^{q-r}}{\sqrt{\nu!} (N_a - q)! (q - r)! r!} |\nu\rangle_F |N_a, q, r\rangle_{GT}. \end{aligned} \quad (9)$$

In the last expression, we expand the variational state in terms of the Fock states for the field, $|\nu\rangle_F$ and the Gelfand-Tsetlin basis [30, 31] for the matter, $|N_a, q, r\rangle_{GT}$. Notice that $a|0\rangle = 0$ defines the vacuum state of the field and $|\alpha|^2$ measures the average number of photons. We defined $\Gamma^\dagger \equiv b_1^\dagger + \gamma_3 b_2^\dagger + \gamma_2 b_3^\dagger$ as a combination of three boson creation operators, this operator was normalized by asking that $[\Gamma, \Gamma^\dagger] = 1$, with $\Gamma \equiv b_1 + \gamma_3^* b_2 + \gamma_2^* b_3$.

For the totally symmetric representation, the $U(3)$ generators can be realized in the form $A_{ij} = b_i^\dagger b_j$. It is straightforward to check that they satisfy the appropriate commutation relations and that the action of the weight generators A_{ii} on the GT basis is given by

$$\begin{aligned} A_{11}|N_a, q, r\rangle_{GT} &= r |N_a, q, r\rangle_{GT}, \\ A_{22}|N_a, q, r\rangle_{GT} &= (q - r) |N_a, q, r\rangle_{GT}, \\ A_{33}|N_a, q, r\rangle_{GT} &= (N_a - q) |N_a, q, r\rangle_{GT}, \end{aligned} \quad (10)$$

while for the lowering generators one has that

$$\begin{aligned} A_{12}|N_a, q, r\rangle_{GT} &= \sqrt{(q - r)(r + 1)} |N_a, q, r + 1\rangle_{GT}, \\ A_{13}|N_a, q, r\rangle_{GT} &= \sqrt{(N_a - q)(r + 1)} |N_a, q + 1, r + 1\rangle_{GT}, \\ A_{23}|N_a, q, r\rangle_{GT} &= \sqrt{(N_a - q)(q - r + 1)} |N_a, q + 1, r\rangle_{GT}, \end{aligned} \quad (11)$$

where it is easy to check that the weight of the states in the right hand side is lower than that of the states on the left hand side. The action of the raising generators can be obtained from the previous results.

Because it is convenient to have an intensive quantity that describes the energy per particle, in units of the photon energy, the expectation value of (7) with respect to (9) is given by

$$\mathcal{E}^c(\alpha, N_a, \gamma_2, \gamma_3) = \frac{1}{N_a \Omega} \langle \alpha; N_a, \gamma_2, \gamma_3 | H | \alpha; N_a, \gamma_2, \gamma_3 \rangle. \quad (12)$$

If we rewrite the complex parameters in their polar form, i.e., $\alpha = \rho e^{i\phi}$ and $\gamma_j = \varrho_j e^{i\varphi_j}$ with $j = 2, 3$, one finds that the energy surface only depends on the angles $\vartheta_3 = \phi - \varphi_3$, $\vartheta_2 = \phi - \varphi_2$ and $\vartheta_1 = \phi - \varphi_2 + \varphi_3$. Minimizing \mathcal{E}^c with respect these phases, the critical values are given by $\vartheta_{kc} = 0, \pi$ with $k = 1, 2, 3$, and it is not difficult to prove that a minimum is always obtained if the energy surface is rewritten as

$$\mathcal{E}^c = r^2 + \frac{\bar{\omega}_1 + \bar{\omega}_2 \varrho_3^2 + \bar{\omega}_3 \varrho_2^2}{1 + \varrho_2^2 + \varrho_3^2} - 2r \frac{\bar{\mu}_{12} \varrho_3 + \bar{\mu}_{13} \varrho_2 + \bar{\mu}_{23} \varrho_2 \varrho_3}{1 + \varrho_2^2 + \varrho_3^2}. \quad (13)$$

where we have made the following definitions: $r \equiv \rho/\sqrt{N_a}$, $\bar{\omega}_k \equiv \omega_k/\Omega$ with $k = 1, 2, 3$, and $\bar{\mu}_{ij} \equiv |\mu_{ij}|/\Omega$. The absolute values for the dipolar strengths appear because the energy surface is invariant under the interchange $\mu_{ij} \rightarrow -\mu_{ij}$.

The critical points are determined by

$$\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \rho_2}, \frac{\partial}{\partial \rho_3} \right) \mathcal{E}^c = \vec{0}. \quad (14)$$

and the stability properties by its corresponding Hessian matrix,

$$\mathcal{H} = \begin{pmatrix} \frac{\partial^2 \mathcal{E}^c}{\partial r^2} & \frac{\partial^2 \mathcal{E}^c}{\partial r \partial \rho_2} & \frac{\partial^2 \mathcal{E}^c}{\partial r \partial \rho_3} \\ \frac{\partial^2 \mathcal{E}^c}{\partial \rho_2 \partial r} & \frac{\partial^2 \mathcal{E}^c}{\partial \rho_2^2} & \frac{\partial^2 \mathcal{E}^c}{\partial \rho_2 \partial \rho_3} \\ \frac{\partial^2 \mathcal{E}^c}{\partial \rho_3 \partial r} & \frac{\partial^2 \mathcal{E}^c}{\partial \rho_3 \partial \rho_2} & \frac{\partial^2 \mathcal{E}^c}{\partial \rho_3^2} \end{pmatrix}_{(r, \rho_2, \rho_3) \rightarrow (r_c, \rho_{2c}, \rho_{3c})},$$

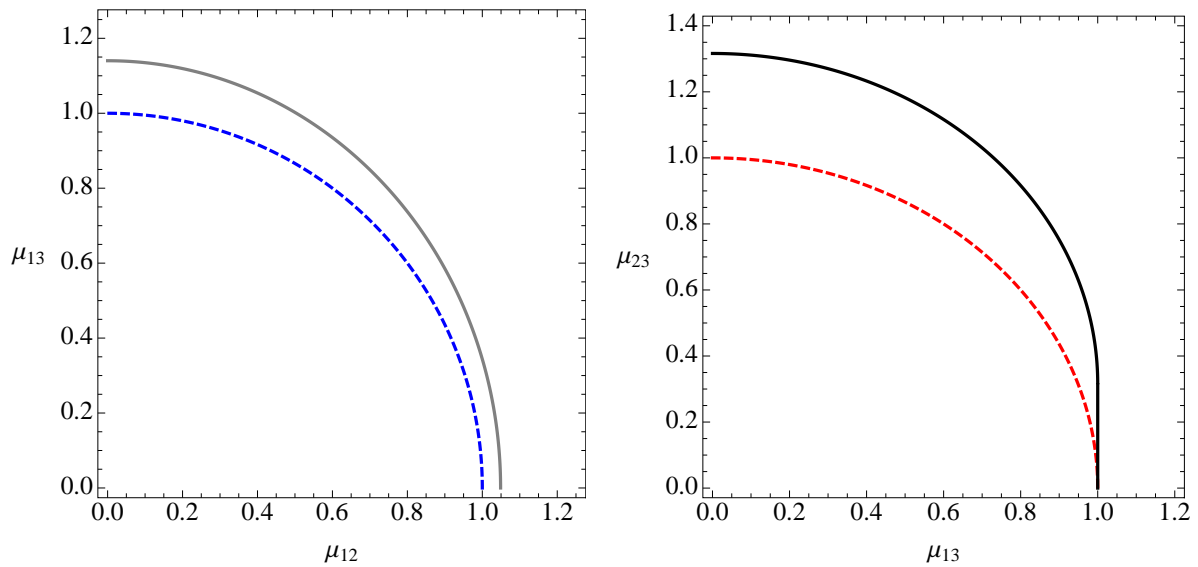


Figure 2. (Color online.) At left the bifurcation set of the V atomic configuration is shown by a continuous line for the frequencies $\bar{\omega}_{31} = 1.3$ and $\bar{\omega}_{21} = 1.1$ and with a dashed line for the double resonance case, $\bar{\omega}_{31} = 1$ and $\bar{\omega}_{21} = 1$. At right the separatrix of the Λ case is shown for the single resonance case $\bar{\omega}_{21} = 0.1$ by a continuous line, and with a dashed line when in double resonance $\bar{\omega}_{21} = 0$.

where r_c , ϱ_{2c} and ϱ_{3c} stand for the critical values of r , ϱ_2 and ϱ_3 , respectively.

From the expression (14) it is immediate that the variable r takes the critical value

$$r_c = \frac{\bar{\mu}_{12} \varrho_{3c} + \bar{\mu}_{13} \varrho_{2c} + \bar{\mu}_{23} \varrho_{2c} \varrho_{3c}}{1 + \varrho_{2c}^2 + \varrho_{3c}^2}. \quad (15)$$

In general we are not able to have analytic solutions for ϱ_{2c} and ϱ_{3c} ; however, the values $\varrho_{2c} = \varrho_{3c} = 0$ are critical points for all values of the parameters of the Hamiltonian. It is then possible to determine the eigenvalues of the Hessian matrix for each one of the atomic configurations and conclude that there are bifurcations sets for each case [32]. They are given by the locus of points

$$(\bar{\mu}_{12})_{\Xi} = \sqrt{\bar{\omega}_{21}}, \quad (\bar{\mu}_{13})_{\Lambda} = \sqrt{\bar{\omega}_{31}}, \quad \left(\frac{\bar{\mu}_{12}^2}{\bar{\omega}_{21}} + \frac{\bar{\mu}_{13}^2}{\bar{\omega}_{31}} \right)_V = 1, \quad (16)$$

where we have used $\bar{\omega}_{ij} = \bar{\omega}_i - \bar{\omega}_j$ to denote the energy shifts between the atomic levels i and j . For these sets the determinant of the Hessian is zero, and we note that the bifurcation sets for the Ξ and Λ configurations are independent of the value of $\bar{\mu}_{23}$. In all the bifurcation sets the energy of the system is given by $\mathcal{E}^c = \bar{\omega}_1$, and there are two degenerate critical points. These sets separate the control parameter space into a region where the total number of excitations and the average number of photons are zero, and a region where they are different from zero. The bifurcation set for the V configuration is displayed in Fig. 2 for the cases with detuning $\Delta_{21} = 0.1$ and $\Delta_{31} = 0.3$, and without detuning ($\Delta_{ij} \equiv \bar{\omega}_{ij} - 1$).

Recently [29] we found the minimum energy surface E^c as a function of the control parameters μ_{ij} . It changes its value from $E^c = 0$ (taking $\bar{\omega}_1 = 0$) to $E^c < 0$, when a transition from $M^c = 0$ (normal regime) to $M^c > 0$ (collective regime) in the total number of excitations takes place.

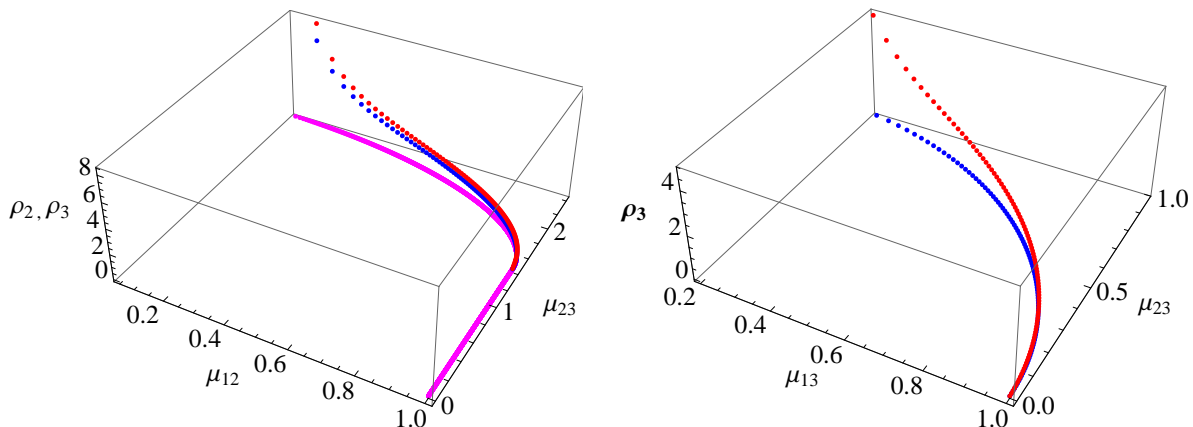


Figure 3. (Color online.) The separatrix of the three-level system is given together with the minima ρ_{2c} and ρ_{3c} as functions of the dipolar strengths, in the double resonance case. At left: the Ξ atomic configuration. At right; the Λ atomic configuration. In this last case the variable $\rho_{2c} = 0$. In both cases, the minima points yield an energy value $\mathcal{E}^c = 0$.

This leads to the existence of a separatrix in parameter space, for which we were able to propose the following ansatz:

$$\begin{aligned}
 \bar{\omega}_{21} &= \bar{\mu}_{12}^2 + [\bar{\mu}_{23} - \sqrt{\bar{\omega}_{31}}]^2 \Theta [\bar{\mu}_{23} - \sqrt{\bar{\omega}_{31}}], & (\Xi \text{ configuration}), \\
 \bar{\omega}_{31} &= \bar{\mu}_{13}^2 + [\bar{\mu}_{23} - \sqrt{\bar{\omega}_{21}}]^2 \Theta [\bar{\mu}_{23} - \sqrt{\bar{\omega}_{21}}], & (\Lambda \text{ configuration}), \\
 1 &= \frac{\bar{\mu}_{12}^2}{\bar{\omega}_{21}} + \frac{\bar{\mu}_{13}^2}{\bar{\omega}_{31}}, & (V \text{ configuration}).
 \end{aligned} \tag{17}$$

Here, $\Theta[x]$ stands for the Heaviside theta function. From these expressions one immediately sees that for values of the dipolar strengths $\bar{\mu}_{23} < \sqrt{\bar{\omega}_{31}}$ (Ξ config.) and $\bar{\mu}_{23} < \sqrt{\bar{\omega}_{21}}$ (Λ config.) one recovers Eq. (16). The separatrix establishes two regions in parameter space: in the internal region the atoms manifest a single state behavior with a total number of excitations equal to zero, while in the external one they have a multicomponent eigenstate with a total number of excitations different from zero.

3.1. Ξ atomic configuration

The Ξ configuration forbids the transition $\omega_1 \longleftrightarrow \omega_3$, and this is introduced in the Hamiltonian by taking $\mu_{13} = 0$. Also the condition $\omega_2 \approx \omega_3/2$ is fulfilled, and the detuning should satisfy $\Delta_{21} \approx \Delta_{32}$ with $|\Delta_{ij}|$ small in order to maintain the RWA. The minima points of the energy surface for the Ξ configuration can be obtained numerically as functions of $\bar{\mu}_{12}$ and $\bar{\mu}_{23}$. They are displayed in Fig. 3 together with the separatrix. For values of $\bar{\mu}_{23} > \sqrt{2}$, on the locus of points of the separatrix, there are two values of the variables with the same minimum energy: $\varrho_{2c} = \varrho_{3c} = 0$, and $\varrho_{2c} \neq 0$ and $\varrho_{3c} \neq 0$. They have different values of the total number of excitations. This shows the existence of a double point, that is, two independent variational states with the same energy but different total number of excitations and average number of photons. The value $(\bar{\mu}_{12}, \bar{\mu}_{23}) = (1, \sqrt{2})$ defines a triple point in parameter space, which is fixed in the sense that it does not move when the number of particles is changed.

3.2. Λ atomic configuration

For atoms in the Λ configuration it is required that the transition $\omega_1 \longleftrightarrow \omega_2$ be forbidden, and so we take $\mu_{12} = 0$. Because of the convention $\omega_1 \leq \omega_2 \leq \omega_3$ used in the labeling of the energy levels, the condition $\omega_1 \approx \omega_2$ requires $\Delta_{31} - \Delta_{32} \approx 0$ with $\Delta_{31} \geq \Delta_{32}$. The minima points can be obtained analytically for the double resonant case $\Delta_{31} = \Delta_{32} = 0$. These are given by $\varrho_{2c} = \varrho_{3c} = 0$ in the normal regime, with $\bar{\mu}_{13}^2 + \bar{\mu}_{23}^2 \leq 1$; while in the collective regime we have

$$\varrho_{2c} = \frac{1}{\bar{\mu}_{13}} \sqrt{\frac{(\bar{\mu}_{13}^2 + \bar{\mu}_{23}^2)(\bar{\mu}_{13}^2 + \bar{\mu}_{23}^2 - 1)}{\bar{\mu}_{13}^2 + \bar{\mu}_{23}^2 + 1}}, \quad \varrho_{3c} = \frac{\bar{\mu}_{23}}{\bar{\mu}_{13}}, \quad (18)$$

where basis states with $\mathcal{M} > 0$ contribute to the ground state. In Fig. 3 the minima are plotted along the separatrix, for which $\varrho_{2c} = 0$ and $\varrho_{3c} = \frac{\sqrt{1-\bar{\mu}_{13}^2}}{\bar{\mu}_{13}}$. This separatrix is displayed in Fig. 2 (right) for the single (one detuning parameter different from zero) and double resonance (both detuning parameters equal to zero) cases.

3.3. V atomic configuration

A system of atoms in the V configuration requires $\mu_{23} = 0$, since transitions between the levels ω_3 and ω_2 are forbidden. Notice that the condition $\omega_2 \approx \omega_3$ on $\omega_1 \leq \omega_2 \leq \omega_3$ reads, in terms of the detuning, as $\Delta_{21} \approx \Delta_{31}$ but satisfying $\Delta_{21} \leq \Delta_{31}$.

In a similar way to the Λ configuration, when the detuning parameters are equal, $\Delta_{21} = \Delta_{31}$, the problem has an analytic solution. The critical points are $\varrho_{2c} = \varrho_{3c} = 0$ for the normal regime implying an energy surface for the ground state equal to zero. For the collective regime, ϱ_{2c} and ϱ_{3c} take the values

$$\varrho_{2c} = \bar{\mu}_{13} \sqrt{\frac{\bar{\mu}_{12}^2 + \bar{\mu}_{13}^2 - 1}{(\bar{\mu}_{12}^2 + \bar{\mu}_{13}^2)(\bar{\mu}_{12}^2 + \bar{\mu}_{13}^2 + 1)}}, \quad \varrho_{3c} = \bar{\mu}_{12} \sqrt{\frac{\bar{\mu}_{12}^2 + \bar{\mu}_{13}^2 - 1}{(\bar{\mu}_{12}^2 + \bar{\mu}_{13}^2)(\bar{\mu}_{12}^2 + \bar{\mu}_{13}^2 + 1)}}. \quad (19)$$

Notice that on the separatrix the critical points coalesce to the values $\rho_c = \rho_2 = \rho_3 = 0$, and all the expectation values of the matter and field observables are given in terms of those values.

4. Quantum solution

The Hamiltonian (7) can be diagonalized in terms of the Fock states $|\nu\rangle$ for the field part and the Gelfand-Tsetlin (GT) states for the matter part

$$|\nu; h_1 h_2 h_3; q_1 q_2; r\rangle = |\nu\rangle \otimes \left| \begin{array}{ccc} h_1 & h_2 & h_3 \\ & q_1 & q_2 \\ & & r \end{array} \right\rangle, \quad (20)$$

where the labels of each row of the GT states denote the irreducible representations (irreps) of the canonical chain of unitary groups $U(3) \supset U(2) \supset U(1)$. The quantum labels of $U(3)$ satisfy the inequalities $h_1 \geq h_2 \geq h_3 \geq 0$; the ones for $U(2)$ satisfy $q_1 \geq q_2 \geq 0$; and for $U(1)$ $r \geq 0$. The branching rules [30] are the following: $h_2 \leq q_1 \leq h_1$, $h_3 \leq q_2 \leq h_2$, and $q_2 \leq r \leq q_1$. The irreps of $U(3)$ are determined by means of the different partitions of the total number of atoms N_a in terms of h_1 , h_2 , and h_3 that satisfy the previous rules. As an example one may consider the three atoms case $N_a = 3$; the $U(3)$ irreps are $[3, 0, 0]$, $[2, 1, 0]$, and $[1, 1, 1]$. These partitions of $N_a = 3$ also indicate the symmetry of the particles under the action of the permutation group S_3 . The first one is symmetric under the interchange of the atoms, and the last one is antisymmetric. The middle irrep has a mixed symmetry.

In this work, we want to consider only the symmetric case and then we set $h_2 = h_3 = q_2 = 0$. Therefore, the exact numerical calculation of the ground state energy may be evaluated using $h_1 = N_a$ and we can simplify the GT states notation as

$$|h_1 0 0; q_1 0; r\rangle_{GT} \equiv |N_a q_1 r\rangle_{GT}. \quad (21)$$

The corresponding matrix elements of the operators A_{ij} (for this particular basis) are easily calculated in the bosonic realization $A_{ij} = b_i^\dagger b_j$, which can be used to evaluate the matrix elements of the Hamiltonian and to obtain numerically its eigenvalues.

For each particular atomic configuration (Ξ , Λ or V) there is an additional constant of motion M , namely the total number of excitations M_Ξ , M_Λ , and M_V given by (8). These can be gathered in a single equation as

$$\hat{M} = a^\dagger a + \lambda_2 A_{22} + \lambda_3 A_{33}, \quad (22)$$

where $(\lambda_2, \lambda_3) = (1, 2)$ for the Ξ configuration; $(\lambda_2, \lambda_3) = (0, 1)$ for the Λ case, and $(\lambda_2, \lambda_3) = (1, 1)$ for V . Then one has two constants of the motion: the total number of atoms $\hat{N} = A_{11} + A_{22} + A_{33}$, and the total number of excitations, and we can use them to write the Hamiltonian (7) in units of the photon energy and per particle, $\mathcal{H} = \frac{H}{N_a \Omega}$, as

$$\begin{aligned} \mathcal{H} = & \frac{M}{N_a} + \frac{\Delta}{N_a} A_{22} + \frac{\bar{\Delta}}{N_a} A_{33} - \frac{\bar{\mu}_{12}}{N_a^{3/2}} (A_{12} a^\dagger + A_{21} a) \\ & - \frac{\bar{\mu}_{13}}{N_a^{3/2}} (A_{13} a^\dagger + A_{31} a) - \frac{\bar{\mu}_{23}}{N_a^{3/2}} (A_{23} a^\dagger + A_{32} a), \end{aligned} \quad (23)$$

where we have taken $\bar{\omega}_1 = 0$, and defined the parameters $\Delta \equiv \bar{\omega}_{21} - \lambda_2$, and $\bar{\Delta} \equiv \bar{\omega}_{32} + \bar{\omega}_{21} - \lambda_3$. Additionally, we have replaced the eigenvalues of \hat{N} and \hat{M} by N_a and M , respectively.

Therefore, the bases of the different configurations can be written in the form

$$\begin{aligned} |\nu N_a q_1 r\rangle_\Xi & \equiv |M - 2N_a + q_1 + r, N_a q_1 r\rangle, & M - 2N_a + q_1 + r \geq 0, \\ |\nu N_a q_1 r\rangle_\Lambda & \equiv |M - N_a + q_1, N_a q_1 r\rangle, & M - N_a + q_1 \geq 0, \\ |\nu N_a q_1 r\rangle_V & \equiv |M - N_a + r, N_a q_1 r\rangle, & M - N_a + r \geq 0. \end{aligned} \quad (24)$$

where we have used the corresponding values of λ_2 and λ_3 .

4.1. Dimensions of the Hilbert space

The Hilbert space is strongly dependent on the values of M and N_a . We start by giving the dimension for the Ξ configuration:

$$d_\Xi(N_a, M) = \begin{cases} (M/2 + 1)^2 & M \text{ even}, M \leq N_a, \\ (M + 1)(M + 3)/4 & M \text{ odd}, M \leq N_a, \\ \{2N_a(2M - N_a + 1) - M(M - 2) + 4\}/4 & M \text{ even}, N_a \leq M \leq 2N_a, \\ \{2N_a(2M - N_a + 1) - (M - 1)^2 + 4\}/4 & M \text{ odd}, N_a \leq M \leq 2N_a, \\ (N_a + 1)(N_a + 2)/2 & M \geq 2N_a. \end{cases} \quad (25)$$

For the Λ configuration, the dimension of the Hilbert space is given by

$$d_\Lambda(N_a, M) = \begin{cases} (M + 1)(2N_a + 2 - M)/2 & M \leq N_a, \\ (N_a + 1)(N_a + 2)/2 & M \geq N_a; \end{cases} \quad (26)$$

while for the V configuration one has

$$d_V(N_a, M) = \begin{cases} (M+1)(M+2)/2 & M \leq N_a, \\ (N_a+1)(N_a+2)/2 & M \geq N_a. \end{cases} \quad (27)$$

These relations are important for the study of the system for a small average number of photons in the cavity, because they imply a small number of total excitations M .

To find the quantum ground energy and its corresponding eigenstate, we proceed as follows. For each configuration of the atom, a given value of M , and for fixed parameters Ω , ω_1 , ω_2 , and ω_3 the eigenvalues and their corresponding eigenstates are evaluated numerically as functions of the control parameters μ_{ij} . This gives us the ground state energy for each corresponding total number of excitations. It is worth mentioning that, for a fixed region of values of the interaction intensity, one may estimate the maximum value of M that is required to find the minimum energy from the semi-classical calculation.

4.2. Single Regime

Next we show the single regime of the atoms for the three atomic configurations. As an example we consider $N_a = 5$ atoms; according to the dimension of the space one has a unique state for the Ξ and V configurations. For the Λ case there are two situations: in the double resonance case, $\bar{\omega}_{21} = 0$ and $\bar{\omega}_{31} = 1$, there are 6 degenerate eigenstates. If one has $\bar{\omega}_{21} \neq \delta$, with $\delta \leq 0.2$ the degeneracy is broken and the ground state of the system has all the atoms in their lowest level and zero photons. However, if $\bar{\omega}_{31} = \bar{\omega}_{32} \neq 1$, the system continues being degenerate. The degeneracy for any number of atoms is equal to $N_a + 1$, and it is present because the first and second levels of the system have the same energy; this degeneracy will be broken when a small energy difference between them is present. Assuming this approximation for the Λ configuration, in the three cases one has all the non-interacting atoms in their lowest energy level and zero photons for the radiation field.

4.3. Collective Regime

In the double resonance case, $\Delta = \bar{\Delta} = 0$, the Hamiltonian for any of the atomic configurations takes the form

$$\mathcal{H}_{DR} = \frac{M}{N_a} - \frac{\bar{\mu}_{12}(A_{12}a^\dagger + A_{21}a)}{N_a^{3/2}} - \frac{\bar{\mu}_{13}(A_{13}a^\dagger + A_{31}a)}{N_a^{3/2}} - \frac{\bar{\mu}_{23}(A_{23}a^\dagger + A_{32}a)}{N_a^{3/2}}, \quad (28)$$

where the first term is a constant of motion. For the Ξ configuration one has $\bar{\mu}_{13} = 0$ and M denotes the eigenvalue of the total excitation number M_Ξ ; for the Λ type of atomic spectra $\bar{\mu}_{12} = 0$ and M is the eigenvalue of M_Λ ; while for the V atomic case one considers $\bar{\mu}_{23} = 0$ and M is the eigenvalue of M_V . Therefore one has

$$\begin{aligned} \mathcal{H}_{DR\Xi} &= \frac{M}{N_a} - \frac{\bar{\mu}_{12}(A_{12}a^\dagger + A_{21}a)}{N_a^{3/2}} - \frac{\bar{\mu}_{23}(A_{23}a^\dagger + A_{32}a)}{N_a^{3/2}}, \\ \mathcal{H}_{DR\Lambda} &= \frac{M}{N_a} - \frac{\bar{\mu}_{13}(A_{13}a^\dagger + A_{31}a)}{N_a^{3/2}} - \frac{\bar{\mu}_{23}(A_{23}a^\dagger + A_{32}a)}{N_a^{3/2}}, \\ \mathcal{H}_{DRV} &= \frac{M}{N_a} - \frac{\bar{\mu}_{12}(A_{12}a^\dagger + A_{21}a)}{N_a^{3/2}} - \frac{\bar{\mu}_{13}(A_{13}a^\dagger + A_{31}a)}{N_a^{3/2}}. \end{aligned} \quad (29)$$

By means of the interaction picture of quantum mechanics one has the evolution equation in terms of the coupling term of the Hamiltonian. In this double resonance case the coupling term

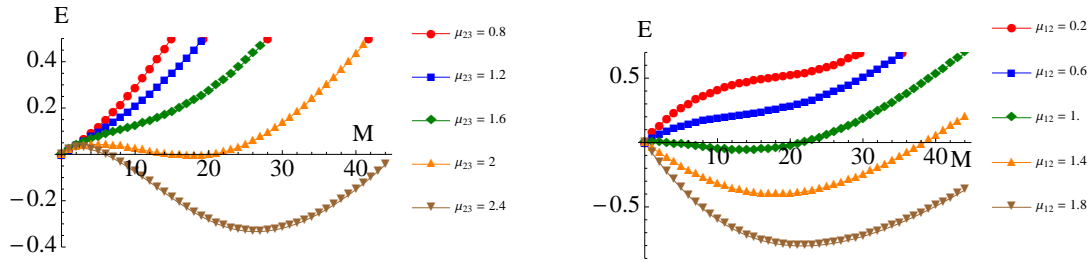


Figure 4. (Color online.) Energy of the atomic configuration Ξ as a function of the total number of excitations, M (horizontal axis). At left, the parameter value $\bar{\mu}_{12} = 0.8$ was used while at right we took $\bar{\mu}_{23} = 1.8$. In both cases the number of atoms is $N_a = 10$.

is time independent, so to find the spectra of the system one has to solve only the corresponding eigenvalue equation. For these atomic configurations, when one of the dipolar strengths are equal to zero the three-level system reduces to a two-level problem and only the levels involved change their properties.

Ξ configuration

For the double resonance case, together with the equality of the dipolar strengths, $\bar{\mu}_{12} = \bar{\mu}_{23}$, it is easy to prove that the system is equivalent to a two-level system with the operators for the matter given by $S_- = A_{12} + A_{23}$, $S_+ = A_{21} + A_{32}$, and $S_0 = (A_{33} - A_{11})/2$. The operator of the total number of excitations takes the form $M_\Xi = a^\dagger a + \hat{N} + 2S_0$.

In the general case, for fixed values of $\bar{\mu}_{12}$ and $\bar{\mu}_{23}$ we explore in Fig. 4 the energy eigenvalues as functions of the total number of excitations. The minimum of each curve determines the ground state of the system. These are calculated for a fixed value of the dipolar strength $\bar{\mu}_{12} = 0.8$ and for several values of $\bar{\mu}_{23}$ to see the transition from the dominance of the state with $M = 0$ to states with $M > 0$. The same was done for a fixed value of $\bar{\mu}_{23} = 1.8$ and changing $\bar{\mu}_{12}$ to see the transition from the normal regime to the collective regime. This is happening because the separatrix of the Hamiltonian model is crossed.

Λ configuration

In the double resonance case the dipolar strengths are associated to transitions from two degenerate levels to a third atomic level. Then, if one proposes

$$\bar{\mu}_{13} = \mu \cos \theta, \quad \bar{\mu}_{23} = \mu \sin \theta \quad (30)$$

together with the unitary transformation for the boson operators

$$c^\dagger = \cos \theta b_1^\dagger + \sin \theta b_2^\dagger, \quad c = \cos \theta b_1 + \sin \theta b_2, \quad (31)$$

it is straightforward to find that the system is equivalent to a two-level case with the following definitions for the matter observables: $S_+ = b_3^\dagger c$, $S_- = c^\dagger b_3$, and $S_0 = (b_3^\dagger b_3 - c^\dagger c)/2$.

In Fig. 5 we study the energy spectra as functions of the total number of excitations. The spectra are calculated for a fixed value of the dipolar strength $\mu_{13} = 0.5$ and detuning $\Delta_{32} = -0.1$, and for several values of μ_{23} to see the transition from the dominance of the state with $M = 0$ to states with $M > 0$. The same study was done in double resonance for a fixed value of $\mu_{23} = 1.0$ and changing μ_{13} to see again the transition to the collective regime.

V configuration

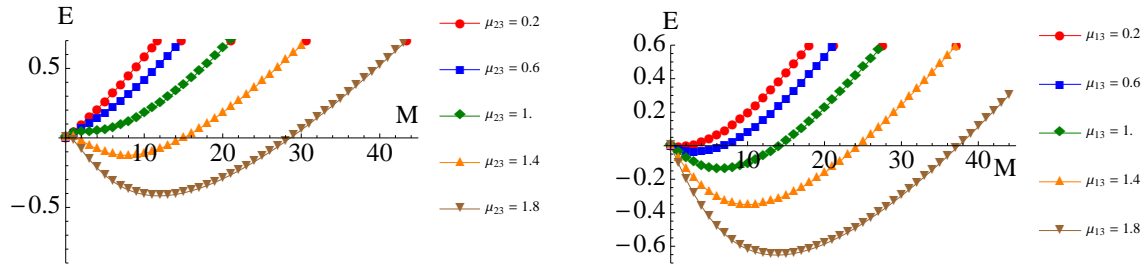


Figure 5. (Color online.) Energy of the atomic configuration Λ as function of the total number of excitations, M (horizontal axis). At left, the parameters $\bar{\mu}_{13} = 0.5$ and $\Delta_{32} = -0.1$ were used while at right we took $\bar{\mu}_{23} = 1.0$ and $\Delta_{32} = 0$. In both cases the number of atoms is $N_a = 10$.

In the double resonance case one has the transition from an atomic energy level to two degenerate levels. One can then proceed in a similar way as for the Λ configuration, defining

$$\bar{\mu}_{12} = \mu \cos \theta, \quad \bar{\mu}_{13} = \mu \sin \theta \quad (32)$$

together with the unitary boson transformation

$$d^\dagger = \cos \theta b_2^\dagger + \sin \theta b_3^\dagger, \quad d = \cos \theta b_2 + \sin \theta b_3. \quad (33)$$

It is straightforward to find that the system is equivalent to a two-level case with the following definitions for the matter observables $S_+ = d^\dagger b_1$, $S_- = b_1^\dagger d$, and $S_0 = (d^\dagger d - b_1^\dagger b_1)/2$.

For this case the results for the energy spectra are very similar to those displayed in Fig. 5 for the Λ configuration.

5. Conclusions

We have established the existence of single and collective regimes for three-level atoms in the Ξ , Λ , and V atomic configurations. These regions, in the dipolar parameter space, are separated by the corresponding separatrix of each configuration. The separatrix of the Ξ and Λ atomic configurations are constituted by bifurcation and Maxwell sets. In the bifurcation sector, the transitions are of second order, two critical points coalesce: one of them yields zero values for the expectation values of the total number of excitations and the photon number operator while the other produces $\langle M \rangle$ and $\langle N_{ph} \rangle$ different from zero. In the Maxwell region the quantum phase transitions are of first order, and there are double critical points yielding the same energy (see Fig.3). These double points are described by two independent states with different distribution functions for M and for the number of photons. For the Ξ configuration there is additionally a triple point, that is, three independent solutions with the same energy for any number of particles. These states have total excitation quantum numbers $M_\Xi = 0$, $M_\Xi = 1$, and $M_\Xi = 2$, noticing that the point is on the border between the bifurcation and Maxwell sets.

For the V atomic configuration we have found that the separatrix is formed by a bifurcation set, where there are quantum phase transitions of second order and two critical degenerate points. One of them implies zero values for the expectation values $\langle M \rangle$ and $\langle N_{ph} \rangle$ and the other yields expectation values of the same observables different from zero.

A connection between the double resonance case of the V and Λ configurations and two-level systems was established by means of unitary transformations. These occupy degenerate energy levels. This result allows for a check of consistency of our separatrix expressions with the localization of the quantum phase transitions in the dipolar parameter space for corresponding

two-level systems. Additionally, we expect that the form of the analytic expressions for the separatrix of each configuration will only be modified by a constant factor if the rotating wave approximation is not considered.

Finally, because the Hamiltonian commutes with M , a new variational state can be proposed if we truncate the series expansion of the variational coherent states given in expression (9) by replacing the photon quantum number as $\nu \rightarrow M - \lambda_3 N_a + (\lambda_3 - \lambda_2)q + \lambda_2 r$, renormalizing the resulting expression, and using the same critical points of the variational coherent states. This procedure has been used to calculate the fluctuation of the number of photons with very good agreement with the exact quantum calculation [29].

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