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SUSY double-well matrix model as 2D IIA superstring on RR background

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Abstract. We discuss an interpretation of a simple supersymmetric matrix model with a double-well potential as two-dimensional type IIA superstrings on a nontrivial Ramond-Ramond background. In particular, we can see direct correspondence between single trace operators in the matrix model and vertex operators in the type IIA theory by computing scattering amplitudes and comparing the results in both sides. Next, we explicitly compute nonperturbative instanton contributions in the matrix model and find that they survive in a double scaling limit realizing the type IIA superstring theory. This suggests that the supersymmetry is spontaneously broken by nonperturbative effect in the target space of the type IIA superstring theory.

1. Introduction

Solvable matrix models for two-dimensional quantum gravity or noncritical string theory were vigorously investigated around 1990, focusing on nonperturbative effects in string theory [1]. While this approach has been successful for bosonic string theory, little has been known for superstring theory, in particular which possesses target-space supersymmetry (SUSY). We would like to consider (solvable) matrix models describing superstring theory with target-space SUSY. In this paper, we discuss correspondence between a simple zero-dimensional SUSY double-well matrix model and two-dimensional type IIA superstring theory on a nontrivial Ramond-Ramond (RR) background. Then, nonperturbative effect of the matrix model is computed in its double scaling limit. As a result, we find that SUSY is spontaneously broken due to instantons in the matrix model. According to the correspondence, this suggests spontaneous SUSY breaking at the nonperturbative level in the type IIA superstring theory. We hope our analysis is helpful to understand nonperturbative dynamics of matrix models of super Yang-Mills type for critical superstring theory [2, 3, 4].

This article is mainly based on the work [5, 6, 7].

2. Double-well SUSY matrix model

Ref. [8] discussed a following simple matrix model:

$$S = N\,\mathrm{tr}\left[\frac{1}{2}B^2 + iB(\phi^2 - \mu^2) + \bar{\psi}(\phi\psi + \psi\phi)\right], \tag{1}$$

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where B and ϕ are $N \times N$ hermitian matrices, and ψ and $\bar{\psi}$ are $N \times N$ Grassmann-odd matrices. The action S is invariant under SUSY transformations generated by Q and $\bar{Q}$:

$$Q\phi = \psi, \quad Q\psi = 0, \quad Q\bar{\psi} = -iB, \quad QB = 0, \tag{2}$$

$$\bar{Q}\phi = -\bar{\psi}, \quad \bar{Q}\bar{\psi} = 0, \quad \bar{Q}\psi = -iB, \quad \bar{Q}B = 0, \tag{3}$$

from which one can see that they are nilpotent: $Q^2 = \bar{Q}^2 = \{Q, \bar{Q}\} = 0$. After integrating out B, we have a scalar potential of a double-well shape: $\frac{1}{2}(\phi^2 - \mu^2)^2$. A large-$N$ saddle point equation for the eigenvalue distribution of the matrix ϕ: $\rho(x) \equiv \frac{1}{N}\,\mathrm{tr}\,\delta(x - \phi)$ reads

$$\int dy\,\rho(y)\,\mathrm{P}\frac{1}{x-y} + \int dy\,\rho(y)\,\mathrm{P}\frac{1}{x+y} = x^3 - \mu^2 x. \tag{4}$$

Its solution with filling fraction (ν_+, ν_-) is given by

$$\rho(x) = \begin{cases} \frac{\nu_+}{\pi}\, x\,\sqrt{(x^2-a^2)(b^2-x^2)} & (a < x < b) \\ \frac{\nu_-}{\pi}\, |x|\,\sqrt{(x^2-a^2)(b^2-x^2)} & (-b < x < -a), \end{cases} \tag{5}$$

where $a = \sqrt{\mu^2 - 2}$ and $b = \sqrt{\mu^2 + 2}$. The filling fractions satisfying $\nu_+ + \nu_- = 1$ indicate that $\nu_+ N$ ($\nu_- N$) eigenvalues are around the right (left) minimum of the double-well. The solution exists for $\mu^2 > 2$. The large-N free energy and the expectation values $\left\langle \frac{1}{N}\,\mathrm{tr}\, B^n \right\rangle$ ($n = 1, 2, \cdots$) evaluated at the solution turn out to all vanish [8]. This strongly suggests that the solution preserves SUSY. Thus, we conclude that the SUSY minima are infinitely degenerate and parametrized by (ν_+, ν_-) at large N. Note that the edges of the support a and b are independent of $\nu_\pm$. It is considered to be a characteristic feature of SUSY matrix models, not observed in bosonic double-well matrix models [9, 10].

There exists a solution having support of a single interval $x \in [-c, c]$ for $\mu^2 < 2$ [11]:

$$\rho(x) = \frac{1}{2\pi}\left(x^2 - \mu^2 + \frac{c^2}{2}\right)\sqrt{c^2 - x^2} \tag{6}$$

with $c = \sqrt{\frac{2}{3}}\left(\mu^2 + \sqrt{\mu^4 + 12}\right)^{1/2}$. Positivity of $\rho(x)$ yields the condition $\mu^2 < 2$. This solution gives nonzero values of $\left\langle \frac{1}{N}\,\mathrm{tr}\, B \right\rangle$ and of the large-N free energy, showing that SUSY is broken. We observed that the third derivative of the free energy with respect to μ^2 is not continuous at $\mu^2 = 2$. The transition between the SUSY phase ($\mu^2 > 2$) and the SUSY broken phase ($\mu^2 < 2$) is of the third order.

In the next section, we will compute various correlation functions at the saddle point (5) and find new logarithmic critical behavior as $\mu^2 \to 2 + 0$. Based on the result, we will discuss correspondence between the matrix model and two-dimensional type IIA superstring theory on a nontrivial Ramond-Ramond (RR) background in sections 4 and 5. The logarithmic critical behavior is somewhat reminiscent of the $c = 1$ matrix model which is a matrix quantum mechanics of a single matrix variable [12]. The Penner model is known as a zero-dimensional matrix model exhibiting the same critical behavior as the matrix quantum mechanics [13]. [1] It describes noncritical string theory propagating on a two-dimensional target space: (Liouville direction) $\times$ (S^1 with self-dual radius). So, it is expected that our matrix model can be regarded as a SUSY version of the Penner model and describes

[1] Also is the normal matrix model [14], which corresponds to $c = 1$ noncritical strings on S^1 with a general radius.

two-dimensional superstring theory with SUSY on the target space (Liouville direction) $\times$ (S^1 with self-dual radius). Indeed, two-dimensional type II superstring theory with the identical target space is constructed [15, 16, 17, 18], where target space SUSY exists only at the self-dual radius of the circle.

Our matrix model is interpreted as the $O(n)$ model on a random surface [19] with $n = -2$, whose critical behavior is described by the $c = -2$ topological gravity [20]. The partition function after B, ψ and $\bar{\psi}$ are integrated out is expressed as a Gaussian one-matrix model by the Nicolai mapping $H = \phi^2$, where the H-integration is over the *positive definite* hermitian matrices, not over all the hermitian matrices. Ref. [21] discusses that the difference of the integration region has only effects which are nonperturbative in $1/N$, and the model can be regarded as the standard Gaussian matrix model at each order of genus expansion.

The Nicolai mapping changes the operators $\frac{1}{N}\,\mathrm{tr}\,\phi^{2n}$ $(n = 1, 2, \cdots)$ to regular operators $\frac{1}{N}\,\mathrm{tr}\,H^n$. Hence, the behavior of their correlators is expected to be described by the Gaussian one-matrix (the $c = -2$ topological gravity) at least perturbatively in $1/N$. However, the operators $\frac{1}{N}\,\mathrm{tr}\,\phi^{2n+1}$ $(n = 0, 1, 2, \cdots)$ are mapped to $\pm\frac{1}{N}\,\mathrm{tr}\,H^{n+1/2}$ that are singular at the origin. They are not observables in the $c = -2$ topological gravity, while they are natural observables as well as $\frac{1}{N}\,\mathrm{tr}\,\phi^{2n}$ in the original setting (1). In the next section, we will see that correlation functions among operators

$$\frac{1}{N}\,\mathrm{tr}\,\phi^{2n+1}, \qquad \frac{1}{N}\,\mathrm{tr}\,\psi^{2n+1}, \qquad \frac{1}{N}\,\mathrm{tr}\,\bar{\psi}^{2n+1} \qquad (n = 0, 1, 2, \cdots) \tag{7}$$

exhibit logarithmic singular behavior of powers of $\ln(\mu^2 - 2)$ at the planar topology.

In considering correspondence of the matrix model to superstring theory, the following observation will be helpful. Suppose ψ and $\bar{\psi}$ are regarded as target-space fermions in the corresponding superstring theory. Namely, ψ is interpreted as an operator in the (NS, R) sector and $\bar{\psi}$ in the (R, NS) sector in the RNS formalism. Then, under the so-called $(-1)^{\mathbf{F}_L}$ and $(-1)^{\mathbf{F}_R}$ transformations changing the signs of operators in the left-moving Ramond sector and those in the right-moving Ramond sector respectively, they transform as

$$(-1)^{\mathbf{F}_L}: \qquad \psi \to \psi, \qquad \bar{\psi} \to -\bar{\psi}, \tag{8}$$
$$(-1)^{\mathbf{F}_R}: \qquad \psi \to -\psi, \qquad \bar{\psi} \to \bar{\psi}. \tag{9}$$

In order for the matrix model action (1) to be invariant under the transformations, B and ϕ should transform as

$$(-1)^{\mathbf{F}_L}: \qquad B \to B, \qquad \phi \to -\phi, \tag{10}$$
$$(-1)^{\mathbf{F}_R}: \qquad B \to B, \qquad \phi \to -\phi. \tag{11}$$

This indicates that B corresponds to an operator in the (NS, NS) sector, and ϕ in the (R, R) sector.

3. Correlation functions

3.1. Planar one-point functions

The planar one-point function $\left\langle \frac{1}{N}\,\mathrm{tr}\,\phi^n \right\rangle_0$ $(n = 1, 2, \cdots)$ are computed as

$$\begin{aligned} \left\langle \frac{1}{N}\,\mathrm{tr}\,\phi^n \right\rangle_0 &= \int dx\, x^n \rho(x) \\ &= (\nu_+ + (-1)^n \nu_-)(2+\mu^2)^{n/2} F\left(-\frac{n}{2}, \frac{3}{2}, 3; \frac{4}{2+\mu^2}\right), \end{aligned} \tag{12}$$

where the suffix "0" in the left hand side indicates the planar contribution. For n even, the expression reduces to a polynomial of μ^2 giving nonsingular behavior as expected from the $c = -2$ topological gravity. On the other hand, when μ^2 is odd, it exhibits logarithmic singular behavior as $\mu^2 \to 2 + 0$:

$$\left\langle \frac{1}{N} \mathrm{tr}\, \phi^{2k+1} \right\rangle_0 \sim (\nu_+ - \nu_-) \frac{2^{k+2}}{\pi} \frac{(2k+1)!!}{(k+2)!} \omega^{k+2} \ln \omega \tag{13}$$

with $\omega \equiv \frac{1}{4}(\mu^2 - 2)$. The symbol "$\sim$" denotes equality up to additive less singular terms. Explicit form for a first few expectation values reads

$$\begin{aligned}
\left\langle \frac{1}{N} \mathrm{tr}\, \phi \right\rangle_0 &= (\nu_+ - \nu_-) \left[\frac{64}{15\pi} + \frac{16}{3\pi}\omega + \frac{2}{\pi}\omega^2 \ln \omega + \mathcal{O}(\omega^2) \right], \\
\left\langle \frac{1}{N} \mathrm{tr}\, \phi^3 \right\rangle_0 &= (\nu_+ - \nu_-) \left[\frac{1024}{105\pi} + \frac{128}{5\pi}\omega + \frac{16}{\pi}\omega^2 + \frac{4}{\pi}\omega^3 \ln \omega + \mathcal{O}(\omega^3) \right], \\
\left\langle \frac{1}{N} \mathrm{tr}\, \phi^5 \right\rangle_0 &= (\nu_+ - \nu_-) \left[\frac{8192}{315\pi} + \frac{2048}{21\pi}\omega + \frac{128}{\pi}\omega^2 + \frac{160}{3\pi}\omega^3 + \frac{10}{\pi}\omega^4 \ln \omega + \mathcal{O}(\omega^4) \right], \\
\cdots &.
\end{aligned} \tag{14}$$

Matrix models can be seen as a sort of "lattice models" for string theory. In the hypergeometric function $F\left(-\frac{n}{2}, \frac{3}{2}, 3; \frac{1}{1+\omega}\right)$ for n being odd, the logarithmic singular terms can be regarded as universal parts relevant to "continuum physics", whereas polynomials of ω as nonuniversal "lattice artifacts".

3.2. Eigenvalue distribution with source

In computing higher-point correlators $\left\langle \prod_{i=1}^K \frac{1}{N} \mathrm{tr}\, \phi^{n_i} \right\rangle_{C,0}$ at the vacuum with general filling fraction (ν_+, ν_-), it is useful to reduce them to those at the vacuum with $(\nu_+, \nu_-) = (1, 0)$. We can show

$$\left\langle \prod_{i=1}^K \frac{1}{N} \mathrm{tr}\, \phi^{n_i} \right\rangle_{C,0}^{(\nu_+,\nu_-)} = (\nu_+ - \nu_-)^\sharp \left\langle \prod_{i=1}^K \frac{1}{N} \mathrm{tr}\, \phi^{n_i} \right\rangle_{C,0}^{(1,0)} \tag{15}$$

up to $K = 3$, by explicit calculations. Here, the suffix "C" means taking a connected part. The superscripts (ν_+, ν_-) and $(1, 0)$ are put to clarify the filling fractions of the vacua at which the expectation values are evaluated, and $\sharp$ counts the number of odd integers in $\{n_1, \cdots, n_K\}$.

In order to obtain higher-point correlators of $\frac{1}{N} \mathrm{tr}\, \phi^p$ $(p = 1, 2, \cdots)$, we introduce source terms $\sum_{p=1}^\infty j_p \,\mathrm{tr}\, \phi^p$ to the partition function:

$$Z_{j_k} = \int d^{N^2}\phi \, e^{-N \,\mathrm{tr}\left[\frac{1}{2}(\phi^2 - \mu^2)^2 - \sum_{p=1}^\infty j_p \phi^p\right]} \det(\phi \otimes \mathbf{1} + \mathbf{1} \otimes \phi) . \tag{16}$$

In the large-N limit, the eigenvalue distribution $\rho_j(x)$ satisfies the saddle point equation

$$\int dy\, \rho_j(y) \left(\mathrm{P}\frac{1}{x-y} + \mathrm{P}\frac{1}{x+y} \right) = x^3 - \mu^2 x - \sum_{p=1}^\infty \frac{p j_p}{2} x^{p-1}. \tag{17}$$

Let us consider the case of the filling fractions $(1, 0)$ with the support of $\rho_j(x)$ $[a_j, b_j]$ $(0 < a_j < b_j)$. We change variables as

$$x^2 = A + B\xi, \quad y^2 = A + B\eta \qquad \text{with} \qquad A \equiv \frac{a_j^2 + b_j^2}{2}, \quad B \equiv \frac{b_j^2 - a_j^2}{2}, \tag{18}$$

and put $\tilde{\rho}(\eta) \equiv \frac{B}{2y}\rho_j(y)$, to simplify (17) as

$$\frac{1}{B}\int_{-1}^{1} d\eta\,\tilde{\rho}(\eta)\,\mathrm{P}\frac{1}{\xi-\eta} = \frac{1}{2}(A-\mu^2+B\xi) - \sum_{p=1}^{\infty}\frac{pj_p}{4}\,(A+B\xi)^{\frac{p}{2}-1} \tag{19}$$

for $\xi \in [-1,1]$, where $\tilde{\rho}$ is normalized by $\int_{-1}^{1} d\eta\tilde{\rho}(\eta) = 1$.

We act $\int_{-1}^{1} d\xi\,\sqrt{1-\xi^2}\,\mathrm{P}\frac{1}{\zeta-\xi}$ to both sides of (19), and apply the formula

$$\int_{-1}^{1} dy\,\sqrt{1-y^2}\,\mathrm{P}\frac{1}{x-y}\,\mathrm{P}\frac{1}{u-y} = -\pi + \pi^2\sqrt{1-u^2}\,\delta(u-x) \tag{20}$$

for $x, u \in [-1,1]$. Consequently,

$$\begin{aligned}\tilde{\rho}(\zeta) \;=\; & \frac{1}{2\pi}\frac{1}{\sqrt{1-\zeta^2}}\left[2-(A-\mu^2)B\zeta - B^2\left(\zeta^2-\frac{1}{2}\right)\right.\\ & \left. +\sum_{p=1}^{\infty} pj_p\,\frac{B}{2\pi}\int_{-1}^{1} d\xi\,\sqrt{1-\xi^2}\,\mathrm{P}\frac{1}{\zeta-\xi}\,(A+B\xi)^{\frac{p}{2}-1}\right].\end{aligned} \tag{21}$$

The condition $\tilde{\rho}(\zeta = \pm 1) = 0$ determines A and B as

$$A \;=\; \mu^2 + \sum_{p=1}^{\infty} j_p\,\frac{p}{2}(A+B)^{\frac{p}{2}-1}F\left(-\frac{p}{2}+1,\frac{1}{2},1;\frac{2B}{A+B}\right), \tag{22}$$

$$B \;=\; 2\left[1+\sum_{p=1}^{\infty}\frac{j_p}{4}\,\frac{p}{2}\left(\frac{p}{2}-1\right)B^2(A+B)^{\frac{p}{2}-2}F\left(-\frac{p}{2}+2,\frac{3}{2},3;\frac{2B}{A+B}\right)\right]^{1/2}, \tag{23}$$

from which A and B are obtained iteratively with respect to $\{j_p\}$. Up to the first order of $\{j_p\}$,

$$A \;=\; \mu^2 + \sum_{p=1}^{\infty} j_p\,\frac{p}{2}(2+\mu^2)^{\frac{p}{2}-1}F\left(-\frac{p}{2}+1,\frac{1}{2},1;\frac{4}{2+\mu^2}\right) + \mathcal{O}(j^2), \tag{24}$$

$$B \;=\; 2 + \sum_{p=1}^{\infty} j_p\,\frac{p}{2}\left(\frac{p}{2}-1\right)(2+\mu^2)^{\frac{p}{2}-2}F\left(-\frac{p}{2}+2,\frac{3}{2},3;\frac{4}{2+\mu^2}\right) + \mathcal{O}(j^2), \tag{25}$$

where $\mathcal{O}(j^2)$ means a quantity of the quadratic order of $\{j_p\}$.

3.3. Planar two-point functions (bosons)

Let us express the planar expectation value of $\mathcal{O}$ under the partition function with the source terms (16) as $\langle\mathcal{O}\rangle_0^{(j)}$. The cylinder amplitude at the vacuum with the filling fraction $(1,0)$ is given as

$$\left\langle \frac{1}{N}\,\mathrm{tr}\,\phi^p\,\frac{1}{N}\,\mathrm{tr}\,\phi^q\right\rangle_{C,0}^{(1,0)} = \frac{\partial}{\partial j_p}\left\langle\frac{1}{N}\,\mathrm{tr}\,\phi^q\right\rangle_0^{(j)}\Bigg|_{\{j_p\}=0} = \frac{\partial}{\partial j_p}\int_{-1}^{1} d\zeta\,(A+B\zeta)^{\frac{q}{2}}\tilde{\rho}(\zeta)\Bigg|_{\{j_p\}=0}. \tag{26}$$

Combining (26) and (15) leads to the result for general filling fraction. In what follows, we omit the superscript (ν_+, ν_-) of the correlators when there is no possible confusion. It turns out that the amplitudes are quadratic forms of the hypergeometric functions. For both of p and q even, they are polynomials of ω independent of $(\nu_+ - \nu_-)$, which is expected from the $c = -2$ topological gravity. When p and q are odd and even respectively,

$$\left\langle \Phi_{2k+1} \frac{1}{N} \operatorname{tr} \phi^{2\ell} \right\rangle_{C,0} \sim (\nu_+ - \nu_-)(\text{const.})\omega^{k+1} \ln \omega. \tag{27}$$

When both of p and q are odd,

$$\langle \Phi_{2k+1} \Phi_{2\ell+1} \rangle_{C,0} \sim -(\nu_+ - \nu_-)^2 \frac{1}{2\pi^2} \frac{1}{k+\ell+1} \frac{(2k+1)!}{(k!)^2} \frac{(2\ell+1)!}{(\ell!)^2} \omega^{k+\ell+1} (\ln \omega)^2. \tag{28}$$

Here, in order to subtract nonuniversal contributions appearing in the form (nonuniversal part) $\times$ (universal part), we took a basis of the odd-power operators mixed with lower even-power operators:

$$\Phi_{2k+1} = \frac{1}{N} \operatorname{tr} \phi^{2k+1} + (\nu_+ - \nu_-) \sum_{i=1}^{k} \alpha_{2k+1,2i}(\omega) \frac{1}{N} \operatorname{tr} \phi^{2i} \tag{29}$$

with $\alpha_{2k+1,2i}(\omega)$ being a regular function at $\omega = 0$. For example, we can explicitly construct the basis for the first three operators by considering $\langle \Phi_1 \Phi_1 \rangle_{C,0}$, $\langle \Phi_1 \Phi_3 \rangle_{C,0}$, $\cdots$, $\langle \Phi_5 \Phi_5 \rangle$:

$$\begin{aligned}
\Phi_1 &= \frac{1}{N} \operatorname{tr} \phi, \\
\Phi_3 &= \frac{1}{N} \operatorname{tr} \phi^3 - (\nu_+ - \nu_-) \frac{4}{\pi} \left(1 + \bar{\alpha}^{(1)}_{3,2} \omega + \mathcal{O}(\omega^2)\right) \frac{1}{N} \operatorname{tr} \phi^2, \\
\Phi_5 &= \frac{1}{N} \operatorname{tr} \phi^5 - (\nu_+ - \nu_-) \frac{4}{\pi} \left(1 + \bar{\alpha}^{(1)}_{5,4} \omega + \mathcal{O}(\omega^2)\right) \frac{1}{N} \operatorname{tr} \phi^4, \\
&\quad -(\nu_+ - \nu_-) \frac{8}{3\pi} \left(1 + 3(1 - \bar{\alpha}^{(1)}_{5,4}) \omega + \mathcal{O}(\omega^2)\right) \frac{1}{N} \operatorname{tr} \phi^2,
\end{aligned} \tag{30}$$

where $\bar{\alpha}^{(1)}_{3,2}$ and $\bar{\alpha}^{(1)}_{5,4}$ are undertermined constants. They would be determined by considering higher operators.

Note that $(\nu_+ - \nu_-)$ corresponds to an RR charge from the observation at the end of section 2: Φ_{2k+1} measuring an RR charge.

3.4. Planar three-point functions (bosons)

Similar procedure to the case of the two-point functions can be used in computing three-point correlation functions. It turns out that the result is expressed as cubic forms of the hypergeometric functions. The first two amplitudes become

$$\begin{aligned}
\left\langle (\Phi_1)^3 \right\rangle_{C,0} &= (\nu_+ - \nu_-)^3 \left[\frac{1}{16\pi^3} (\ln \omega)^3 + \mathcal{O}((\ln \omega)^2) \right], \\
\left\langle (\Phi_1)^2 \Phi_3 \right\rangle_{C,0} &= (\nu_+ - \nu_-)^3 \left[\frac{2}{\pi^3} + \frac{3}{8\pi^3} \omega (\ln \omega)^3 + \mathcal{O}(\omega (\ln \omega)^2) \right].
\end{aligned} \tag{31}$$

3.5. Planar higher-point functions (bosons)

The results obtained for the one-, two- and three-point functions of operators Φ_{2k+1} ($k = 0, 1, 2, \cdots$) naturally suggest the form of higher-point functions as

$$\left\langle \prod_{i=1}^{n} \Phi_{2k_i+1} \right\rangle_{C,0} \sim (\nu_+ - \nu_-)^n \,(\text{const.})\, \omega^{2-\gamma+\sum_{i=1}^{n}(k_i-1)} (\ln \omega)^n \tag{32}$$

with $\gamma = -1$. Besides the power of logarithm $(\ln\omega)^n$, it has the standard scaling behavior with the string susceptibility $\gamma = -1$ (the same as in the $c = -2$ topological gravity) and the gravitational scaling dimension k of Φ_{2k+1}, if we identify ω with "the cosmological constant" coupled to the lowest dimensional operator on a random surface [22, 23, 24].

3.6. Planar two-point functions (fermions)
The simplest two-point correlator of fermions is computed as

$$\begin{aligned} \left\langle \frac{1}{N}\,\mathrm{tr}\,\psi\,\frac{1}{N}\,\mathrm{tr}\,\bar{\psi}\right\rangle_{C,0} &= \frac{1}{2}\int_\Omega dx\,\frac{1}{x}\,\rho(x) = (\nu_+ - \nu_-)\,\frac{1}{2}\,(4(1+\omega))^{-1/2}\,F\left(\frac{1}{2},\frac{3}{2},3;\frac{1}{1+\omega}\right) \\ &= (\nu_+ - \nu_-)\left[\frac{4}{3\pi} + \frac{1}{\pi}\,\omega\ln\omega + \mathcal{O}(\omega)\right] \qquad (\omega\to+0), \end{aligned} \tag{33}$$

exhibiting singular behavior of $\ln\omega$. SUSY invariance implies that this is equal to $\left\langle \frac{1}{N}\,\mathrm{tr}\,(iB)\,\frac{1}{N}\,\mathrm{tr}\,\phi\right\rangle_{C,0} = \frac{1}{4}\frac{\partial}{\partial\omega}\left\langle\frac{1}{N}\,\mathrm{tr}\,\phi\right\rangle_0$, interestingly which can be seen from (14).

Next, for $\left\langle \frac{1}{N}\,\mathrm{tr}\,\psi^3\,\frac{1}{N}\,\mathrm{tr}\,\bar{\psi}^3\right\rangle_{C,0}$, we should consider an operator mixing similar to the bosonic case (29). Let us take a new basis as

$$\begin{aligned} \Psi_1 &\equiv \tfrac{1}{N}\,\mathrm{tr}\,\psi, & \bar{\Psi}_1 &\equiv \frac{1}{N}\,\mathrm{tr}\,\bar{\psi}, \\ \Psi_3 &\equiv \tfrac{1}{N}\,\mathrm{tr}\,\psi^3 + (\text{mixing}), & \bar{\Psi}_3 &\equiv \frac{1}{N}\,\mathrm{tr}\,\bar{\psi}^3 + (\text{mixing}), \\ \Psi_5 &\equiv \tfrac{1}{N}\,\mathrm{tr}\,\psi^5 + (\text{mixing}), & \bar{\Psi}_5 &\equiv \frac{1}{N}\,\mathrm{tr}\,\bar{\psi}^5 + (\text{mixing}), \\ &\cdots, & &\cdots, \end{aligned} \tag{34}$$

where "mixing" means operators to be added so that

$$\left\langle \Psi_{2k+1}\bar{\Psi}_{2\ell+1}\right\rangle_{C,0} \sim \delta_{k,\ell}\,v_k\,(\nu_+ - \nu_-)^{2k+1}\omega^{2k+1}\ln\omega \tag{35}$$

with v_k constants holds for $k, \ell = 0, 1$. It turns out that the choice

$$\begin{aligned} \Psi_3 &= \frac{1}{N}\,\mathrm{tr}\,\psi^3 + \frac{3}{\sqrt{2}}\,(1+\omega+\mathcal{O}(\omega^2))\frac{1}{N}\,\mathrm{tr}\,\{(iB - \phi^2 + \mu^2)\psi\}, \\ \bar{\Psi}_3 &= \frac{1}{N}\,\mathrm{tr}\,\bar{\psi}^3 + \frac{3}{\sqrt{2}}\,(1+\omega+\mathcal{O}(\omega^2))\frac{1}{N}\,\mathrm{tr}\,\{(iB - \phi^2 + \mu^2)\bar{\psi}\} \end{aligned} \tag{36}$$

or

$$\begin{aligned} \Psi_3 &= \frac{1}{N}\,\mathrm{tr}\,\psi^3 - \frac{3}{\sqrt{2}}\,(1+\omega+\mathcal{O}(\omega^2))\frac{1}{N}\,\mathrm{tr}\,\{(iB - \phi^2 + \mu^2)\psi\}, \\ \bar{\Psi}_3 &= \frac{1}{N}\,\mathrm{tr}\,\bar{\psi}^3 - \frac{3}{\sqrt{2}}\,(1+\omega+\mathcal{O}(\omega^2))\frac{1}{N}\,\mathrm{tr}\,\{(iB - \phi^2 + \mu^2)\bar{\psi}\} \end{aligned} \tag{37}$$

does the job (35) with $v_0 = \frac{1}{\pi}$ and $v_1 = \frac{6}{\pi}$.

The result (35) tells us that Ψ_{2k+1} and $\bar{\Psi}_{2k+1}$ have the gravitational scaling dimension k same as Φ_{2k+1} besides the logarithmic factor.

4. 2D type IIA superstring

The type II superstring theory discussed in Refs. [15, 16, 17] has the target space $(\varphi, x) \in$ (Liouville direction) $\times$ (S^1 with self-dual radius). The holomorphic energy-momentum tensor on the string world-sheet is

$$T = -\frac{1}{2}(\partial x)^2 - \frac{1}{2}\psi_x \partial \psi_x - \frac{1}{2}(\partial\varphi)^2 + \partial^2\varphi - \frac{1}{2}\psi_\ell \partial\psi_\ell \tag{38}$$

excluding ghosts' part. ψ_x and ψ_ℓ are superpartners of x and φ, respectively. Target-space supercurrents in the type IIA theory

$$q_+(z) = e^{-\frac{1}{2}\phi(z) - \frac{i}{2}H(z) - ix(z)}, \qquad \bar{q}_-(\bar{z}) = e^{-\frac{1}{2}\bar{\phi}(\bar{z}) + \frac{i}{2}\bar{H}(\bar{z}) + i\bar{x}(\bar{z})} \tag{39}$$

exist only for the S^1 target space of the self-dual radius. ϕ ($\bar{\phi}$) is the holomorphic (anti-holomorphic) bosonized superconformal ghost, and the fermions are bosonized as $\psi_\ell \pm i\psi_x = \sqrt{2}\, e^{\mp iH}$, $\bar{\psi}_\ell \pm i\bar{\psi}_x = \sqrt{2}\, e^{\mp i\bar{H}}$. In addition, we should care about cocycle factors in order to realize the anticommuting nature between q_+ and $\bar{q}_-$. Supercurrents with the cocycle factors are

$$\hat{q}_+(z) = e^{\pi\beta(\frac{1}{2}p_{\bar{\phi}} - i\frac{1}{2}p_{\bar{h}} - ip_{\bar{x}})}\, q_+(z), \qquad \hat{\bar{q}}_-(\bar{w}) = e^{-\pi\beta(\frac{1}{2}p_\phi + i\frac{1}{2}p_h + ip_x)}\, \bar{q}_-(\bar{w}), \tag{40}$$

where $\beta \in \mathbf{Z} + \frac{1}{2}$, and p_ϕ, p_h and p_x ($p_{\bar{\phi}}$, $p_{\bar{h}}$ and $p_{\bar{x}}$) are momentum modes of holomorphic part (anti-holomorphic part) of free bosons [6]. Then the supercharges

$$\hat{Q}_+ = \oint \frac{dz}{2\pi i}\, \hat{q}_+(z), \qquad \hat{\bar{Q}}_- = \oint \frac{d\bar{z}}{2\pi i}\, \hat{\bar{q}}_-(\bar{z}) \tag{41}$$

are nilpotent $\hat{Q}_+^2 = \hat{\bar{Q}}_-^2 = \{\hat{Q}_+, \hat{\bar{Q}}_-\} = 0$, which indeed matches the property of the supercharges Q and $\bar{Q}$ in the matrix model.

The spectrum except special massive states is represented by the NS "tachyon" [2] vertex operator (in (-1) picture):

$$T_k = e^{-\phi + ikx + p_\ell \varphi}, \qquad \bar{T}_{\bar{k}} = e^{-\bar{\phi} + i\bar{k}\bar{x} + p_\ell \bar{\varphi}}, \tag{42}$$

and by the R vertex operator (in $(-\frac{1}{2})$ picture):

$$V_{k,\epsilon} = e^{-\frac{1}{2}\phi + \frac{i}{2}\epsilon H + ikx + p_\ell\varphi}, \qquad \bar{V}_{\bar{k},\bar{\epsilon}} = e^{-\frac{1}{2}\bar{\phi} + \frac{i}{2}\bar{\epsilon}\bar{H} + i\bar{k}\bar{x} + p_\ell\bar{\varphi}} \tag{43}$$

with $\epsilon, \bar{\epsilon} = \pm 1$. Cocycle factors for vertex operators are introduced as

$$\begin{aligned} \hat{T}_k(z) &= e^{\pi\beta(p_{\bar{\phi}} + ikp_{\bar{x}})}\, T_k(z), \qquad & \hat{\bar{T}}_{\bar{k}}(\bar{z}) &= e^{-\pi\beta(p_\phi + i\bar{k}p_x)}\, \bar{T}_{\bar{k}}(\bar{z}), \\ \hat{V}_{k,\epsilon}(z) &= e^{\pi\beta(\frac{1}{2}p_{\bar{\phi}} + i\frac{\epsilon}{2}p_{\bar{h}} + ikp_{\bar{x}})}\, V_{k,\epsilon}(z), \qquad & \hat{\bar{V}}_{\bar{k},\bar{\epsilon}}(\bar{z}) &= e^{-\pi\beta(\frac{1}{2}p_\phi + i\frac{\bar{\epsilon}}{2}p_h + i\bar{k}p_x)}\, \bar{V}_{\bar{k},\bar{\epsilon}}(\bar{z}). \end{aligned} \tag{44}$$

Locality with the supercurrents, mutual locality, superconformal invariance (including the Dirac equation constraint) and the level matching condition determine physical vertex operators. As discussed in [17], there are two consistent sets of physical vertex operators - "momentum background" and "winding background". Let us consider the "winding background". [3] The

[2] In two dimensions, "tachyon" turns out to be not truely tachyonic but massless.

[3] We can repeat the parallel argument for "momentum background" in the type IIB theory, which is equivalent to the "winding background" in the type IIA theory through T-duality with respect to the S^1 direction.

physical spectrum in the “winding background” is given by

$$
\begin{aligned}
&(\text{NS, NS}): && \hat{T}_k\,\hat{\bar{T}}_{-k} && (k \in \mathbf{Z} + \tfrac{1}{2}),\\
&(\text{R}+,\ \text{R}-): && \hat{V}_{k,+1}\,\hat{\bar{V}}_{-k,-1} && (k = \tfrac{1}{2}, \tfrac{3}{2}, \cdots),\\
&(\text{R}-,\ \text{R}+): && \hat{V}_{-k,-1}\,\hat{\bar{V}}_{k,+1} && (k = 0, 1, 2, \cdots),\\
&(\text{NS},\ \text{R}-): && \hat{T}_{-k}\,\hat{\bar{V}}_{-k,-1} && (k = \tfrac{1}{2}, \tfrac{3}{2}, \cdots),\\
&(\text{R}+,\ \text{NS}): && \hat{V}_{k,+1}\,\hat{\bar{T}}_{k} && (k = \tfrac{1}{2}, \tfrac{3}{2}, \cdots),
\end{aligned}
\tag{45}
$$

where we take a branch of $p_\ell = 1 - |k|$ satisfying the locality bound $p_\ell \le Q/2 = 1$ [25]. We can see that the vertex operators

$$
\hat{V}_{\frac{1}{2},+1}\,\hat{\bar{V}}_{-\frac{1}{2},-1}, \qquad \hat{T}_{-\frac{1}{2}}\,\hat{\bar{V}}_{-\frac{1}{2},-1}, \qquad \hat{V}_{\frac{1}{2},+1}\,\hat{\bar{T}}_{\frac{1}{2}}, \qquad \hat{T}_{-\frac{1}{2}}\,\hat{\bar{T}}_{\frac{1}{2}}
\tag{46}
$$

form a quartet under $\hat{Q}_+$ and $\hat{\bar{Q}}_-$:

$$
\begin{aligned}
[\hat{Q}_+, \hat{V}_{\frac{1}{2},+1}\,\hat{\bar{V}}_{-\frac{1}{2},-1}] &= \hat{T}_{-\frac{1}{2}}\,\hat{\bar{V}}_{-\frac{1}{2},-1}, & \{\hat{Q}_+, \hat{T}_{-\frac{1}{2}}\,\hat{\bar{V}}_{-\frac{1}{2},-1}\} &= 0,\\
\{\hat{Q}_+, \hat{V}_{\frac{1}{2},+1}\,\hat{\bar{T}}_{\frac{1}{2}}\} &= \hat{T}_{-\frac{1}{2}}\,\hat{\bar{T}}_{\frac{1}{2}}, & [\hat{Q}_+, \hat{T}_{-\frac{1}{2}}\,\hat{\bar{T}}_{\frac{1}{2}}] &= 0,
\end{aligned}
\tag{47}
$$

$$
\begin{aligned}
[\hat{\bar{Q}}_-, \hat{V}_{\frac{1}{2},+1}\,\hat{\bar{V}}_{-\frac{1}{2},-1}] &= -\hat{V}_{\frac{1}{2},+1}\,\hat{\bar{T}}_{\frac{1}{2}}, & \{\hat{\bar{Q}}_-, \hat{V}_{\frac{1}{2},+1}\,\bar{T}_{\frac{1}{2}}\} &= 0,\\
\{\hat{\bar{Q}}_-, \hat{T}_{-\frac{1}{2}}\,\hat{\bar{V}}_{-\frac{1}{2},-1}\} &= \hat{T}_{-\frac{1}{2}}\,\hat{\bar{T}}_{\frac{1}{2}}, & [\hat{\bar{Q}}_-, \hat{T}_{-\frac{1}{2}}\,\bar{T}_{\frac{1}{2}}] &= 0.
\end{aligned}
\tag{48}
$$

Notice that (47) and (48) are isomorphic to (2) and (3), respectively. It leads to correspondence of single-trace operators in the matrix model to integrated vertex operators in the type IIA theory:

$$
\begin{aligned}
\Phi_1 = \frac{1}{N}\,\mathrm{tr}\,\phi \quad &\Leftrightarrow \quad \mathcal{V}_\phi(0) \equiv g_s^2 \int d^2z\, \hat{V}_{\frac{1}{2},+1}(z)\,\hat{\bar{V}}_{-\frac{1}{2},-1}(\bar{z}),\\
\Psi_1 = \frac{1}{N}\,\mathrm{tr}\,\psi \quad &\Leftrightarrow \quad \mathcal{V}_\psi(0) \equiv g_s^2 \int d^2z\, \hat{T}_{-\frac{1}{2}}(z)\,\hat{\bar{V}}_{-\frac{1}{2},-1}(\bar{z}),\\
\bar{\Psi}_1 = \frac{1}{N}\,\mathrm{tr}\,\bar{\psi} \quad &\Leftrightarrow \quad \mathcal{V}_{\bar{\psi}}(0) \equiv g_s^2 \int d^2z\, \hat{V}_{\frac{1}{2},+1}(z)\,\hat{\bar{T}}_{\frac{1}{2}}(\bar{z}),\\
\frac{1}{N}\,\mathrm{tr}\,(-iB) \quad &\Leftrightarrow \quad \mathcal{V}_B(0) \equiv g_s^2 \int d^2z\, \hat{T}_{-\frac{1}{2}}(z)\,\hat{\bar{T}}_{\frac{1}{2}}(\bar{z}),
\end{aligned}
\tag{49}
$$

where the bare string coupling g_s put in the right hand sides is to count the number of external lines of amplitudes in the IIA theory. Note (49) is consistent with the identification in (8)–(11). Furthermore, it can be naturally extended as

$$
\begin{aligned}
\Phi_{2k+1} = \frac{1}{N}\,\mathrm{tr}\,\phi^{2k+1} + (\text{mixing}) \quad &\Leftrightarrow \quad \mathcal{V}_\phi(k) \equiv g_s^2 \int d^2z\, \hat{V}_{k+\frac{1}{2},+1}(z)\,\hat{\bar{V}}_{-k-\frac{1}{2},-1}(\bar{z}),\\
\Psi_{2k+1} = \frac{1}{N}\,\mathrm{tr}\,\psi^{2k+1} + (\text{mixing}) \quad &\Leftrightarrow \quad \mathcal{V}_\psi(k) \equiv g_s^2 \int d^2z\, \hat{T}_{-k-\frac{1}{2}}(z)\,\hat{\bar{V}}_{-k-\frac{1}{2},-1}(\bar{z}),\\
\bar{\Psi}_{2k+1} = \frac{1}{N}\,\mathrm{tr}\,\bar{\psi}^{2k+1} + (\text{mixing}) \quad &\Leftrightarrow \quad \mathcal{V}_{\bar{\psi}}(k) \equiv g_s^2 \int d^2z\, \hat{V}_{k+\frac{1}{2},+1}(z)\,\hat{\bar{T}}_{k+\frac{1}{2}}(\bar{z})
\end{aligned}
\tag{50}
$$

for higher $k (= 1, 2, \cdots)$. Since the “tachyons” of the negative winding $\int d^2z\, \hat{T}_{-k-\frac{1}{2}}(z)\,\hat{\bar{T}}_{k+\frac{1}{2}}(\bar{z})$ $(k = 0, 1, 2, \cdots)$ are invariant under $\hat{Q}_+$ and $\hat{\bar{Q}}_-$, they are expected to be mapped to

$\{\frac{1}{N}\,\mathrm{tr}\,(-iB)^{k+1}\}$ $(k = 0, 1, 2, \cdots)$ perhaps with some mixing terms. We see in (50) that the powers of matrices are interpreted as windings or momenta in the S^1 direction of the type IIA theory. This interpretation is similar to what is discussed in refs. [26, 27]: a positive power k of a matrix variable in the Penner model correctly describe the "tachyons" with negative momentum $-k$ in the $c = 1$ string on S^1. Furthermore, in the literature the positive momentum "tachyons" are represented by introducing source terms of an external matrix via the Kontsevich-Miwa transformation in the Penner model. In turn, it is natural to expect in our case that the positive winding "tachyons" $\int d^2z\, \hat{T}_{-k-\frac{1}{2}}(z)\,\hat{\bar{T}}_{k+\frac{1}{2}}(\bar{z})$ $(k = -1, -2, \cdots)$ in the type IIA theory are expressed in a similar manner in the SUSY double-well matrix model.

Note that (R$-$, R$+$) operators are singlets under the target-space SUSYs $\hat{Q}_+, \hat{\bar{Q}}_-$, and appear to have no counterpart in the matrix model side. Since the expectation value of operators measuring an RR charge $\langle\Phi_{2k+1}\rangle_0$ does not vanish as seen in (13), the matrix model is considered to correspond to the type IIA theory on a nontrivial background of the (R$-$, R$+$) fields. We may introduce the (R$-$, R$+$) background in the form of vertex operators, when the strength of the background $(\nu_+ - \nu_-)$ is small.

5. Correspondence between the matrix model and the type IIA theory

Correlation functions among integrated vertex operators in the type IIA theory on the trivial background are given by

$$\left\langle \prod_i \mathcal{V}_i \right\rangle = \frac{1}{\mathrm{Vol.(CKV)}} \int \mathcal{D}(x, \varphi, H, \mathrm{ghosts})\, e^{-S_{\mathrm{CFT}}} e^{-S_{\mathrm{int}}} \prod_i \mathcal{V}_i, \tag{51}$$

where

$$\begin{aligned} S_{\mathrm{CFT}} &= \frac{1}{2\pi}\int d^2z \left[\partial x\bar{\partial}x + \partial\varphi\bar{\partial}\varphi + \frac{1}{2}\sqrt{\hat{g}}\hat{R}\varphi + \partial H\bar{\partial}H + (\mathrm{ghosts})\right], \\ S_{\mathrm{int}} &= \mu_1 \mathcal{V}_B^{(0,0)}(0) \equiv \mu_1 \int d^2z\, \hat{T}^{(0)}_{-\frac{1}{2}}(z)\hat{\bar{T}}^{(0)}_{\frac{1}{2}}(\bar{z}). \end{aligned} \tag{52}$$

The 0-picture (NS, NS) "tachyon" is given by

$$\begin{aligned} \hat{T}^{(0)}_{-\frac{1}{2}}(z) &= e^{\pi\beta(ip_{\bar{h}} - i\frac{1}{2}p_{\bar{x}})}\,\frac{i}{\sqrt{2}}\,e^{iH - i\frac{1}{2}x + \frac{1}{2}\varphi}(z), \\ \hat{\bar{T}}^{(0)}_{\frac{1}{2}}(\bar{z}) &= e^{-\pi\beta(-ip_h + i\frac{1}{2}p_x)}\,\frac{i}{\sqrt{2}}\,e^{-i\bar{H} + i\frac{1}{2}\bar{x} + \frac{1}{2}\bar{\varphi}}(\bar{z}). \end{aligned} \tag{53}$$

We consider correlation functions in the IIA theory on a nontrivial (R$-$, R$+$) background as a form

$$\left\langle\!\!\left\langle \prod_i \mathcal{V}_i \right\rangle\!\!\right\rangle \equiv \left\langle \left(\prod_i \mathcal{V}_i\right) e^{W_{\mathrm{RR}}} \right\rangle. \tag{54}$$

The background W_{RR} is invariant under the target-space SUSYs:

$$\begin{aligned} W_{\mathrm{RR}} &= (\nu_+ - \nu_-)\sum_{k\in\mathbf{Z}} a_k\, \mu_1^{k+1}\mathcal{V}_k^{\mathrm{RR}}, \\ \mathcal{V}_k^{\mathrm{RR}} &\equiv \begin{cases} \int d^2z\, \hat{V}_{k,-1}(z)\hat{\bar{V}}_{-k,+1}(\bar{z}) & (p_\ell = 1 - |k|,\ k \le 0) \\ \int d^2z\, \hat{V}^{(\mathrm{nonlocal})}_{-k,-1}(z)\hat{\bar{V}}^{(\mathrm{nonlocal})}_{k,+1}(\bar{z}) & (p_\ell = 1 + |k|,\ k \ge 1) \end{cases} \end{aligned} \tag{55}$$

with a_k being numerical constants. Although the nonlocal operators in (55) with $p_\ell > 1$ do not satisfy the Dirac equation constraint on the trivial background, these operators are necessary to make correspondence with the matrix model as we see later. Since the RR background possibly change the on-shell condition, it would be acceptable. We treat the RR background for $(\nu_+ - \nu_-)$ small as

$$\left\langle\!\!\left\langle \prod_i \mathcal{V}_i \right\rangle\!\!\right\rangle \equiv \left\langle \left(\prod_i \mathcal{V}_i\right) e^{W_{\rm RR}} \right\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle \left(\prod_i \mathcal{V}_i\right) (W_{\rm RR})^n \right\rangle, \tag{56}$$

and the picture is adjusted by hand so that the total picture is equal to -2.

In computation of amplitudes in the type IIA theory, we consider the so-called $s = 0$ amplitude in the Liouville theory, which is interpreted as a bulk amplitude insensitive to details of the Liouville wall [28]. It is somewhat similar to considering the leading nontrivial contribution for small $(\nu_+ - \nu_-)$, because higher orders of $(\nu_+ - \nu_-)$ seems to detect a cigar geometry deformed from the two-dimensional target space (Liouville direction)$\times$(S^1 with self-dual radius) [16]. The direction to the Liouville wall corresponds to the direction to the tip of the cigar. Computation in the Liouville theory [6] yields

$$\left\langle \mathcal{V}_B(0)\, \mathcal{V}_\phi(k)\, \mathcal{V}_\ell^{\rm RR} \right\rangle = -g_s^4\, \delta_{k,\ell}\, (2\ln\mu_1)\, e^{i2\pi\beta(-k^2-\frac{1}{2}k+\frac{1}{4})}, \tag{57}$$

$$\begin{aligned} \left\langle \mathcal{V}_\phi(k_1), \mathcal{V}_\phi(k_2)\, \mathcal{V}_{\ell_1}^{\rm RR}\, \mathcal{V}_{\ell_2}^{\rm RR} \right\rangle &= g_s^4\, \left(\delta_{\ell_1,k_1+k_2}\delta_{\ell_2,-1} + (\ell_1 \leftrightarrow \ell_2)\right)\, c_L (2\ln\mu_1)^2 \\ &\quad \times \frac{\pi}{2} \left(\frac{(k_1+k_2)!}{k_1! k_2!}\right)^2 e^{-i\pi\beta\{\sum_{i=1}^2 (k_i+\frac{1}{2})^2 + \sum_{i=1}^2 \ell_i^2\}}. \end{aligned} \tag{58}$$

In the derivation of (58), we encountered the integral

$$\int d^2z\, z^\alpha \bar{z}^{\bar\alpha} (1-z)^\beta (1-\bar z)^{\bar\beta} = \pi \frac{\Gamma(\bar\alpha+1)\Gamma(\bar\beta+1)}{\Gamma(\bar\alpha+\bar\beta+2)} \frac{\Gamma(-\alpha-\beta-1)}{\Gamma(-\alpha)\Gamma(-\beta)} \tag{59}$$

with

$$\alpha = \bar\alpha = k_1 + k_2, \qquad \beta = \bar\beta = -k_1 - 1, \qquad (k_1, k_2 = 0, 1, 2, \cdots). \tag{60}$$

This expression is indefinite. We computed it by regularizing as

$$\alpha \to \alpha + \epsilon, \qquad \bar\alpha \to \bar\alpha + \epsilon, \qquad \beta \to \beta + \epsilon, \qquad \bar\beta \to \bar\beta + \epsilon, \tag{61}$$

where $\epsilon = \frac{1}{c_L V_L}$. $V_L \equiv 2\ln\frac{1}{\mu_1}$ is the Liouville volume, and c_L is a numerical constant. The point of the regularization is preserving mutual locality of vertex operators due to the homogeneous shifts.

Let us identify the coupling μ_1 of the Liouville interaction $S_{\rm int}$ in (52) with the "cosmological constant" ω by an appropriate shift of the Liouville coordinate. Then, it leads to the identification

$$N\ {\rm tr}(-iB) \cong \frac{1}{4}\mathcal{V}_B^{(0,0)}(0), \tag{62}$$

which is consistent to the last line in (49) (up to the choice of the picture) with

$$\frac{1}{N} \cong g_s. \tag{63}$$

Also, introducing coefficients c_k, d_k, $\bar d_k$, we precisely express the correspondence in (49) and (50) as

$$\Phi_{2k+1} \cong c_k \mathcal{V}_\phi(k), \qquad \Psi_{2k+1} \cong d_k \mathcal{V}_\psi(k), \qquad \bar\Psi_{2k+1} \cong \bar d_k \mathcal{V}_{\bar\psi}(k). \tag{64}$$

We put the overall normalization factor $\mathcal{N}$ in identifying the amplitudes in the matrix-model side and those in the IIA theory side:

$$\langle N\, {\rm tr}(-iB)\, \Phi_{2k+1} \rangle_{C,0} \cong \mathcal{N} g_s^{-2} \left\langle\!\!\left\langle \frac{1}{4}\mathcal{V}_B^{(0,0)}(0)\, c_k \mathcal{V}_\phi(k) \right\rangle\!\!\right\rangle. \tag{65}$$

The left hand side is calculated by using (13):

$$(\mathrm{LHS}) = -\frac{1}{4}\partial_\omega \langle \Phi_{2k+1}\rangle_0 \sim -(\nu_+ - \nu_-)\frac{2^k}{\pi}\frac{(2k+1)!!}{(k+1)!}\,\omega^{k+1}\ln\omega. \tag{66}$$

On the other hand, under a suitable choice of the picture, leading nontrivial contribution for $(\nu_+ - \nu_-)$ small to the right hand side is

$$\begin{aligned}\frac{1}{4}\mathcal{N} g_s^{-2} c_k \langle \mathcal{V}_B(0)\,\mathcal{V}_\phi(k)\,W_{\mathrm{RR}}\rangle &= \frac{1}{4}\mathcal{N} g_s^{-4} c_k (\nu_+ - \nu_-)\sum_{\ell\in\mathbf{Z}} a_\ell\,\omega^{\ell+1}\left\langle \mathcal{V}_B(0)\,\mathcal{V}_\phi(k)\,\mathcal{V}_\ell^{\mathrm{RR}}\right\rangle \\ &= -\frac{1}{2}(\nu_+ - \nu_-)\mathcal{N} c_k\, a_k\,\omega^{k+1}(\ln\omega)\,e^{i2\pi\beta(-k^2-\frac{1}{2}k+\frac{1}{4})}, \end{aligned}\tag{67}$$

where (57) was used. The identification (65) leads to

$$\mathcal{N}\,\hat{c}_k\hat{a}_k\,e^{i\pi\beta\frac{3}{4}} = \frac{2}{\pi}\frac{(2k+1)!}{k!(k+1)!} \tag{68}$$

with

$$\hat{c}_k \equiv c_k\,e^{-i\pi\beta(k+\frac{1}{2})^2}, \qquad \hat{a}_k \equiv a_k\,e^{-i\pi\beta k^2}. \tag{69}$$

Next, let us consider the correspondence

$$\langle \Phi_{2k_1+1}\Phi_{2k_2+1}\rangle_{C,0} \cong \mathcal{N} g_s^{-2}\,\langle\!\langle\, c_{k_1}\mathcal{V}_\phi(k_1)\,c_{k_2}\mathcal{V}_\phi(k_2)\,\rangle\!\rangle\,. \tag{70}$$

Leading nontrivial contribution to the right hand side is obtained from (58) as

$$\begin{aligned}&\mathcal{N} g_s^{-2} c_{k_1}\,c_{k_2}\left\langle \mathcal{V}_\phi(k_1)\,\mathcal{V}_\phi(k_2)\,\frac{1}{2!}(W_{\mathrm{RR}})^2\right\rangle \\ &= \frac{1}{2}\mathcal{N} g_s^{-2} c_{k_1}\,c_{k_2}\,(\nu_+ - \nu_-)^2\sum_{\ell_1,\ell_2\in\mathbf{Z}} a_{\ell_1}\,a_{\ell_2}\,\omega^{\ell_1+\ell_2+2}\left\langle \mathcal{V}_\phi(k_1)\,\mathcal{V}_\phi(k_2)\,\mathcal{V}_{\ell_1}^{\mathrm{RR}}\,\mathcal{V}_{\ell_2}^{\mathrm{RR}}\right\rangle \\ &= (\nu_+ - \nu_-)^2\mathcal{N} g_s^2 c_L\,\hat{c}_{k_1}\,\hat{c}_{k_2}\,\hat{a}_{k_1+k_2}\,\hat{a}_{-1}\,2\pi\left(\frac{(k_1+k_2)!}{k_1!k_2!}\right)^2\omega^{k_1+k_2+1}\,(\ln\omega)^2, \end{aligned}\tag{71}$$

while the result of the left hand side is given by (28). Comparing these, we find the same dependence on $\nu_\pm$ and ω for any k_1 and k_2. In addition, we have an equation:

$$\left(\frac{\hat{c}_{k_1}}{(2k_1+1)!}\right)\left(\frac{\hat{c}_{k_2}}{(2k_2+1)!}\right)\left(\hat{a}_{k_1+k_2}(k_1+k_2)!(k_1+k_2+1)!\right) = -\frac{1}{4\pi^3}\frac{1}{\mathcal{N} c_L\hat{a}_{-1}}, \tag{72}$$

which is solved as

$$\hat{c}_k = \hat{c}_0\,e^{\gamma k}\,(2k+1)!, \qquad \hat{a}_k = \frac{\hat{a}_0\,e^{-\gamma k}}{k!(k+1)!} \qquad (k=0,1,2,\cdots) \tag{73}$$

with γ being a numerical constant and

$$\hat{c}_0^2\hat{a}_0 = -\frac{1}{4\pi^3}\frac{1}{\mathcal{N} c_L\,\hat{a}_{-1}}. \tag{74}$$

Remarkably, (68) is consistent to (73). It serves a quite nontrivial check of the correspondence.

Furthermore, the correspondence of the amplitudes containing fermions

$$\begin{aligned} \langle \Psi_1 \bar{\Psi}_1 \rangle_{C,0} &\cong \mathcal{N} g_s^{-2} \left\langle\!\!\left\langle\, d_0 \mathcal{V}_\psi(0)\, \bar{d}_0 \mathcal{V}_{\bar{\psi}}(0) \,\right\rangle\!\!\right\rangle, \\ \langle \Psi_3 \bar{\Psi}_3 \rangle_{C,0} &\cong \mathcal{N} g_s^{-2} \left\langle\!\!\left\langle\, d_1 \mathcal{V}_\psi(1)\, \bar{d}_1 \mathcal{V}_{\bar{\psi}}(1) \,\right\rangle\!\!\right\rangle \end{aligned} \tag{75}$$

yields

$$d_0 \bar{d}_0 = \frac{1}{4} c_0, \qquad d_1 \bar{d}_1 = -\frac{3}{\pi^2} \frac{c_0}{a_0^2}. \tag{76}$$

It leads to the precise correspondence between the supercharges:

$$Q \cong \frac{d_0}{c_0} \hat{Q}_+, \qquad \bar{Q} \cong \frac{\bar{d}_0}{c_0} \hat{\bar{Q}}_-. \tag{77}$$

So far, the correspondence seems consistent at the level of planar or tree amplitudes. Furthermore, the consistency is checked in the torus partition function [6].

6. Nonperturbative SUSY breaking in the matrix model

In this section, we compute instanton effects in the matrix model which are nonperturbative in $1/N$. Although such effects are of the order e^{-N} and vanish in the simple large N limit, we will see that they are nonvanishing in a double scaling limit

$$N \to \infty, \quad \omega \to 0 \quad \text{with } t \equiv N^{2/3}\omega \text{ fixed.} \tag{78}$$

The partition function of the matrix model given by the action (1) is expressed as

$$\begin{aligned} Z &= \int d^{N^2}\phi\, e^{-N\frac{1}{2}\operatorname{tr}(\phi^2-\mu^2)^2} \det(\phi \otimes \mathbf{1} + \mathbf{1} \otimes \phi) \\ &= \tilde{C}_N \int \Big(\prod_{i=1}^N d\lambda_i \Big) \triangle(\lambda)^2 \prod_{i,j=1}^N (\lambda_i + \lambda_j)\, e^{-N\sum_{i=1}^N \frac{1}{2}(\lambda_i^2-\mu^2)^2}, \end{aligned} \tag{79}$$

after integrating out matrices other than ϕ. Here, $\mathbf{1}$ is an $N \times N$ unit matrix, λ_i $(i = 1, \cdots, N)$ are eigenvalues of ϕ, and $\triangle(\lambda)$ denotes the Vandermonde determinant $\triangle(\lambda) = \prod_{i>j}(\lambda_i - \lambda_j)$. $\tilde{C}_N$ is an numerical factor depending only on N given by

$$\frac{1}{\tilde{C}_N} = \int \Big(\prod_{i=1}^N d\lambda_i \Big) \triangle(\lambda)^2\, e^{-N\sum_{i=1}^N \frac{1}{2}\lambda_i^2} = (2\pi)^{\frac{N}{2}} \frac{\prod_{k=0}^N k!}{N^{\frac{N^2}{2}}}. \tag{80}$$

Contributions to the partition function are divided by sectors labeled by the filling fraction (ν_+, ν_-) as

$$Z = \sum_{\nu_- N=0}^{N} \frac{N!}{(\nu_+ N)!(\nu_- N)!} Z_{(\nu_+,\nu_-)} \tag{81}$$

with

$$\begin{aligned} Z_{(\nu_+,\nu_-)} \equiv \tilde{C}_N \int_0^\infty \left(\prod_{i=1}^{\nu_+ N} d\lambda_i \right) \int_{-\infty}^0 \left(\prod_{j=\nu_+ N+1}^{N} d\lambda_j \right) \left(\prod_{n=1}^N 2\lambda_n \right) \left\{ \prod_{n>m} (\lambda_n^2 - \lambda_m^2)^2 \right\} \\ \times e^{-N\sum_{i=1}^N \frac{1}{2}(\lambda_i^2-\mu^2)^2}. \end{aligned} \tag{82}$$

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Here, it is easy to see

$$Z_{(\nu_+,\nu_-)} = (-1)^{\nu_- N} Z_{(1,0)}, \tag{83}$$

which leads to the vanishing partition function:

$$Z = (1 + (-1))^N Z_{(1,0)} = 0. \tag{84}$$

In order for expectation values normalized the partition function to be well-defined, we regularize the partition function by introducing a factor $e^{-i\alpha\nu_- N}$ with small α in front of $Z_{(\nu_+,\nu_-)}$. The regularized partition function becomes

$$Z_\alpha \equiv \sum_{\nu_- N=0}^{N} \frac{N!}{(\nu_+ N)!(\nu_- N)!} e^{-i\alpha\nu_- N} Z_{(\nu_+,\nu_-)} = (1 - e^{-i\alpha})^N Z_{(1,0)}. \tag{85}$$

Notice that calculations in perturbation theory of $1/N$ in sections 2 and 3 concern the partition function in a single sector ($Z_{(\nu_+,\nu_-)}$), in which such a regularization was not needed. On the other hand, since nonperturbative contributions to be computed here possibly communicate among various sectors of filling fractions, we should consider the total partition function (81) and its vanishing value requires the regularization.

The expectation value of $\frac{1}{N}\operatorname{tr}(iB)$ under the regularization (85) is expressed as

$$\left\langle \frac{1}{N}\operatorname{tr}(iB) \right\rangle_\alpha = \left\langle \frac{1}{N}\operatorname{tr}(\phi^2 - \mu^2) \right\rangle_\alpha = \frac{1}{N^2}\frac{1}{Z_\alpha}\frac{\partial}{\partial(\mu^2)} Z_\alpha = \left\langle \frac{1}{N}\operatorname{tr}(\phi^2 - \mu^2) \right\rangle^{(1,0)} \tag{86}$$

due to a cancellation of the factor $(1 - e^{-i\alpha})^N$ in (85) between the numerator and the denominator. The regularized expectation value $\left\langle \frac{1}{N}\operatorname{tr}(iB) \right\rangle_\alpha$ is independent of α and well-defined in the limit $\alpha \to 0$, and thus serves as an order parameter for spontaneous SUSY breaking.

6.1. Orthogonal polynomials

Under the change of variables $x_i = \lambda_i^2 - \mu^2$, the partition function $Z_{(1,0)}$ defined in (82) reduces to Gaussian matrix integrals

$$Z_{(1,0)} = \tilde{C}_N \int_{-\mu^2}^{\infty} \left(\prod_{i=1}^{N} dx_i \right) \triangle(x)^2 \, e^{-N\sum_{i=1}^{N} \frac{1}{2}x_i^2}. \tag{87}$$

It seems almost trivial, but a nontrivial effect possibly arises from the boundary of the integration region. Ref. [21] mentions that the boundary effect is nonperturbative in $1/N$.

Let us consider polynomials

$$P_n(x) = x^n + \sum_{i=0}^{n-1} p_n^{(i)} x^i \qquad (n = 0, 1, 2, \cdots) \tag{88}$$

with the coefficient of the top degree (x^n) fixed to 1. The coefficients $p_n^{(i)}$ are uniquely determined so that the orthogonality relation

$$(P_n, P_m) \equiv \int_{-\mu^2}^{\infty} dx \, e^{-\frac{N}{2}x^2} P_n(x) \, P_m(x) = h_n \delta_{n,m} \tag{89}$$

is satisfied. Similar to the case without a boundary [29], we have recursion relations of the form

$$xP_m(x) = P_{m+1}(x) + S_m P_m(x) + R_m P_{m-1}(x), \tag{90}$$

$$h_m = R_m h_{m-1}. \tag{91}$$

The one-point function is expressed as a sum of the coefficients S_n:

$$\left\langle \frac{1}{N} \operatorname{tr}(\phi^2 - \mu^2) \right\rangle^{(1,0)} = \frac{1}{N} \sum_{n=0}^{N-1} S_n = \frac{1}{N^2} e^{-\frac{N}{2}\mu^4} \sum_{n=0}^{N-1} \frac{1}{h_n} P_n(-\mu^2)^2, \tag{92}$$

where the last equality follows from the identity

$$\int_{-\mu^2}^{\infty} dx \, \frac{d}{dx} \left(e^{-\frac{N}{2}x^2} P_n(x)^2 \right) = -e^{-\frac{N}{2}\mu^4} P_n(-\mu^2)^2. \tag{93}$$

6.2. One-instanton contribution

In what follows, we take into account the boundary effects in an iterative manner. First, we simply neglect the boundary effect by changing the lower bound of the integral (89) to $-\infty$. The orthogonal polynomials in this case, denoted by $P_n^{(H)}(x)$, are given by the Hermite polynomials:

$$P_n^{(H)}(x) = \frac{1}{(2N)^{n/2}} H_n \left(\sqrt{\frac{N}{2}} \, x \right) \tag{94}$$

with

$$H_n(x) \equiv (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \tag{95}$$

and coefficients

$$S_n^{(H)} = 0, \qquad R_n^{(H)} = \frac{n}{N}, \qquad h_n^{(H)} = \sqrt{2\pi} \, \frac{n!}{N^{n+\frac{1}{2}}}. \tag{96}$$

Note that the boundary $x = -\mu^2$ corresponds to the local maximum of the original double-well potential $\frac{1}{2}(\lambda^2 - \mu^2)^2$, and k eigenvalues sitting at the local maximum give k-instanton contribution as discussed in [30]. Computing the one-point function (92) by using the orthogonal polynomials (94) gives rise to one-instanton contribution:

$$\left. \left\langle \frac{1}{N} \operatorname{tr}(\phi^2 - \mu^2) \right\rangle^{(1,0)} \right|_{1-\text{inst.}} = \frac{e^{-z^2}}{\sqrt{2\pi}\, N^{3/2}} \frac{1}{2^N (N-1)!} \left[H_N(z)^2 - H_{N-1}(z) \, H_{N+1}(z) \right], \tag{97}$$

where

$$z \equiv \sqrt{\frac{N}{2}} \, \mu^2 = \sqrt{2N} \, (1 + 2\omega), \tag{98}$$

and the relation

$$\sum_{k=0}^{n-1} \frac{1}{2^k k!} H_k(x)^2 = \frac{1}{2^n (n-1)!} \left[H_n(x)^2 - H_{n-1}(x) \, H_{n+1}(x) \right] \tag{99}$$

was used. Upon taking the double scaling limit in (97), the following asymptotic formula plays a relevant role [7]:

$$e^{-x^2/2} H_n(x) = \pi^{\frac{1}{4}} 2^{\frac{n}{2}+\frac{1}{4}} n^{-\frac{1}{12}} \sqrt{n!} \left[\mathrm{Ai}(s) + \mathcal{O}(n^{-2/3}) \right] \tag{100}$$

which is valid for large n with

$$x = \sqrt{2n+1} + \frac{s}{\sqrt{2}\, n^{1/6}}, \tag{101}$$

and the Airy function is defined by

$$\mathrm{Ai}(s) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} dz\, e^{-isz - \frac{i}{3} z^3}. \tag{102}$$

Applying (100) to (97) yields the result in the double scaling limit (78):

$$\left. \left\langle \frac{1}{N} \mathrm{tr}\,(\phi^2 - \mu^2) \right\rangle^{(1,0)} \right|_{1-\mathrm{inst.}} = N^{-4/3} \left[\mathrm{Ai}'(4t)^2 - 4t\mathrm{Ai}(4t)^2 + \mathcal{O}(N^{-2/3}) \right] \tag{103}$$

$$= N^{-4/3} \frac{1}{32\pi t} e^{-\frac{32}{3} t^{3/2}} \left[1 + \sum_{n=1}^{\infty} a_n^{(1)}\, t^{-\frac{3}{2} n} \right] \tag{104}$$

with $a_1^{(1)} = -\frac{17}{192}$, $a_2^{(1)} = \frac{1225}{73728}$, $a_3^{(1)} = -\frac{199115}{42467328}, \cdots$. The double scaling limit (78) is expected from the $c = -2$ topological gravity with the string susceptibility $\gamma = -1$ because the free energy at the spherical topology behaves as $N^2 \omega^{2-\gamma} = t^{2-\gamma}$. However, use of this expectation is established in perturbative computations as pointed out in section 2, while it is nontrivial that the nonperturbative contribution (103) obeys this scaling. The exponential factor $e^{-\frac{32}{3} t^{3/2}}$ in (104) indicates the weight of one-instanton configurations which remains finite in the double scaling limit. Also, the power series with respect to $t^{-3/2}$ can be regarded as perturbative contributions to all orders around the one-instanton background.

6.3. Leading order two-instanton contribution

In order to compute effects from higher instantons, we consider corrections to replacing P_n by $P_n^{(H)}$ in the previous subsection:

$$\begin{aligned} &P_n(x) = P_n^{(H)}(x) + \tilde{P}_n(x), \\ &S_n = S_n^{(H)} + \tilde{S}_n, \qquad R_n = R_n^{(H)} + \tilde{R}_n, \qquad h_n = h_n^{(H)} + \tilde{h}_n. \end{aligned} \tag{105}$$

Treating quantities with tildes at the linearized level gives rise to two-instanton effects. As a result of the computation in [7], we have the two-instanton contribution:

$$\left. \left\langle \frac{1}{N} \mathrm{tr}\,(\phi^2 - \mu^2) \right\rangle^{(1,0)} \right|_{2-\mathrm{inst.}} = N^{-4/3} \frac{1}{(64\pi)^2\, t^{5/2}} e^{-\frac{64}{3} t^{3/2}} \left[1 + \mathcal{O}(t^{-3/2}) \right]. \tag{106}$$

Both effects from one instanton (104) and from two instantons (106) are of the same order in N and equally contribute in the double scaling limit to the quantity

$$N^{4/3} \left\langle \frac{1}{N} \mathrm{tr}\,(\phi^2 - \mu^2) \right\rangle^{(1,0)} = N^{4/3} \frac{1}{N} \sum_{n=0}^{N-1} \left\{ \tilde{S}_n^{(1)} + \tilde{S}_n^{(2)} + \cdots \right\}. \tag{107}$$

Thus, we can conclude that the nonperturbative effect dynamically breaks the supersymmetry under wave function renormalization absorbing the factor $N^{-4/3}$. As (86) suggests, the renormalization factor can be understood from the fact that the renormalized one-point function is obtained by the t-derivative of the free energy $F_{(1,0)} = -\ln Z_{(1,0)}$ with the factor $(-\frac{1}{4})$ multiplied. The weight of the exponential in (106) is twice that of (104), as it should be from the interpretation of a two-instanton contribution.

7. Summary and Discussion

We computed planar correlation functions in the double-well SUSY matrix model, and discussed its correspondence to two-dimensional type IIA superstring theory on (R−,R+) background by comparing amplitudes in both sides. This is an interesting example of matrix models for superstrings with target-space SUSY, in which various amplitudes are explicitly calculable.

It is interesting to examine the correspondence at deeper level in higher genus or higher point amplitudes and in amplitudes containing special massive operators. Also, it is important to discuss the correspondence in the off-shell formulation such as the hybrid formalism [18].

Next, we explicitly calculated nonperturbative instanton effects in the matrix model. In particular, a closed form expression was obtained for full one-instanton contribution to the one-point function $\left\langle \frac{1}{N}\,\mathrm{tr}(\phi^2-\mu^2)\right\rangle^{(1,0)}$ including all perturbative fluctuations around the one-instanton background. Also, presented was its analytic expression for the leading two-instanton effect with respect to finite but large t. The result shows that the supersymmetry is spontaneously broken by nonperturbative effects due to instantons. In particular, the instanton effects survive in the double scaling limit, which implies that supersymmetry breaking takes place by nonperturbative dynamics in the target space of the type IIA superstring theory. Corresponding Nambu-Goldstone fermions are identified with $\frac{1}{N}\,\mathrm{tr}\,\bar{\psi}$ and $\frac{1}{N}\,\mathrm{tr}\,\psi$ associated with the breaking of Q and $\bar{Q}$, respectively. It is interesting to investigate dynamics of D-branes in the type IIA theory and to reproduce the instanton contributions from the type IIA theory side.

Moreover, numerical results for full nonperturbative contribution to the one-point function and the free energy $F_{(1,0)}$ are obtained in [7], where the numerical results up to $N = 1,000,000$ are extrapolated to $N = \infty$ in the double scaling limit. The result for the free energy seems to be smooth even at $t = 0$ which corresponds to the strongly coupled limit of the type IIA superstring theory. It might suggest the existence of an S-dual theory. It would be intriguing to obtain an analytic expression for the full nonperturbative contribution and to identify the S-dual theory.

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