

Formal Proof of Flowers & Ruderman's Instability Mechanism in Magnetic Stars

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Abstract. In 1977, Flowers and Ruderman described a perturbation that destabilized a purely dipolar magnetic field in a star. They considered the effect of cutting the star in half along a plane parallel to the symmetry axis and rotating each half 90 degrees in opposite directions, which would cause the energy of the magnetic field in the exterior of the star to be greatly reduced, just as it happens with a pair of aligned magnets. We formally solve for the energy of the external magnetic field and check that it decreases monotonously along the entire rotation.

We also describe the instability using perturbation theory, and see that it happens due to the work done by the interaction of the magnetic field with surface currents.

Finally, we consider the stabilizing effect of adding a toroidal field by studying the internal energy perturbation when the rotation is not done along a sharp cut, but with a continuous displacement field that switches the direction of rotation across a region of small but finite width. Using these results, we estimate the relative strengths of the toroidal and poloidal field needed to make the star stable to this displacement and see that the energy of the toroidal field required for this is much smaller than the energy of the poloidal field.

1. Introduction

Large-scale magnetic fields are present in many stellar objects, and they appear to be long-lived since they do not evolve in a timescale accessible to observations.

These objects are mostly stably stratified so dynamo effects are expected to be irrelevant in keeping the strength of the magnetic field, which should be in a state of stable equilibrium.

Even though these long-lived fields have been known to exist for more than half a century, it has not been possible to find an analytic model for a field that has been shown to be in stable equilibrium. However, stable configurations have been found to exist via numerical calculations, where an initially random field usually evolves into an approximately axisymmetric configuration that is a combination of toroidal and poloidal components of similar energies. In these simulations, once a stable configuration has been achieved, the decay of the field is driven by Ohmic dissipation, and it can be seen to evolve in a timescale comparable to the lifetime of the star.

With regard to the stability of purely poloidal fields, Flowers & Ruderman 1977 argued that any poloidal field that has the shape of a dipole outside of the star should be unstable. In an analogous way to a couple of aligned magnets, one half of the star could turn with respect to the other producing a quadrupole (as shown in figures 1 and 2).



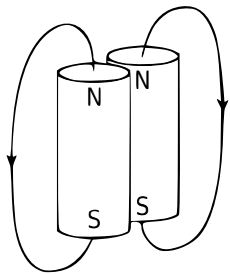


Figure 1. A couple of aligned magnets is in an unstable state of equilibrium.

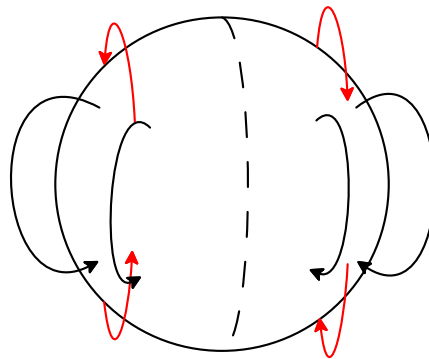
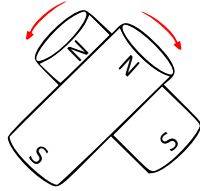


Figure 2. In a similar way to the aligned magnets, each half of a star with a dipole field could rotate in opposite directions, until it reaches a state of stable equilibrium.

We formally prove Flowers and Ruderman's instability for a pure dipole, and study the stabilizing effect of a toroidal field.

2. Formal proof of the instability

To prove Flowers and Ruderman's instability, the following simplifying assumptions are made:

- The star is perfectly spherical, and all hydrodynamic quantities are functions only of r .
- Magnetic flux is completely frozen in the fluid.
- Outside the star there is a perfect vacuum.

Under these conditions, only the energy of the magnetic field outside the star changes, and this was calculated as

$$E = \frac{1}{8\pi} \int_{r>R} \mathbf{B}^2 dV = E_0 [1 + \sin^2 \Omega (A - 1)] , \quad (1)$$

where E_0 is the initial energy of the dipole outside the star, Ω is the angle of rotation of each half of the star, and A is a numerical constant that results from an infinite series that could not be summed analytically. Also, a quantity Υ defined as

$$\Upsilon = \frac{R^3}{8\pi} \int_{4\pi} (B_r^2)_{r=R} d\Omega, \quad (2)$$

was considered. Since the field is only displaced on the surface, Υ is equal to a constant along the entire rotation. It was shown that due to the conservation of Υ , the final energy is smaller than the initial one, which implies that the constant A in (1) is smaller than 1, and the energy decreases monotonically with the angle of rotation.

Using perturbation theory, it was seen that all the net work done was due to surface currents produced by the perturbation.

3. Previous Misconception

Some works, for instance, Roberts 1981 and Braithwaite & Nordlund 2006, mentioned that a second cut in a direction perpendicular to the first one would produce a configuration which resembles a quadrupole. Additionally, they expected that this process could be repeated *ad*

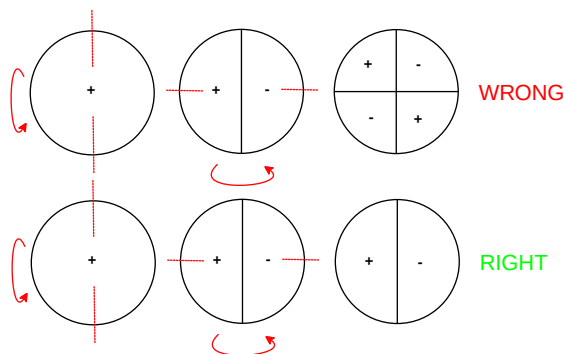


Figure 3. Performing two cuts to the dipole. The top image shows the wrong picture, where an octupole is produced.

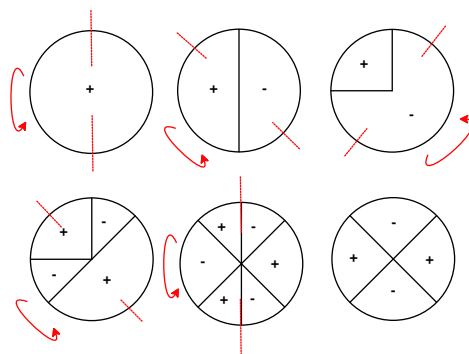


Figure 4. Performing several cuts to the dipole in order to obtain an octupole configuration.

infinitum to produce a configuration that resembles an arbitrary multipole. However, this process breaks down just at the second cut.

In this figure, the star is observed from the top of the symmetry axis, and the plus and minus signs indicate field coming out or into the star. The second cut does not produce the octupole, but simply returns the star to the quadrupole configuration.

It is possible to obtain the quadrupole with a significant number of cuts, however, through this process the energy is not monotonically reduced, so the star cannot actually follow these displacements.

We therefore expect this mechanism to affect only the initial axisymmetric field.

4. Stabilizing effect of a toroidal field

With a toroidal field, a cut sharp is not possible. However, one can consider each side of the star to rotate in opposite directions, with a thin region in which the direction of rotation switches direction (see figure 5).

In this case, the “bending” of toroidal field lines opposes the displacement, so for a sufficiently strong toroidal field we would expect the field to be stable to Flowers & Ruderman’s instability mechanism. We considered particular models for the poloidal and toroidal components with variable strength, which resulted in a dipole field outside the star. The poloidal field was given by

$$\mathbf{B}_P = \nabla\alpha \times \nabla\phi, \quad \alpha(r) = \frac{35B_p}{16} \left[r^2 - \frac{6}{5} \frac{r^4}{R^2} + \frac{3}{7} \frac{r^6}{R^4} \right] \sin^2 \theta, \quad (3)$$

and the toroidal field was modeled as contained in a torus, as shown in figure 6, and the strength of the field is given by

$$\mathbf{B}_T = B_T \cos^2 \left(\frac{\rho\pi}{2\mu R} \right) \hat{\phi} \quad (4)$$

where the meaning of ρ and μ is illustrated in figure 6.

Under these conditions, we estimated that the system is stable when the ratio of poloidal to total magnetic energy satisfies $E_P/E_T \lesssim 0.99$. This means that a very weak toroidal field is sufficient to stabilize the star.

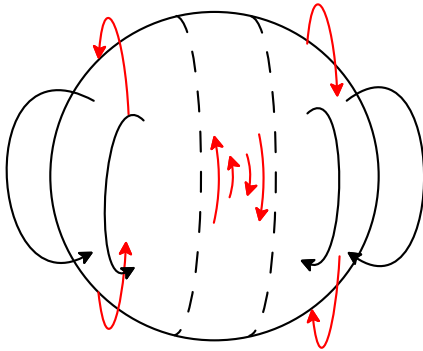


Figure 5. Performing a smooth cut to the star due to the presence of a toroidal field.

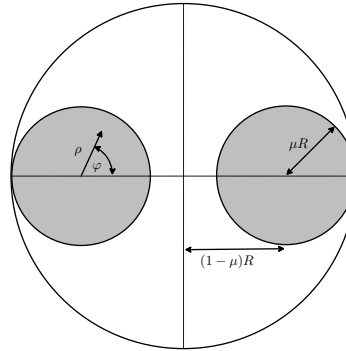


Figure 6. The toroidal field was modeled as contained in a torus of radius μR .

5. Conclusions

We showed that a pure dipole field is unstable to Flowers & Ruderman's instability mechanism by directly evaluating the external energy of the magnetic field for an arbitrary angle of rotation for each half of the star, and proving that it decreases monotonously along the entire rotation.

It was also demonstrated that a second cut could not produce an octupole, as has been said in previous works. To reach an octupole, several cuts must be made, however, in doing so, the external magnetic energy does not decrease monotonically, so this process is not physically possible.

To study how a toroidal field could stabilize the star against the perturbation described by Flowers & Ruderman's instability, I considered particular models for the toroidal and poloidal components of the field. In this case, the cut had to be performed smoothly, as shown in figure 5. Under these conditions, the field was stable to Flowers & Ruderman's instability when the ratio of poloidal to total magnetic energy satisfies $E_P/E_T \lesssim 0.99$, so a very weak toroidal field is sufficient to stabilize the star.

Acknowledgments

This project is supported by FONDECYT Regular Project 1060644, FONDAP Center for Astrophysics (15010003), Proyecto Basal PFB-06/2007 and Proyecto Límite VRI 2010-15.

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