

# Transversal dynamics of relativistic charged particle beams

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**Abstract.** The idea behind this work is to analyse the transversal dynamics of a relativistic charged particle beam. The beam is azimuthally symmetric, focused by a constant magnetic field and supposed initially cold. While mismatched, nonrelativistic and homogeneous beams oscillate with an invariant cold density profile, it is shown that relativistic homogeneous beams progressively heat and lose an important amount of constituents during its magnetic confinement. This heating process starts with phase-space wave breaking, a mechanism observed before in initially inhomogeneous beams. The results have been obtained with full self-consistent  $N$ -particle beam numerical simulations.

## 1. Introduction

Beams composed by charged particles usually evolve to its equilibrium state with the ejection of some representative amount of its constituents as propagate inside the magnetic focusing channel. From the perspective of the beam phase-space, these ejected particles form a rarefied population around a dense one, which congregates the remaining particles. While the first population is recognized as the beam halo, the second one is denoted as beam core [1][2][3][4].

Halo is a well-known issue in beam physics. It has many undesired implications over the accelerator structure, reducing its lifetime and making hard and increasing the costs of its maintenance [5]. Also, there are many applications in which beam halo has to be, if not suppressed, at least minimized [6].

Although beam halo has been identified and measured during these last years [7], its causes remains not satisfactorily understood yet. A beam is a complex system composed by a very large amount of charged particles submitted to forces generated by the internal and external electromagnetic fields. The internal fields are that generated self-consistently by the beam evolution and the external fields are that produced by the focusing system to control the beam dynamics. The way by which beam particles interact causing halo formation directly depends of the beam initial distribution. For initially cold, quasi-homogeneous and mismatched beams, it has been found that, while the initial spurious inhomogeneity is the forerunner mechanism that allows particles to be progressively excited, in fact the envelope mismatch is the mechanism effectively responsible for halo formation [8]. On the other hand, for initially cold, non-homogeneous but envelope matched beams, it has been found that the forerunner mechanism by which particles are ejected from the core is through phase-space wave-



breaking [9]. Once out of the beam core, the ejected particles are permanently excited by a process called charge redistribution [8]. If in this last situation the mismatch is also present, then envelope oscillation acts as other source of energy to excite beam particles [10]. The only difference is that now the coupling is resonant [8].

In the cases mentioned above, nothing has been commented about the relativity of the beam dynamics. In fact, a paraxial approximation has been considered, which implies that, although the beam could be relativistic in the longitudinal direction, its transversal dynamics was purely classic, nonrelativistic. However, notwithstanding the many situations in which this is an adequate approximation, in many others one can assure that this is not. The mass correction introduced by the relativistic effects can be more or less impacting in the dynamics of beam particles and must be understood. For this propose, hereafter an analytical description for the relativistic dynamics of beam particles will be presented.

## 2. The theory

Suppose an initially homogeneous beam of charged particles evolving inside a linear channel. A conducting pipe with a circular transversal section encapsulates the channel. Solenoids provide the constant magnetic field that permeates the channel. For this system, from Newton 2<sup>nd</sup> Law, the dynamics of beam particles could be described by the following equation

$$\frac{d}{dt} \mathbf{P} = \mathbf{F}_{\text{lorentz}} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \quad (1)$$

in which  $\mathbf{P} = \gamma m \mathbf{v}$  is the relativistic linear momentum and  $\mathbf{v}$  is the velocity of the particle,  $\gamma$  is the Lorentz factor,  $q$  is the charge and  $m$  is the rest mass of the particle, and  $\mathbf{E}$  and  $\mathbf{B}$  are respectively the electric and magnetic flux density fields inside the accelerator structure.

Since the beam here is azimuthally symmetric, equation (1) could be expressed in cylindrical coordinates. Consider  $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$  as the unit-vector base in cylindrical coordinates. In this case,  $\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta + \dot{z}\mathbf{e}_z$ . The fields can be expressed as  $\mathbf{E} = E_r\mathbf{e}_r + E_\theta\mathbf{e}_\theta + E_z\mathbf{e}_z$  and  $\mathbf{B} = B_r\mathbf{e}_r + B_\theta\mathbf{e}_\theta + B_z\mathbf{e}_z$ . Then, in this coordinate system, the equation (1) has the following form

$$\begin{aligned} \frac{d}{dt}(\gamma m \dot{r}) - \gamma m r \dot{\theta}^2 &= q(E_r + r\dot{\theta}B_\theta - \dot{z}B_\theta) \\ \frac{1}{r} \frac{d}{dt}(\gamma m r^2 \dot{\theta}) &= -q\dot{r}B_\theta \\ \frac{d}{dt}(\gamma m \dot{z}) &= q\dot{r}B_\theta. \end{aligned} \quad (2)$$

in which, due to the symmetry of the problem, one has that  $E_\theta = E_z = B_r = 0$ ,  $B_z = B_0$ , in which  $B_0$  is the density of magnetic flux produced by the solenoids.  $E_r$  and  $B_\theta$  are the fields generated self-consistently by the particle distribution assigned to the beam. The notation  $\dot{x}$  means  $dx/dt$  with  $x = \{r, \theta, z\}$ . In the limit  $\gamma \rightarrow 1$ , the system of EDOs in equation (1) reduces to that of a nonrelativistic case.

Solving the ordinary differential equation (ODE) for  $\mathbf{e}_\theta$  component, one obtains

$$\gamma m \dot{\theta} = -\frac{qB_0}{2} = \text{constant}. \quad (3)$$

Note that  $\gamma m \dot{\theta}$  is now a conserved quantity. The Larmor frequency  $\dot{\theta}$  of particles in the external density of magnetic field is a function of the absolute value of velocity  $\mathbf{v}$ .

With the aid of equation (2) and supposing a changing of variables of  $t$  to  $z$  one can rewrite the ODE for  $\mathbf{e}_r$  component of equation (1) as follows

$$\frac{d^2 r}{dz^2} + \frac{\dot{\theta}^2}{\dot{z}^2} r = \frac{q}{\gamma m \dot{z}^2} \left[ E_r - \dot{z} B_\theta - \dot{z} B_\theta \left( \frac{dr}{dz} \right)^2 \right], \quad (4)$$

in which also has been use the ODE for  $\mathbf{e}_z$  component of equation (1).

To turn equation (4) completely soluble, it is necessary to determine both  $E_r$  and  $B_\theta$ . Considering that all charged particles have initially negligible velocity, and being  $n_b$  the initial density of charged particles, from the Ampère-Maxwell Law

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}. \quad (5)$$

Bearing mind the symmetry of the problem and the stationary characteristic of the fields, the equation above reduces to

$$B_\theta = \frac{\mu_0 q \dot{z}}{2\pi r} Q(r), \quad (6)$$

in which  $\mu_0$  is the vacuum magnetic permeability.  $Q(r)$  is the charged trapped by a Gauss surface at  $r$

$$Q(r) = \int_0^r n_b(r^*) r^* dr^* d\theta. \quad (7)$$

For the  $E_r$  field, solving the Gauss Law

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad (8)$$

one obtains

$$E_r = \frac{q}{2\pi r \epsilon_0} Q(r), \quad (9)$$

in which  $\rho$  is charge density and  $\epsilon_0$  is the electric permmissivity of the vacuum.

Observe that in fact  $B_\theta = B_\theta(Q)$  and  $E_r = E_r(Q)$ , being possible to express  $B_\theta$  as a function of  $E_r$ . Proceeding in this way

$$B_\theta = \frac{\dot{z}}{c^2} E_r, \quad (10)$$

and inserting this expression in equation (4) with the help provided by equation (3), in view of that  $\gamma = \gamma(v)$  by definition, after some extensive algebra, it is possible to achieve the equation

$$\frac{d^2 \bar{r}}{d\bar{z}^2} + k_0 \frac{1 - \left(\frac{d\bar{r}}{d\bar{z}}\right)^2}{1 + k_0 \bar{r}^2} \bar{r} = K \frac{\left[1 - \left(\frac{d\bar{r}}{d\bar{z}}\right)^2\right]^{3/2}}{(1 + k_0 \bar{r}^2)^{1/2}} \frac{Q(\bar{r})}{\bar{r}}. \quad (11)$$

in which

$$k_0 = \frac{q^2 B_0^2}{4m^2 \dot{z}^2} \quad (12)$$

is the coefficient of magnetic focusing proportionate by the solenoids and

$$K = \frac{q^2}{2\pi \epsilon_0 m \gamma_z \dot{z}^2} \quad (13)$$

is conventionally denominated the perveance of the beam, with  $\gamma_z \equiv 1/\sqrt{1 - \beta_z^2}$  and  $\beta_z = \dot{z}/c$ . To reach at equation (11), also the following change of variables has been also applied

$$\bar{z} = \gamma_z z \quad e \quad \bar{r} = \beta_z r. \quad (14)$$

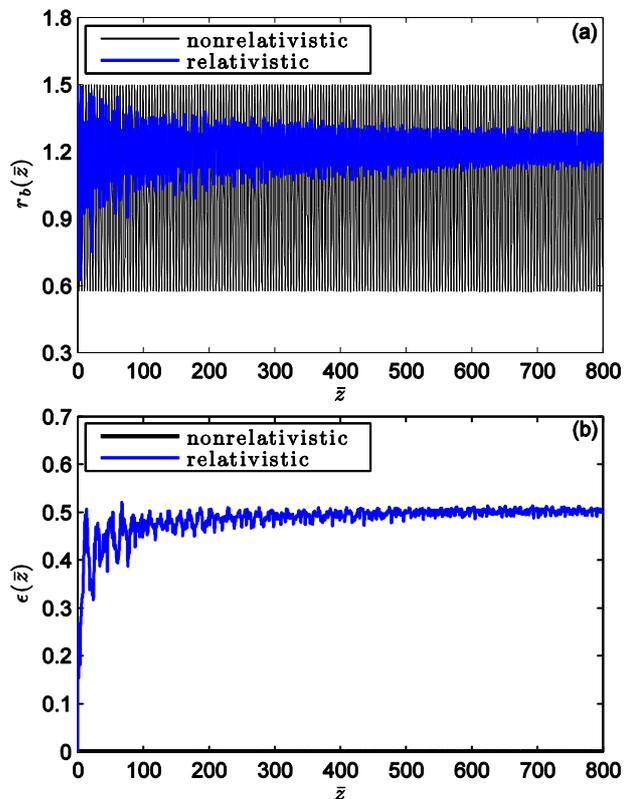
Once in the present case  $\dot{z} = \text{constant}$ , from the point of view of the equations of motion, the axial length and time has the same meaning. Note that equation (11) is a completely nonlinear equation for the transversal dynamics of beam particles. The second order derivative depends not only of nonlinear functions of  $\bar{r}$  but also of  $d\bar{r}/d\bar{z}$ . The term left to the right of equation (11) can be identified as the source for the particles transversal dynamics, since the fields autoconsistently generated by the evolution of density  $n_b$  excite each beam constituent.

### 3. The results, conclusions, and future works

To explore this new information introduced by relativity, a full self-consistent  $N$ -particle beam numerical simulation has been performed. The orbit of each particle that composes the beam has been

evaluated with numerical integration of equation (11). As initial condition, the beam particle density has been considered completely homogenous  $n_b(\bar{r}, \bar{z} = 0) = N_b/\pi r_b^2$ , in which  $r_b$  is the beam border, the envelope. For each particle  $i$  at the radial coordinate  $\bar{r} = \bar{r}_i$ ,  $Q(\bar{r}_i)$  is self-consistently computed. Indeed,  $Q(\bar{r}_i)$  represents the influences of the remaining particles that compose the beam over the one at  $\bar{r}_i$ . The beam particles just interact by the means of the fields inside the accelerator channel. Thus just collective effects are accounted, which is expected in the case of a space-charge dominated beam such is of interest here in this work. The total number of beam particles adopted was  $N_b = 10000$ . The equation (11) has suffered additional scaling to allow particles at the beam border to be in equilibrium ( $d\bar{r}/d\bar{z}|_{\text{border}} = 0$ ).

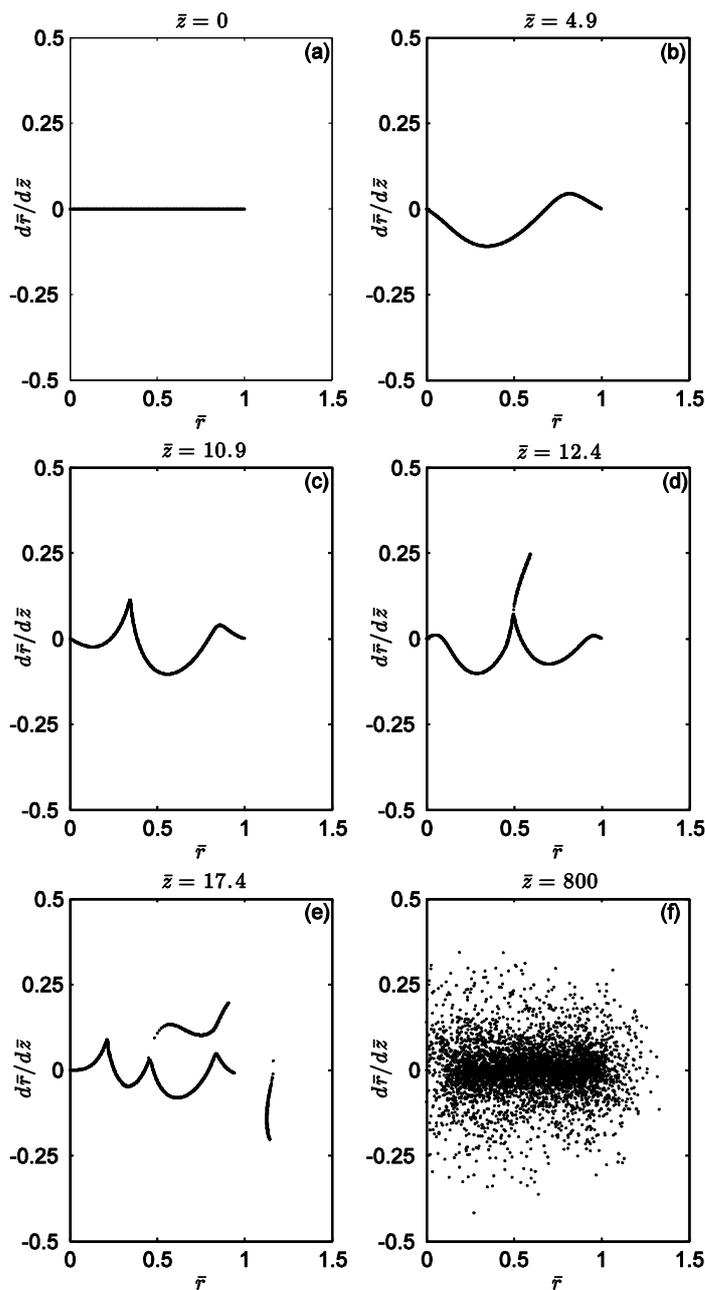
The Figure 1 shows the results obtained with the numerical simulations for two macroscopic beam quantities of interest: beam envelope, in panel (a), and beam emittance, defined as  $\epsilon = \sqrt{4(\langle \bar{r}'^2 \rangle \langle \bar{r}^2 \rangle - \langle \bar{r}' \bar{r} \rangle)}$  with  $\bar{r}' = d\bar{r}/d\bar{z}$ , in panel (b). The angle brackets  $\langle \rangle$  represents phase-space averaging. The beam has been supposed initially mismatched in 50%, that is  $r_0 \equiv r_b(\bar{z} = 0) = 1.5$ , since  $r_{\text{eq}} = 1$ . As expected, beam envelope for the nonrelativistic case oscillates permanently by an undeterminable axial length of the focusing channel. By the other side, the same does not occur if, for this same beam, the relativistic effects are considered and formally accounted. The beam envelope oscillates by a very short axial distance  $\bar{z}$  until it suffers a sharp decay. In panel (b), it is possible to observe that, at this same coordinate, beam emittance experience a suddenly growth. The emittance growth is a macroscopic indicative that microscopically particles are increasing their velocities and improving the amplitude of their orbit in the phase-space. The particles with these attributes are potentially the ones that compose the beam halo and must be investigated.



**Figure 1.** The results of numerical simulations for an initially 50% mismatched, cold and homogeneous beam. In panel (a) the beam envelope  $r_b$  and in panel (b) the beam emittance  $\epsilon$ . While nonrelativistic beams oscillate with fixed amplitude, relativistic beams suffer a strong decay. The emittance growth associated with the relativistic case is pretty perceptible. The beam is composed by  $N_b = 10000$ .

For this purpose, the beam phase-space dynamics is shown in Figure 2, which has been composed by a set of snapshots at adequate axial lengths  $\bar{z}$ . In this case, the beam has been considered initially matched for practical purposes. The beam is initially cold, implying that at  $\bar{z} = 0$  its

appearance at the phase-space is a horizontal line over the  $\bar{r}$  axis (panel (a)). However, as the beam propagates inside the focusing channel, a process of redistribution of charge is observed as panel (b) shows. Although initially particles are disposed homogeneously, during its dynamics, particles can accumulate differently along the beam. This gives rise to the propagation of waves inside the beam density. These phase-space waves are self-consistently amplified until a breaking is observed. Some particles are violently ejected from the beam core. Panel (c) and (d) respectively show the axial length right before and right after the first breaking. Panel (e) shows the beam phase-space picture after the second breaking. Many other wave-breakings occur until the beam reaches its equilibrium, which is presented in panel (f).



**Figure 2.** A sequence of snapshots of the beam phase-space dynamics of an initially cold and relativistic beam with matched envelope. In panel (a), the initial beam picture at the phase-space. During its evolution inside the focusing channel, beam charge starts to be redistributed as Panel (b) shows. This process is amplified and the phase-space density wave across the beam breaks. Panel (c) shows the axial coordinate just before and panel (d) just after the first breaking. Many other wave-breakings occur during the beam confinement process (panel (e) shows the second) until the reaching of its equilibrium (panel (f)).

This is pretty interesting since wave-breaking has been observed before in just initially inhomogeneous beams. However, it is shown that also even initially homogeneous beams can undergo

wave-breaking if the relativistic effects cannot be neglected. If important, the relativistic effects unconditionally induce the breaking of the density waves inside the beam independently of the character of the initial beam density. And this is of great importance for engineering purposes once particles expelled with the wave-breaking will be the ones that will form the beam halo.

In fact, wave-breaking is associated with the nonlinear behavior of the dynamical equation that describes the motion of each particle that composes the beam. The nonlinearity can be introduced by inhomogeneity or by the inclusion of the relativistic effects to the description of the beam dynamics. Considering that both effects can act together inside the beam, in some particular situation, their superposition can eliminate the breaking of density waves. For some particular kind of initial beam density, the inhomogeneity can compensate the relativistic effects and the wave-breaking phenomenon can be suppressed. This kind of particle density can be readily obtained from equation (11). Imposing that  $d\bar{r}/d\bar{z} = 0$  in this equation, solving for the fraction of charge  $Q(\bar{r})$ , and inserting this result in equation (7), one obtains

$$\tilde{n}_b = \frac{1}{2\pi} \frac{\kappa}{K} \frac{2 + \kappa\bar{r}^2}{(1 + \kappa\bar{r}^2)^{3/2}}. \quad (15)$$

If one employs as initial beam density the function above, no wave-breaking will be observed. The beam spatial inhomogeneity produces forces that, when superposed with the ones introduced by the relativistic correction, cancel each other.

Future works will dedicate time to better understand the effects of the envelope mismatch over the beam relativistic dynamics.

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