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Model Kinetic Description For Many-Component Plasma

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Abstract. A consistent derivation of the model linearized collision operator for a multicomponent system is presented. In these results an ambiguity in the choice of coefficients is eliminated, in contrast to the BGK type models. A technique for reconstruction of the model collision integral form based on a known expression for the model linearized operator is proposed. It is shown that the model collision integral in the local (not complete) equilibrium approximation does not contain a complicated exponential, that is common for the BGK type integrals. Boltzmann's H-theorem is proved for our model.

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1. Introduction

As it is well known the Balescu collision operator for plasma

$$I_a^{BALU}(\mathbf{p}) = \sum_b 2e_a^2 e_b^2 n_b \frac{\partial}{\partial p_i} \int \frac{k_i k_j \delta(\mathbf{kv}-\mathbf{kv}')}{k^4 \left|\varepsilon(\mathbf{kv},\mathbf{k})\right|^2} \{\frac{\partial f_a}{\partial p_j} f_b - f_a \frac{\partial f_b}{\partial p'_j}\} d\mathbf{k} d\mathbf{p}' \tag{1}$$

satisfy general properties:
conservation laws:

$$\sum_a \int \varphi_a(\mathbf{p}) I_a(\mathbf{p}) d\mathbf{p} = 0, \text{ if } \varphi_a(\mathbf{p}) = 1,\ \mathbf{p},\ \frac{\mathbf{p}^2}{2m_a} \tag{2}$$

and Boltzmann's H-theorem:

$$\frac{\partial}{\partial t} H(t) = -\frac{\partial}{\partial t} S(t) = \frac{\partial}{\partial t} \sum_a \int f_a(\mathbf{p},t) \ln f_a(\mathbf{p},t) d\mathbf{p} \leq 0, \tag{3}$$

where $S(t)$ - is the entropy of the system.

$$\frac{\partial}{\partial t} H^{BALU}(t) = -\sum_{ab} e_a^2 e_b^2 n_a n_b \int \frac{k_i k_j \delta(\mathbf{kv}-\mathbf{kv}')}{k^4 \left|\varepsilon(\mathbf{kv},\mathbf{k})\right|^2} (\mathbf{k}\frac{\partial \ln f_a}{\partial \mathbf{p}} f_b - f_a \mathbf{k}\frac{\partial \ln f_b}{\partial \mathbf{p}'})^2 f_a f_b d\mathbf{k} d\mathbf{p}' d\mathbf{p} \leq 0. \tag{4}$$

The Balescu kinetic equations cannot be resolved exactly except for Maxwell molecules. In this case the eigenfuctions of the collision operator are Hermite polynomials. In all other cases,

the modeling approach is used. The most widely used model kinetic equation, especially in the case of discrete simulation of gas and plasma dynamics, is the Bhatnagar, Gross and Krook (BGK) model equation [1]. We recall that in the BGK model the collision term for the one-component system is the deviation of the distribution function (d.f.) from the Maxwellian d.f. whose parameters are the moments of the d.f.:

$$I^{BGK}\{f\} = -\nu(f - f^0) \tag{5}$$

where

$$f^0 = \frac{n}{(2\pi m T)^{3/2}} \exp -\frac{m(\mathbf{v}-\mathbf{V})^2}{2T}; \tag{6}$$

and $n(\mathbf{r},t) = \int f d\mathbf{p}$; $\mathbf{V}(\mathbf{r},t) = \frac{1}{n}\int \mathbf{v} f d\mathbf{p}$; $\mathbf{T}(\mathbf{r},t) = \int \frac{m(\mathbf{v}-\mathbf{V})^2}{3n} f d\mathbf{p}$ are the local density, the mean velocity and the temperature in energy units, respectively. This model term vanishes at equilibrium and satisfies the conservation laws:

$$\int \varphi(\mathbf{p}) I^{BGK}\{f\} d\mathbf{p} = 0, \text{ if } \varphi(\mathbf{p}) = 1,\ \mathbf{p},\ \frac{\mathbf{p}^2}{2m} \tag{7}$$

and the Boltzmann's H-theorem:

$$\frac{\partial}{\partial t} H^{BGK}(t) = \nu \int (f - f^0) \log \frac{f}{f^0} d\mathbf{p} \le 0, \tag{8}$$

The advantage of the BGK model is that the solution of the kinetic equation reduces to that of a system of algebraic equations [2]. A weak point is that the model implies that the Prandtl number (Pr) equals 1.

Holway and Cercignani [2, 3] independently introduced the so-called ellipsoidal statistical model in order to take into account real Prandtl numbers by substituting the local anisotropic Gaussian distribution for the local Maxwellian distribution:

$$f^0 = n\pi^{-3/2}(\det A)^{1/2} \exp -\sum_{i,j=1}^{3} \alpha_{ij}(\mathbf{v}_i - \mathbf{V}_i)(\mathbf{v}_j - \mathbf{V}_j) \tag{9}$$

$$A = \|\alpha_{ij}\| = \left\|(\mathrm{Pr})^{-1}(2T/m)\delta_{ij} - 2(1-\mathrm{Pr})\mathrm{p}_{ij}/n\,\mathrm{Pr}\right\|^{-1}$$

In the problems of linear transport and fluctuations one usually uses the linearized form of the BGK collision operator:

$$\delta\widehat{I}\,|h\rangle = -\nu\left(|h\rangle - \sum_{\alpha=1}^{5} |\Psi_\alpha\rangle\langle\Psi_\alpha|h\rangle\right), \tag{10}$$

where $|h\rangle$ is defined by $f = f^0 + \delta f = f^0\,(1+h)$, and $|\Psi_\alpha\rangle$ are the first five Hermite polynomials.

One of first works dedicated to derivation of the linearized collision integral is the paper by Gross and Jackson (GJ) [4]. Later, extension of the BGK kinetic model for the inclusion of higher order matrix elements was discussed and it was applied to investigate the generalized Enskog equation and the dynamic structure factor for gas and fluids [5-11]. The approximation

consisted in taking into account exactly a finite part of the matrix operator, while the remaining part was only represented by the diagonal matrix elements. In the present paper we do not consider spatial inhomogeneities and assume the wave vector $\mathbf{k} = \mathbf{0}$. But we do take into account the non-diagonal components arising in the collision operator expansion with respect to the complete system of polynomials in the quadratic approximation. Making use of these non-diagonal elements allows us to obtain a new form for the model integral with the Spitzer corrections taken into account. In the case of a Boltzmann gas of hard spheres these corrections are insignificant.

A more grave situation arises in the case of many-component systems. According to the Gross and Krook (GK) model [12], the collision operator has the form of the deviation of the d.f. from a "mythical" exponent:

$$I_a^{GK}\{f_a\} = -\sum_b \nu_{ab}[f_a - \frac{n_a}{(2\pi m_a T_{ab})^{3/2}} \exp -\frac{m_a(\mathbf{v}-\mathbf{V}_{ab})^2}{2T_{ab}}], \tag{11}$$

were the parameters$\mathbf{V}_{ab}$ and T_{ab} are related linearly to the d.f. moments:

$$\mathbf{V}_{ab} = \alpha_{aa}\mathbf{V}_a + \alpha_{ab}\mathbf{V}_b,\ T_{ab} = \beta_{aa}T_a + \beta_{ab}T_b, \tag{12}$$

Coefficients α_{aa}, α_{ab}, β_{aa} and β_{ab} are chosen in such manner that both conservation laws and balance equations for the momenta and energy for each component hold valid. Since the number of equations that should be satisfied by the parameters of the model (for a five-moment description of a two-component system there are four equations: 2 for the balance of moments and 2 for the balance of temperature) is less than the number of unknown parameters (in this approximation there are 5:ν_{ab}; α_{aa}; α_{ab}; β_{aa}; β_{ab}), there is an arbitrariness in the choice of parameters. Therefore there exist various modifications (see, for example, [13]) of the collision model which correctly describe relaxation of the five moments. But, probably, the most dubious point of the GK model is the complicated exponential dependence on the d.f. Recently, a new type of G-K model for gas mixtures [10, 14] was proposed and Boltzmann's H- theorem was proven for this model.

2. Model Construction

2.1. One-component systems

To correct the BGK model, following Sirovich [15] we introduce two projection operators $\widehat{H}$ and $\widehat{N}$ satisfying

$$\widehat{H}\widehat{N} = \widehat{N}\widehat{H} = 0;\ \widehat{H} + \widehat{N} = \widehat{Id}. \tag{13}$$

Here $\widehat{Id}$ is the identity operator, $\widehat{H}$ is the operator of projection onto the 'hydrodynamical subspace' spanned by kets corresponding to the polynomials of the lowest order in the moment variable. In the BGK model these kets are the first five polynomials which correspond to the collisional invariants: density, momentum and kinetic energy. However, one may include in this subspace polynomials of higher order. Their number and order depend on the physical processes that one wishes to treat "exactly". Thus, one may take into account non-invariant values like those of the pressure tensor and heat flux. The projection operator $\widehat{N}$ maps the state vector onto the remaining "non-hydrodynamical subspace". Since we are interested in a model operator describing the first 13 moments correctly we take the operator $\widehat{H}$ in the following form:

$$\widehat{H} = \sum_{i=1}^{13} |\Psi_i\rangle \langle \Psi_i| . \tag{14}$$

The linearized collision operator is the following:

$$\delta\widehat{I} = \widehat{H}\delta\widehat{I}\widehat{H} + \widehat{H}\delta\widehat{I}\widehat{N} + \widehat{N}\delta\widehat{I}\widehat{H} + \widehat{N}\delta\widehat{I}\widehat{N}. \tag{15}$$

Since the first five Hermite polynomials are the eigenfunctions of the collision operator for identical particles corresponding to the zero eigenvalue, it follows that for $1 \leq i \leq 5$

$$\widehat{H}\delta\widehat{I}\widehat{H} = \widehat{H}\delta\widehat{I}\widehat{N} = \widehat{N}\delta\widehat{I}\widehat{H} = 0. \tag{16}$$

The higher Hermite polynomials are eigenfunctions of the collision operator only for a Maxwell molecule. In this case, non-diagonal matrix elements $\widehat{H}\delta\widehat{I}\widehat{N}$ and $\widehat{N}\delta\widehat{I}\widehat{H}$ vanish:

$$\widehat{H}\delta\widehat{I}\widehat{N} = \widehat{N}\delta\widehat{I}\widehat{H} = 0. \tag{17}$$

For any other interaction potentials the Hermite polynomials are not the eigenfunctions of the collision operator, the equality(17) does not hold and the collision operator matrix elements contain non-diagonal elements. Our first approximation is that we accept (17) as a valid formula for the Boltzmann gas of hard spheres and Coulomb plasma. However, the first approximation is not sufficient for describing real gas and plasma. In the second approximation we take into account only the non-diagonal terms closest to the diagonal. As we will show below, the corrections for a Boltzmann gas of hard spheres turns out to be small, but for Coulomb systems they are not small and play a major role in the Spitzer corrections to transport coefficients. We can continue this process and take into account in the third approximation the next, non-diagonal, terms more distant from the diagonal elements. We performed these calculations and found that the third approximation yields very small corrections (compared to the Spitzer approximation), that can be neglected.

Since for one-component systems the operator is Hermitian and isotropic, Wigner-Ekkart theorem implies

$$\left\langle Y_l^m h_1(u^2) \right| \delta\widehat{I} \left| Y_{l'}^{m'} h_2(u^2) \right\rangle = \delta_{ll'}\delta_{mm'} \left\langle \langle Y_l^m h_1 | \, \delta\widehat{I} \, | Y_l^m h_2 \rangle \right\rangle , \tag{18}$$

where $h_1(u^2)$,$h_2(u^2)$ are arbitrary functions , $Y_l^m-$are spherical harmonics and $\langle\langle | \, | \rangle\rangle$ means the reduced matrix element independent of m. From (18) the selection rule follows: the contribution to the non-diagonal matrix elements $\widehat{H}\delta\widehat{I}\widehat{N}$ and $\widehat{N}\delta\widehat{I}\widehat{H}$ is given only by polynomials with identical pairs of orbital numbers l and m. For example, for the polynomial $|\Psi_6\rangle = \frac{\sqrt{3}}{2}(u_x u_x - \frac{1}{3}u^2)$ defining the xx component of the pressure tensor, the non-zero contribution to the non-diagonal matrix elements is given by non-hydrodynamical polynomials of higher order in u^2 but with the same values of l and m ($l = 2; m = 2$). For example, $\left|\Psi_6^{(2)}\right\rangle =$

$\sqrt{\frac{3}{14}}\frac{1}{2}(u^2-7)(u_x u_x - \frac{1}{3}u^2)$.The main modeling procedure consists of approximating the non-hydrodinamical contribution. If the operator $\widehat{H}$ involves the first 13 Hermite polynomials, then the neglect of the term $\widehat{N}\delta\widehat{I}\widehat{N}$ does not affect calculations for such transport coefficients as viscosity and heat conductivity. Nevertheless the approximation

$$\widehat{N}\delta\widehat{I}\widehat{N} = -\nu\widehat{N} \tag{19}$$

allows one to describe at least qualitatively the 'tails' of neglected 'non-hydrodynamical' terms (ν corresponds to the longest non-hydrodynamical relaxation time). An account of these 'tails' may be important at the kinetic level of fluctuation description. Using this approximation one may rewrite, in a first approximation, that corresponds to the Maxwell's molecule, the model operator as follows:

$$\delta\widehat{I} = -\nu\widehat{Id} + \widehat{H}(\delta\widehat{I} + \nu)\widehat{H}. \tag{20}$$

For a 13 moment basis for $\widehat{H}$ in the first approximation

$$\delta\widehat{I}\,|h\rangle = -\nu\,|h\rangle + \nu\sum_{i=1}^{5}|\Psi_i\rangle\,\langle\Psi_i|h\rangle + \sum_{i=6}^{13}|\Psi_i\rangle\,(\langle\Psi_i|\,\delta\widehat{I}\,|\Psi_i\rangle + \nu)\,\langle\Psi_i|h\rangle\,. \tag{21}$$

In the second approximation, in (21) appear the nearest non-diagonal entries:

$$\sum_{i=6}^{13}|\Psi_i\rangle\,\langle\Psi_i|\,\delta\widehat{I}\left|\Psi_i^{(2)}\right\rangle\left\langle\Psi_i^{(2)}|h\right\rangle, \tag{22}$$

where the non-hydrodynamical polynomials, which we take into account, are

$$\left|\Psi_i^{(2)}\right\rangle = \frac{1}{\sqrt{14}}(u^2-7)\,|\Psi_i\rangle\,,\ 6\le i\le 10 \tag{23}$$

$$\left|\Psi_{r+10}^{(2)}\right\rangle = \frac{1}{\sqrt{280}}(u^4-14u^2+35)\,|\Psi_r\rangle\,,\ 1\le r\le 3. \tag{24}$$

To close the terms (22) we use the following equation for the non-hydrodynamical moments in the Fourier presentation

$$(-i\omega - \left\langle\Psi_i^{(2)}\right|\delta\widehat{I}\left|\Psi_i^{(2)}\right\rangle)\left\langle\Psi_i^{(2)}|h\right\rangle_\omega = \left\langle\Psi_i^{(2)}\right|\delta\widehat{I}\,|\Psi_i\rangle\,\langle\Psi_i|h\rangle_\omega \tag{25}$$

Thus in the second approximation, linearized model collision operator has the form

$$\delta\widehat{I}\,|h\rangle_\omega = -\nu\,|h\rangle_\omega + \nu\sum_{i=1}^{5}|\Psi_i\rangle\,\langle\Psi_i|h\rangle_\omega - \sum_{i=6}^{13}|\Psi_i\rangle\,({}_i\Lambda_i^{(2)}(\omega)-\nu)\,\langle\Psi_i|h\rangle_\omega\,, \tag{26}$$

where

$${}_i\Lambda_i^{(2)}(\omega) = -\,\langle\Psi_i|\,\delta\widehat{I}\,|\Psi_i\rangle - \frac{\langle\Psi_i|\,\delta\widehat{I}\left|\Psi_i^{(2)}\right\rangle^2}{-i\omega - \left\langle\Psi_i^{(2)}\right|\delta\widehat{I}\left|\Psi_i^{(2)}\right\rangle} \tag{27}$$

contains the square of the non-diagonal entries and the projection of the kinetic equation resolvent to the non-hydrodynamical subspace. Here we take into account non-stationarity of non-hydrodynamical moments. Thus, although the original collision integral is Markovian,

the part projected onto the supspace of 13 moments becomes, in the second approximation, a frequency dependent operator. A similar situation occurs in quantum-mechanical perturbation theory. Note that in the Markov approximation the second order corrections in (27)are negative for any interaction potentials.

Calculate now matrix elements of the operator for a concrete interaction potential, namely for Coulomb plasma and a Boltzmann hard sphere gas. For the Coulomb interaction take the linearized collision operator in the Balescu-Lenard form. In this case the matrix elements are of the form

$$\langle\Psi_i|\,\delta\widehat{I}\,|\Psi_j\rangle = -\int d\mathbf{p}d\mathbf{p}'d\mathbf{k}\frac{e^4k_rk_s\delta(\mathbf{kv}-\mathbf{kv}')}{k^4\left|\varepsilon(\mathbf{kv},\mathbf{k})\right|^2}f^0(\mathbf{p})f^0(\mathbf{p}') \tag{28}$$

$$\times(\frac{\partial\Psi_i}{\partial\mathbf{p}_r}-\frac{\partial\Psi_i}{\partial\mathbf{p}'_r})(\frac{\partial\Psi_j}{\partial\mathbf{p}_s}-\frac{\partial\Psi_j}{\partial\mathbf{p}'_s}).$$

For i=j the entries are

$$\langle\Psi_i|\;\delta\widehat{I}\,|\Psi_i\rangle \le 0 \tag{29}$$

for any polynomials. Equality to zero corresponds to the five-time-degenerate zero eigenvalue. It is easy to see that the matrix elements of the operator $\widehat{H}\delta\widehat{I}\widehat{H}$ have following values:

$$\langle\Psi_i|\;\delta\widehat{I}\,|\Psi_j\rangle = -\,\delta_{ij}[\;\Lambda_1\sum_{k=6}^{10}\;\delta_{ik}+\;\Lambda_2\sum_{k=11}^{13}\;\delta_{ik}], \tag{30}$$

where Λ_1,Λ_2 are the relaxation frequencies of the pressure tensor and the heat flux vector which in terms of plasma parameters are as follows:

$$\Lambda_1=\frac{8}{5}\frac{ne^4\sqrt{\pi}}{m^{1/2}T^{3/2}}Ln;\quad\Lambda_2=\frac{2}{3}\;\Lambda_1. \tag{31}$$

The matrix elements of the operators $\widehat{N}\delta\widehat{I}\widehat{N}$ and $\widehat{H}\delta\widehat{I}\widehat{N}$ defining 'tails' and the second approximation for the Coulomb plasma are equal to [16]

$$\langle\Psi_{14}|\;\delta\widehat{I}\,|\Psi_{14}\rangle=-\frac{2}{3}\;\Lambda_1;\;\Psi_{14}=\frac{1}{\sqrt{120}}(u^4-10u^2+15)$$

$$\langle\Psi_{15}|\;\delta\widehat{I}\,|\Psi_{15}\rangle=...=\langle\Psi_{21}|\;\delta\widehat{I}\,|\Psi_{21}\rangle=-\frac{3}{2}\;\Lambda_1;\;\Psi_{15}=u_xu_yu_z$$

$$\langle\Psi_{22}|\;\delta\widehat{I}\,|\Psi_{22}\rangle=...=\langle\Psi_{35}|\;\delta\widehat{I}\,|\Psi_{35}\rangle=-\frac{191}{16}\;\Lambda_1;$$

$$\Psi_{22}=\frac{1}{\sqrt{105}}\frac{1}{8}(35u_x^4-30u^2u_x^2+3u^4);$$

$$\left\langle\Psi_i^{(2)}\right|\delta\widehat{I}\left|\Psi_i^{(2)}\right\rangle=-\frac{201}{168}\Lambda_1;\;\langle\Psi_i|\,\delta\widehat{I}\left|\Psi_i^{(2)}\right\rangle=\frac{3}{2\sqrt{14}}\Lambda_1;\;6\le i\le 10$$

$$\left\langle\Psi_{10+r}^{(2)}\right|\delta\widehat{I}\left|\Psi_{10+r'}^{(2)}\right\rangle=-\frac{15}{14}\delta_{rr'}\Lambda_1;\;\langle\Psi_{10+r}|\,\delta\widehat{I}\left|\Psi_{10+r'}^{(2)}\right\rangle=\frac{1}{7}\delta_{rr'}\Lambda_1;\quad 1\le r\le 3. \tag{32}$$

This estimate implies that the higher tensor character of the polynomial leads to higher values of the matrix elements and for polynomials with the same tensor character the matrix elements are greater for polynomials of higher order in u^2. The smallest value of the diagonal matrix elements for the non-hydrodynamical polynomial is achieved for the Ψ_{14} polynomial and equals Λ_2 (the

heat flux relaxation frequency). The same holds for Maxwell's molecule and the Boltzmann gas hard sphere at least. Thus in the first approximation the collision operator is of the form:

$$\delta\widehat{I}\,|h\rangle = -\frac{2}{3}\Lambda_1\,|h\rangle + \frac{2}{3}\Lambda_1\sum_{i=1}^{5}|\Psi_i\rangle\,\langle\Psi_i|h\rangle - \frac{1}{3}\Lambda_1\sum_{i=6}^{10}|\Psi_i\rangle\,\langle\Psi_i|h\rangle\,. \tag{33}$$

We see that the polynomials corresponding to the heat flux disappear. In the second approximation

$$\delta\widehat{I}\,|h\rangle_\omega = -\nu(|h\rangle_\omega - \sum_{i=1}^{13}|\Psi_i\rangle\,\langle\Psi_i|h\rangle_\omega)$$

$$-\sum_{i=6}^{10}\Lambda_1^{(2)}(\omega)\,|\Psi_i\rangle\,\langle\Psi_i|h\rangle_\omega - \sum_{i=11}^{13}\Lambda_2^{(2)}(\omega)\,|\Psi_i\rangle\,\langle\Psi_i|h\rangle_\omega\,, \tag{34}$$

where

$$\Lambda_1^{(2)}\,(\omega) = \Lambda_1(1 - \frac{\Lambda_1 9/56}{-i\,\omega + \Lambda_1 205/168});\ \Lambda_2^{(2)}\,(\omega) = \Lambda_2(1 - \frac{\Lambda_1 3/14}{-i\,\omega + \Lambda_1 15/14}). \tag{35}$$

In the Markov approximation ($\omega = 0$) this leads to Spitzer values [17] of transport coefficients.

$$\Lambda_1^{(2)} = \Lambda_1(1 - \frac{27}{205});\quad \Lambda_2^{(2)} = \Lambda_2(1 - \frac{1}{5}). \tag{36}$$

This correction are rather significant. In the third approximation the relaxation frequencies vary by no more than one per cent. Thus our model leads to Spitzer kinetic coefficients.

The second-order corrections for Boltzmann gas are one order less than for Coulomb plasma and one may stop at the first approximation. Thus in the case of a Boltzmann gas of hard sphere the collision operator may be represented in the form [18]

$$I\{f\} = -\nu\{f - f^0(1 - P_{ij}\frac{\delta v_i\delta v_j}{4PT}m)\}, \tag{37}$$

where

$$P_{ij} = m\int d\mathbf{p}f(\delta v_i\delta v_j - \delta_{ij}\frac{\delta v^2}{3}) \tag{38}$$

is the pressure tensor.

In the equilibrium state, $P_{ij} = 0$ and $I\{f^0\} = 0$. In the equation for the heat flux only the first term in (37) contributes. The relaxation of the pressure tensor is determined by both the first and the last terms in this equation.

Now we will prove the H theorem for our model (37):

$$\frac{\partial}{\partial t}H(t) = \nu\int(f-\Phi)\log\frac{f}{\Phi}d\mathbf{p} + \nu\int(f-\Phi)\log\Phi d\mathbf{p},\quad \text{where}\ \ \Phi = f^0(1 - P_{ij}\frac{\delta v_i\delta v_j}{4PT}m)$$

$$= \nu\int(f-\Phi)\log\frac{f}{\Phi}d\mathbf{p} + \nu\int[f - f^0(1 - P_{ij}\frac{\delta v_i\delta v_j}{4PT}m)][\log f^0 + \log(1 - P_{ij}\frac{\delta v_i\delta v_j}{4PT}m)]d\mathbf{p}, \tag{39}$$

The first term in the second square bracket in (39) vanishes, while the second term can be expended as: $\log(1+x) = x - x^2/2 + x^3/3...$; for $-1 < x \leq 1$. Hear only the first term contributes after the integration.

Therefore:

$$\frac{\partial}{\partial t}H(t)=\nu\int(f-\Phi)\log\frac{f}{\Phi}d\mathbf{p}+\nu\int[f-f^0(1-P_{ij}\frac{\delta v_i\delta v_j-\delta_{ij}\frac{\delta v^2}{3}}{4PT}m)](-P_{ij}\frac{\delta v_i\delta v_j-\delta_{ij}\frac{\delta v^2}{3}}{4PT}m)d\mathbf{p}. \tag{40}$$

Taking into account that

$$\int f^0(\delta v_i\delta v_j-\delta_{ij}\frac{\delta v^2}{3})(\delta v_k\delta v_l-\delta_{kl}\frac{\delta v^2}{3})d\mathbf{p}=\frac{T^2}{m^2}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}-\frac{2}{3}\delta_{ij}\delta_{kl}), \tag{41}$$

we obtain the H theorem:

$$\frac{\partial}{\partial t}H(t)=\nu\int(f-\Phi)\log\frac{f}{\Phi}d\mathbf{p}-\frac{\nu}{8P^2}P_{ij}P_{ij}\leq\nu\int(f-\Phi)\log\frac{f}{\Phi}d\mathbf{p}\leq\mathbf{0} \tag{42}$$

Thus our collision integral in the form (37) possesses all necessary properties and is free from the drawbacks of the one-component model of the BGK model mentioned above. The linearized form of (37) is congruent with the linearized ellipsoidal statistical model [2-3].

Earlier, another model correctly describing the viscosity and thermal conductivity relaxation was proposed ad hoc [19]:

$$I\{f\}=-\nu\{f-f^0[1-\Pr\frac{\mathbf{J}\delta\mathbf{v}}{5P}(\frac{\delta v^2}{mT}-5)]\}, \tag{43}$$

where $\mathbf{J}$ is the heat flux. But this model does not give a correct description of non-hydrodynamic 'tails'.

2.2. Many-component systems

Using the technique described above for a one-component system, one may get the following expression for the linearized model collision operator of a many-component system in the five-moment approximation [18]:

$$\delta\widehat{I}_{a\mathbf{p}}\delta f_a(\mathbf{p})=-\nu_a\delta f_a(\mathbf{p})+\sum_{j=1}^{5}\nu_a f_a^0(\mathbf{p})\Psi_j^a(\mathbf{p})\int\Psi_j^a(\mathbf{p}')\delta f_a(\mathbf{p}')d\mathbf{p}'$$
$$+\sum_b\sum_{i,j=1}^{5}f_a^0(\mathbf{p})\Psi_i^a(\mathbf{p})\left\langle\Psi_i^a\right|\delta\widehat{I}\left|\Psi_j^b\right\rangle\int\Psi_j^b(\mathbf{p}')\delta f_b(\mathbf{p}')d\mathbf{p}', \tag{44}$$

where $f_a^0(\mathbf{p})$ -is the local equilibrium distribution function (with different temperatures and mean velocities), ν_a -is the inverse time of the heat flux relaxation of component a, and $\left\langle\Psi_i^a\right|\delta\widehat{I}\left|\Psi_j^b\right\rangle$ represents the matrix elements of the linearized collision integral (of the Balescu-Lenard integral, for example).

In order to recover the form of the model collision integral from its linearized form it suffices to use conservation of the number of elastically interacting particles. This property, as well as total momentum and energy conservation, is valid for both mean and fluctuating quantities. Consequently, the expression for Langevin's source intensity in the kinetic equation for the distribution function fluctuation [20]

$$(\frac{\partial}{\partial t}+\mathbf{v}\frac{\partial}{\partial\mathbf{r}}+\delta\widehat{I}_{a\mathbf{p}})\delta f_a(\mathbf{x},t)+\delta\mathbf{F}\frac{\partial}{\partial\mathbf{p}}f_a(\mathbf{x},t)=y_a(\mathbf{x},t) \tag{45}$$

should satisfy the conditions

$$\sum_b \int \Psi^b(\mathbf{p}_2)(y_a y_b)_{\omega,\mathbf{k},\mathbf{p}_1,\mathbf{p}_2} d\mathbf{p}_2 = 0 \text{ for } \Psi^b(\mathbf{p}_2) = 1,\ \mathbf{p}_2,\ p_2^2/2m_b\ . \tag{46}$$

The spectral function of the Langevin source in a non-equilibrium state is given [21, 22] by following form:

$$(y_a y_b)_{\omega,\mathbf{k},\mathbf{p}_1,\mathbf{p}_2} = -(\delta\widehat{I}_{a\mathbf{p}_1} + \delta\widehat{I}_{b\mathbf{p}_2})\delta_{ab}\delta(\mathbf{p}_1-\mathbf{p}_2)f_a(\mathbf{p}_1) + \delta_{ab}\delta(\mathbf{p}_1-\mathbf{p}_2)I_a(\mathbf{p}_1) + I_{ab}(\mathbf{p}_1,\mathbf{p}_2)\ , \tag{47}$$

where$I_{ab}(\mathbf{p}_1,\mathbf{p}_2)-$ is the so-called "not integrated"' collision operator:$\sum_b \int I_{ab}(\mathbf{p}_1,\mathbf{p}_2)d\mathbf{p}_2 = I_a(\mathbf{p}_1)$.

In the case of plasma the "not integrated" collision operator has the form:

$$I_{ab}(\mathbf{p}_1,\mathbf{p}_2) = 2e_a^2 e_b^2 n_b (\frac{\partial}{\partial p_i} - \frac{\partial}{\partial p_i'}) \int \frac{k_i k_j \delta(\mathbf{kv}-\mathbf{kv}')}{k^4 \left|\varepsilon(\mathbf{kv},\mathbf{k})\right|^2} (\frac{\partial}{\partial p_j} - \frac{\partial}{\partial p_j'}) f_a f_b d\mathbf{k}. \tag{48}$$

Summing (47) over b and integrating over $\mathbf{p}_2$, and taking into account (46), we get

$$I_a(\mathbf{p}_1) = -\frac{1}{2}(\delta\widehat{I}_{a\mathbf{p}_1} + \delta\widehat{I}_{b\mathbf{p}_2})\delta_{ab}\delta(\mathbf{p}_1-\mathbf{p}_2)f_a(\mathbf{p}_1)d\mathbf{p}_2. \tag{49}$$

Since(46) and(47) are of a general character, the relation (49) is valid both for "exact" and model collision integrals. Substituting (44) into (49), we obtain a quite simple and at the same time sufficiently rigorous form of the model collision integral for many-component plasma:

In the local equilibrium state the collision integral assumes the fairly simple form:

$$I_a(\mathbf{p}_1) = -\sum_b \nu_{ab} f_a^0(\mathbf{p}) \left[\delta\mathbf{v}_a m_a \frac{\mathbf{V}_a - \mathbf{V}_b}{T_a} + \left(\frac{m_a}{T_a}\delta\mathbf{v}_a^2 - 3\right)(T_a - T_b)\frac{m_a}{m_a+m_b}\right], \tag{50}$$

where ν_{ab} is the momentum relaxation frequency for plasma:

$$\nu_{ab} = \frac{4}{3}\sqrt{2\pi}e_a^2 e_b^2 n_b L \sqrt{\frac{m_a}{m_b}} \frac{(m_a+m_b)^{1/2}}{(m_a T_b + m_b T_a)^{3/2}}, \text{ and } \delta\mathbf{v}_a = \mathbf{v}_a - \mathbf{V}_a. \tag{51}$$

In the second term in (50) we took into account the isothermal case and in the third term we neglected corrections of the square mean velocities. The first term in (50) describe relaxation to the local equilibrium state, the second term describes relaxation of the momenta and the last term the temperature relaxation. In this case it is easy to verify the Boltzmann H-theorem for this form of the collision integral.

$$\frac{\partial}{\partial t}H(t) = -\sum_a \int \frac{\delta\mathbf{p}^2}{2m_a T_a} I_a(\mathbf{p}_1)d\mathbf{p} = -\frac{3}{4}\sum_{ab}\nu_{ab}\frac{m_a n_a}{m_a+m_b}\frac{(T_a-T_b)^2}{T_a T_b} \le 0, \tag{52}$$

Thus, the complicated exponential dependency typical for the GK model appears to be unfounded and does not hold for states remote from the full equilibrium.

$$I_a(\mathbf{p}_1) = -\nu_a[f_a(\mathbf{p}) - f_a^0(\mathbf{p})(1 - P_{ij}\frac{\delta v_i \delta v_j}{4PT}m)]$$
$$-\sum_b \nu_{ab} f_a^0(\mathbf{p}) \left[\delta \mathbf{v}_a m_a \frac{\mathbf{V}_a - \mathbf{V}_b}{T_a} + \left(\frac{m_a}{T_a}\delta \mathbf{v}_a^2 - 3\right)(T_a - T_b)\frac{m_a}{m_a + m_b}\right] \quad (53)$$

Thus, the time evolution of the two-component gas up to the hydrodynamic stage can be described as follows: first, the lighter component achieves equilibrium, then the balanced state of the heavy component, and finally a balance across the gas mixture is achieved. At all these stages Boltzmann's H-theorem holds.

3. Conclusion

Using the well-known projection technique, a new form of the collision operator for a Boltzmann gas of hard spheres and for Coulomb plasma has been developed. The proposed collision operator takes into account relaxation of the first 13 hydrodynamic moments properly and accounts for the contribution of non-diagonal components in the expansion of the linearized collision operator in the complete system of Hermite polynomials. The non-diagonal components accounted for in this basis in the quadratic approximation contribute to the diagonal components. It is shown that for a system of charged particles with a Coulomb interaction potential, these contributions are essential and lead to Spitzer corrections to the transport coefficients. In the case of a Boltzmann gas of hard spheres these corrections are insignificant. In the case of a many-component system, the nonlinear model collision integral is constructed on the basis of the linearized one. Unlike previous cases, it does not exhibit any complicated exponential dependence and avoids coefficient ambiguity in the many-component collision integral.

4. References

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