

# Relativistic distribution function for particles with spin at local thermodynamical equilibrium

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**Abstract.** We present an extension of relativistic single-particle distribution function for weakly interacting particles at local thermodynamical equilibrium including spin degrees of freedom, for massive spin 1/2 particles. We infer, on the basis of the global equilibrium case, that at local thermodynamical equilibrium particles acquire a net polarization proportional to the vorticity of the inverse temperature four-vector field. The obtained formula for polarization also implies that a steady gradient of temperature entails a polarization orthogonal to particle momentum. The single-particle distribution function in momentum space extends the so-called Cooper-Frye formula to particles with spin 1/2 and allows to predict their polarization, and particularly of Lambda hyperons, in relativistic heavy ion collisions at the freeze-out.

## 1. Introduction

The single-particle distribution function is the main quantity in kinetic theory and its form at local thermodynamical equilibrium for relativistic, weakly interacting, gases is well known. For spinless particles, it is simply the Bose-Einstein distribution with  $x$ -dependent values of temperature and chemical potential (which can be defined as Bose-Jüttner distribution):

$$f(x, p) = \frac{1}{e^{\beta(x) \cdot p - \xi(x)} - 1} \quad (1)$$

where  $\beta = \frac{1}{T_0}u$  is the inverse temperature four-vector,  $T_0$  being the proper temperature measured by a comoving thermometer with the four-velocity  $u$  and  $\xi = \mu_0/T_0$  is the ratio between the proper chemical potential  $\mu_0$  and  $T_0$ . The above formula has, as straightforward consequence, the invariant momentum spectrum at local thermodynamical equilibrium, the so-called Cooper-Frye formula [1]:

$$\varepsilon \frac{dN}{d^3p} = \int_{\Sigma} d\Sigma_{\mu} p^{\mu} f(x, p) \quad (2)$$

where  $\Sigma$  is a space-like 3-dimensional hypersurface. This formula is widely used in e.g. relativistic heavy ion collisions to calculate hadronic spectra at the end of the hydrodynamical stage.

The distribution (1), multiplied by a degeneracy factor  $(2S + 1)$ , is also used for particles with spin (for fermions replacing the  $-1$  with  $+1$  in the denominator) being understood that  $f$



means the *total* particle density in phase space, i.e. summed over polarization states. However, in general, particles may not evenly populate the various polarization states and one may then wonder what is the appropriate extension of (1) in this case. Indeed, in this work, we will answer this question and provide a generalization of (1) and (2) including the spin degrees of freedom. We will argue that a non-even population of the polarization states arises when the inverse temperature four-vector field has a non-vanishing antisymmetric part of its gradient and calculate the polarization vector for massive spin 1/2 particles. Phenomenologically, this extension may have several interesting applications. For instance, it would make possible to predict the value of particle polarization in relativistic heavy ion collisions [2, 3, 4, 5, 6, 7] at the hydrodynamical decoupling, provided that local thermodynamical equilibrium applies to spin degrees of freedom as well. The detailed analysis and calculation can be found in [8].

## 2. Single particle distribution function with spin

At global thermodynamical equilibrium with finite angular momentum density the density operator is well known [9, 10] and reads:

$$\hat{\rho} = \frac{1}{Z} \exp[-\hat{H}/T + \mu\hat{Q}/T + \boldsymbol{\omega} \cdot \hat{\mathbf{J}}/T] P_V \quad (3)$$

where  $\boldsymbol{\omega}$  is a constant fixed vector whose physical meaning is that of an angular velocity and  $T$  is the global temperature, that is the temperature of a thermostat in contact with the system or that measured by a thermometer at rest with respect to the external inertial observer. The  $P_V$  operator is the projector operator onto localized states [11] which is needed in order to avoid the relativistic singularity at  $r = c/\omega$ . For this distribution, it has been shown the single-particle distribution function of an ideal relativistic Boltzmann gas of particles with spin  $S$  can be obtained from just statistical mechanics arguments [11]:

$$f(x, p)_{rs} = e^\xi e^{-\beta \cdot p} \frac{1}{2} \left( D^S([p]^{-1} R_{\hat{\omega}}(i\omega/T)[p]) + D^S([p]^\dagger R_{\hat{\omega}}(i\omega/T)[p]^\dagger)^{-1} \right)_{rs} \quad (4)$$

where  $\beta = \frac{1}{T}(1, \boldsymbol{\omega} \times \mathbf{x})$  is the inverse temperature four-vector;  $\lambda$  is the fugacity;  $[p]$  is the  $SL(2, C)$  matrix corresponding to the Lorentz transformation taking the time unit vector  $\hat{t}$  into  $\hat{p}$  (so-called standard transformation);  $D^S$  stands for the  $(S, 0)$  irreducible representation of  $SL(2, C)$ ;  $R$  is the  $SL(2, C)$  corresponding of a rotation, which is calculated for an imaginary angle  $i\omega/T$ .

The extension to Fermi-Dirac statistics of this formula is not a straightforward calculation like in the non-rotating case, hence we make an *ansatz* about this extension which reproduces the usual Fermi-Dirac distribution in the non-rotating case and, at the same time, has the (4) for  $S = 1/2$  case as its Boltzmann limit.

$$\begin{aligned} f(x, p)_{rs} &= \frac{1}{2m} \bar{u}_r(p) \left( \exp[\beta \cdot p - \xi] \exp \left[ -\frac{1}{2} \varpi_{\mu\nu} \Sigma^{\mu\nu} \right] + I \right)^{-1} u_s(p) \\ \bar{f}(x, p)_{rs} &= -\frac{1}{2m} \bar{v}_s(p) \left( \exp[\beta \cdot p + \xi] \exp \left[ \frac{1}{2} \varpi_{\mu\nu} \Sigma^{\mu\nu} \right] + I \right)^{-1} v_r(p) \end{aligned} \quad (5)$$

whereas  $\varpi$  is the covariant form for the angular velocity:

$$\varpi_{\mu\nu} = (\omega/T)(\delta_\mu^1 \delta_\nu^2 - \delta_\nu^1 \delta_\mu^2) = \sqrt{\beta^2} \Omega_{\mu\nu} \quad (6)$$

and  $\Omega_{\mu\nu}$  turns out to be the acceleration tensor of the Frenet-Serret tetrad of the  $\beta$  field lines [11]. The above second equality holds for a rigid velocity field only [11, 12].

The expression of the single particle distribution function for a fermion field, at local thermodynamical equilibrium, in principle, should be calculated using the density operator for

local thermodynamical equilibrium [13]. At the lowest order of approximation, however, we know that the single particle distribution function must yield the same formal expression at global thermodynamical equilibrium with space-time dependent intensive thermodynamics functions, that is space-time dependent  $\beta$ ,  $\varpi$  and  $\xi$ . Hence, the single-particle distribution functions (4) for  $S = 1/2$  case become:

$$\begin{aligned} f(x, p)_{rs} &= \frac{1}{2m} \bar{u}_r(p) \left( \exp[\beta(x) \cdot p - \xi(x)] \exp \left[ -\frac{1}{2} \varpi(x)_{\mu\nu} \Sigma^{\mu\nu} \right] + I \right)^{-1} u_s(p) \\ \bar{f}(x, p)_{rs} &= -\frac{1}{2m} \bar{v}_s(p) \left( \exp[\beta(x) \cdot p + \xi(x)] \exp \left[ \frac{1}{2} \varpi(x)_{\mu\nu} \Sigma^{\mu\nu} \right] + I \right)^{-1} v_r(p) \end{aligned} \quad (7)$$

where  $\beta$  is the inverse temperature four-vector,  $\xi$  is the ratio between comoving chemical potential and temperature,  $\Sigma^{\mu\nu} = (i/4)[\gamma^\mu, \gamma^\nu]$  are the generators of Lorentz transformations of 4-components spinors,  $u(p)$  and  $v(p)$  are the spinors solutions of the free Dirac equation. While the general physical meaning of the fields  $\beta$  and  $\xi$  can be easily inferred from the equilibrium limit ( $\beta$  is the local inverse temperature four-vector field and  $\xi$  the ratio between the comoving chemical potential  $\mu_0(x)$  and the local comoving temperature  $T_0(x) = 1/\sqrt{\beta^2}$ ),  $\varpi_{\mu\nu}(x)$ 's expression cannot be uniquely obtained from the equilibrium distribution. The reason of this ambiguity is that at rotational equilibrium, the tensor  $\varpi$  is:

$$\varpi_{\mu\nu} = \sqrt{\beta^2} \Omega_{\mu\nu} \quad (8)$$

where  $\Omega_{\mu\nu}$  is the acceleration tensor of the Frenet-Serret tetrad of the  $\beta$  field lines [11] and [12]:

$$\varpi_{\mu\nu} = -\frac{1}{2} (\partial_\mu \beta_\nu - \partial_\nu \beta_\mu) \quad (9)$$

that is the relativistic generalization of angular velocity of fluid. In a nonequilibrium situation, the right hand sides of eqs. (9) and (8) differ and it is not obvious which one applies, perhaps neither. However, if the system is not too far from equilibrium,  $\varpi$  cannot be too distant from the right hand side of (9). Particularly, the difference must be of the 2nd order in the gradients of the  $\beta$  field (for instance:  $(\partial \cdot \beta)(\partial_\mu \beta_\nu - \partial_\nu \beta_\mu)$  or second order gradients like  $\partial_\mu \partial_\nu \beta^\lambda$ ) which vanish at equilibrium [12]. For the lowest-order formulation of relativistic hydrodynamics, the expression (9) is sufficient to determine the expression of spin-related quantities.

### 3. Polarization

The main consequence of the distribution functions (7) is that spin 1/2 particles get polarized at local thermodynamical equilibrium. The polarization four-vector, for a particle with mass  $m$  and four-momentum  $p$  is defined as:

$$\Pi_\mu = -\frac{1}{2} \epsilon_{\mu\rho\sigma\tau} S^{\rho\sigma} \frac{p^\tau}{m} \quad (10)$$

where  $S^{\rho\sigma}$  is the mean value of the *total* angular momentum operator of the single particle. However the Levi-Civita tensor makes the orbital part of the total angular momentum irrelevant, so we are left with the spin-tensor density in the phase space  $\mathcal{S}^{0,\rho\tau}(x, p)$  contribution only:

$$\langle \Pi_\mu(x, p) \rangle = -\frac{1}{2} \frac{1}{\sum_r f_{rs}} \epsilon_{\mu\rho\sigma\tau} \frac{d\mathcal{S}^{0,\rho\sigma}(x, p)}{d^3p} \frac{p^\tau}{m} \quad (11)$$

The *canonical* spin tensor can be computed using (7):

$$\mathcal{S}^{\lambda,\mu\nu}(x) = \frac{1}{2} \int \frac{d^3p}{2\varepsilon} \sum_{rs} \left( f_{rs}(x, p) \bar{u}_s(p) \{ \gamma^\lambda, \Sigma^{\mu\nu} \} u_r(p) \right) - \left( \bar{f}_{rs}(x, p) \bar{v}_r(p) \{ \gamma^\lambda, \Sigma^{\mu\nu} \} v_s(p) \right) \quad (12)$$

Not far from equilibrium, using the (9), expanding for small value of  $\varpi^{\mu\nu}$ , the polarization vector becomes:

$$\langle \Pi_\mu(x, p) \rangle \simeq \frac{1}{16} \epsilon_{\mu\rho\sigma\tau} (1 - n_F) (\partial^\rho \beta^\sigma - \partial^\sigma \beta^\rho) \frac{p^\tau}{m} = \frac{1}{8} \epsilon_{\mu\rho\sigma\tau} (1 - n_F) \partial^\rho \beta^\sigma \frac{p^\tau}{m} \quad (13)$$

where  $n_F = 1/(\exp(\beta(x) \cdot p - \xi(x)) + 1)$  is the Fermi-Dirac distribution function.

It may be of interest, e.g. for relativistic heavy ion collisions, to calculate the space-integrated mean polarization vector. For a three-dimensional spacelike hypersurface  $\Sigma$ , one has:

$$\langle \Pi_\mu(p) \rangle \equiv \frac{\int d\Sigma_\lambda \frac{p^\lambda}{\varepsilon} (-1/2) \epsilon_{\mu\rho\sigma\tau} \frac{dS^{0,\rho\sigma}}{d^3\mathbf{p}} \frac{p^\tau}{m}}{\int d\Sigma_\lambda \frac{p^\lambda}{\varepsilon} \sum_r f_{rr}(x, p)} \simeq \frac{1}{8} \epsilon_{\mu\rho\sigma\tau} \frac{p^\tau}{m} \frac{\int d\Sigma_\lambda p^\lambda n_F (1 - n_F) \partial^\rho \beta^\sigma}{\int d\Sigma_\lambda p^\lambda n_F} \quad (14)$$

For antiparticles, one gets the same formula, with  $\bar{n}_F = 1/(\exp(\beta(x) \cdot p + \xi(x)) + 1)$  replacing  $n_F$ . Here an important comment is in order: the fact that local thermodynamical equilibrium implies the same orientation for the polarization vector of particles and antiparticles (unlike e.g. in the electromagnetic field) is a general outcome and does not depend on the introduced approximations. It stems from the fact that the spin tensor, as well as the angular momentum, is a charge-conjugation even operator, or, more simply stated, that thermal and mechanical effects do not "see" the internal charge of the particles.

This formula (14) may be used to predict the polarization of particles, in particular  $\Lambda$  hyperons, produced at the freeze-out (primary particles) in a relativistic heavy-ion collision, after the hydrodynamical evolution. This polarization is proportional to the thermal vorticity (9) at freeze-out hypersurface and could give an observable contribution [15]. The observation of this effect would be a confirmation of the achievement the local thermodynamical equilibrium also to spin degrees of freedom and would also indicate that vorticity and circulation may persist up to freeze-out.

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