Scaling of pressure spectrum in turbulent boundary layers

Saurabh S Patwardhan and O N Ramesh

Department of Aerospace Engineering, Indian Institute of Science, Bangalore, India

E-mail: psaurabh@aero.iisc.ernet.in

Abstract. Scaling of pressure spectrum in zero-pressure-gradient turbulent boundary layers is discussed. Spatial DNS data of boundary layer at one time instant ($\mathrm{Re}_\theta = 4500$) are used for the analysis. It is observed that in the outer regions the pressure spectra tends towards the -7/3 law predicted by Kolmogorov's theory of small-scale turbulence. The slope in the pressure spectra varies from -1 close to the wall to a value close to -7/3 in the outer region. The streamwise velocity spectra also show a -5/3 trend in the outer region of the flow. The exercise carried out to study the amplitude modulation effect of the large scales on the smaller ones in the near-wall region reveals a strong modulation effect for the streamwise velocity, but not for the pressure fluctuations. The skewness of the pressure follows the same trend as the amplitude modulation coefficient, as is the case for the velocity. In the inner region, pressure spectra were seen to collapse better when normalized with the local Reynolds stress ($-\overline{u'v'}$) than when scaled with the local turbulent kinetic energy ($q^2 = \overline{u'^2} + \overline{v'^2} + \overline{w'^2}$)

1. Introduction

The pressure spectra are not studied and understood as extensively as the velocity spectra. The main reason for this is the difficulty involved in measuring pressure in laboratory experiments. However, pressure is an important variable and, in an incompressible flow, it is governed by a Poisson equation and it is non-local in nature. In recent years, there has been considerable interest in the study of spectra of static pressure. Just as in the case of velocity, it has been hypothesized that the similarity arguments of Kolmogorov [1] may also apply to pressure fluctuations. These arguments yield a $\mathrm{k}^{-7/3}$ law for the longitudinal pressure spectrum in the wavenumber space. However, it is not entirely clear if the application of the local isotropy arguments to pressure spectrum is entirely justified [2] due to its non-local nature. This is because even regions far away from the point under observation make non-negligible contributions to the pressure fluctuations at the point of interest. Albertson *et. al.* [3] carried out static pressure measurements in an atmospheric boundary layer. They found that the large-scale motions exert strain on the inertial-range small-scale motions. This was expected to have an effect on the inertial-range pressure spectrum and hence to induce a deviation from the $\mathrm{k}^{-7/3}$ law. To explain the lack of a -7/3 law and the straining of inertial range pressure fluctuations, they made a connection between the Townsend's "active-inactive motion" hypothesis and the straining of the pressure field. Townsend [4] hypothesized that the turbulent motion close to the wall consists of two parts, namely "active motion" and "inactive motion". Active motion is the one responsible for the Reynolds stress in the wall region, while the inactive part is the motion induced by large-scale-induced fluctuation generated close to the wall [5]. Hence, it is conceivable that significant interaction exists between the large- and small-scale wave numbers in the spectral space, and the local isotropy assumption may not be strictly valid very close to the wall in the case of boundary layers. Tsuji *et. al.* [6] attempted to verify the existence of a -7/3 law in a high-Reynolds-number laboratory experiment ($Re_\theta = 20000$). However, they observed a -1 power law in the low-wavenumber region, and a -7/3 law was not evident at any wavenumber at any location in the boundary layer. George *et. al.* [7] carried out experiments on turbulent jets to measure the static pressure fluctuations. They found a reasonable agreement between the experimental data and theoretical predictions at wavenumbers which correspond to -7/3 ranges. The existence of a -7/3 law in free shear flows and its non-existence in experiments in wall bounded flows warrant a further study of the spectra of relatively less understood variables like pressure. The use of direct numerical simulations of high-Reynolds-number turbulent boundary layers will be extremely useful in the evaluation of such laws, since the main difficulty in the experiments is the accurate measurement of static pressure inside the boundary layer [8]. In this work, we carry out an analysis of a well-resolved DNS data [9] of pressure fluctuations to gain insights into the spectral laws.

2. Details of direct numerical simulation

Although the work did not involve development of the computer algorithm, the details of numerical method for the simulation will be briefly discussed for the sake of completeness. The details of the algorithm can be found elsewhere [10, 11]. The code uses a fractional-step method [12] that solves for velocity using an estimated pressure gradient term, and then corrects it by enforcing the mass conservation condition with the help of a Poisson equation for the pressure. The time stepping involves three sub-steps of a Runge-Kutta third-order method with the wall-normal viscous terms treated in a semi-implicit way using a Crank-Nicholson scheme. The first- and second-order derivatives in streamwise (x) and wall-normal (y) directions are calculated using staggered three-point compact finite differences. A pseudo-spectral scheme is used to calculate derivatives in the homogeneous spanwise (z) direction, using the 2/3 rule to prevent aliasing. The Reynolds number (based on momentum thickness) of the simulated boundary layer range from 2800 to 6600. The number of grid points is $15360 \times 535 \times 4096$ in x, y and z directions respectively.

3. Results and Discussion

Recently, Mathis *et. al.* [13] carried out a study to investigate the relationship between large and small scales in high Reynolds number boundary layer flows. The inner and outer interaction was identified in terms of the imprint of the outer-layer (large-scale) flow on the inner-layer (small-scale) flow. The effect was quantified by calculating the correlation between the large-scale component of the flow close to the wall and the large-scale component of the Hilbert transform of the small-scale component. They observed that a strong correlation between the envelope of the small- and large-scale parts of the same flow close to the wall. This led them to the conclusion that very-large-scale motions from the outer layer tend to modulate the small-scale part of the flow close to the wall. They also found a strong resemblance between the wall-normal profile of amplitude modulation and the skewness of the velocity fluctuations [14].

As mentioned earlier, Albertson *et. al.* [3] found that the large-scale pressure fluctuations strain the small-scale fluctuations, which could cause pressure spectra to deviate from the one predicted by the isotropy arguments. This hypothesis is further supported by observations in jets, where, unlike in boundary layers, there are no eddy hierarchies, and pressure spectra reasonably agree with the -7/3 law [7]. In the following, we will try to assess the DNS data and look for the inner-outer interactions in pressure fluctuations.

Before looking at the results of amplitude modulation of pressure fluctuations, we present some basic data that will allow us to proceed with further analysis [9]. Figure 1 shows the

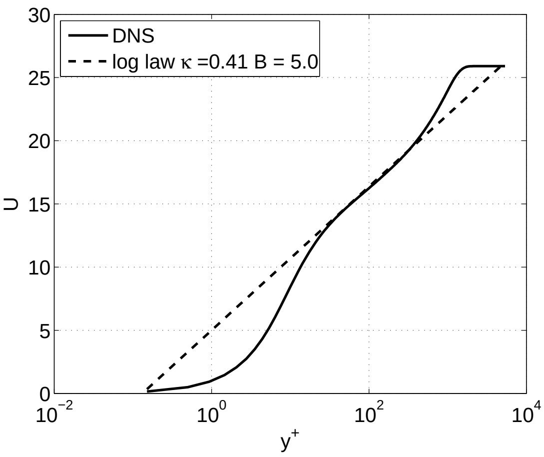

Figure 1. Mean velocity profile in zero-pressure-gradient turbulent boundary layer (Re_θ = 4500) [9].

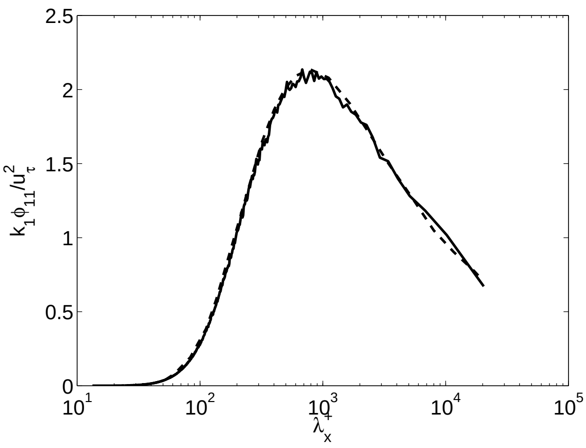

Figure 2. Premultiplied streamwise velocity spectra at $y_+ = 15$, from space series. Solid line: not time averaged. Dashed: time averaged from UPM data [9].

mean velocity profile of turbulent boundary layer flow. Also shown is the logarithmic-law profile with standard zero-pressure-gradient constants. The Reynolds number of the flow based on the momentum thickness is 4500. The spectra which will be presented in this paper are calculated from the spatial series at a given time, averaged in the homogeneous z direction. The streamwise extent of the space series is -7δ to 7δ. We expect the variation in boundary layer parameters, such as boundary layer thickness and skin friction coefficient, to be very small within that distance, allowing us to calculate the spectra assuming a locally parallel evolution of the boundary layer within a streamwise extent of 14δ. Also, we expect the correlations at such distances to be very small so that the flow does not suffer from memory effects. Note that the spectra are not time-averaged but the time averaging does not have a significant effect, as shown in figure 2. Apart from being a little wiggly in nature as expected, there is hardly any difference in shape and peak magnitude and hence it should not matter in the final conclusions of the study. Also, the amplitude modulation calculations are carried out on a space series and not on a time series as in original paper of Mathis *et. al.* [13], and the correlation is averaged in the z direction.

Figures 3 and 7 show pressure spectra at different wall-normal locations in Kolmogorov and outer scaling respectively. The spectra shown in all the figures here extend from $y_+ = 15$ to $y_+ = 1834$. Also shown in those figures are -1, -3/2 and -7/3 variations with wavenumber. Note that the pressure spectra obtained by Tsuji *et. al.* [6] showed k^{-1} to $k^{-3/2}$ like variation. It is interesting to note that the spectra very close to the wall neither behaves like a -1 nor a -3/2 power law, and even less like a -7/3 variation. As we move away from the wall, the -1 scaling region appears between wavenumbers $k_x\delta = 30$ to 60 in figure 7. Further away from wall, the slope continues to decrease and shows a -3/2 like behaviour. The decrease in slope continues further in the outer region and tends towards -7/3 far out in the boundary layer. To visualize this change in slope at different wall-normal locations, the premultiplied pressure spectra are plotted in figures 4-6 in Kolmogorov scaling, and in figures 8-10 in outer scaling. The pressure spectra in figures 4-6, and similarly in figures 8-10, are premultiplied with $(k\delta)^{7/3}$, $(k\delta)^{3/2}$ and $(k\delta)$ respectively. Premultiplication helps in a clear identification of the different slopes in the spectra by identifying the plateau region. In both these scalings, the region very close to the wall has a flat region in the spectra premultiplied with kδ. As we progressively move out in the boundary layer, the plateau region is observed in the spectra premultiplied with $(k\delta)^{3/2}$. Far out in the boundary layer and close to the edge of the boundary layer the plateau is observed

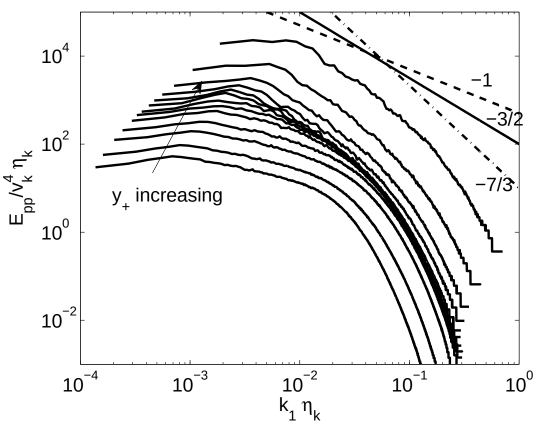

Figure 3. Pressure spectra at different wall-normal locations, normalized with Kolmogorov scale.

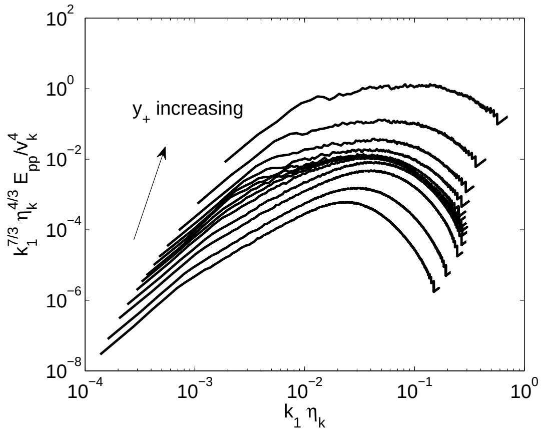

Figure 4. Premultiplied ($k^{7/3}$) pressure spectra at different wall-normal locations, normalized with Kolmogorov scale.

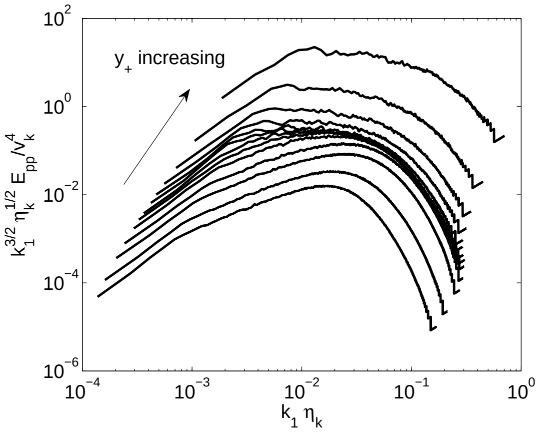

Figure 5. Premultiplied ($k^{3/2}$) pressure spectra at different wall-normal locations in normalized with Kolmogorov scale.

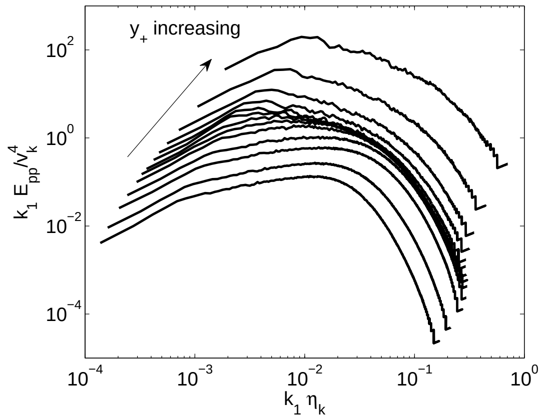

Figure 6. Premultiplied (k) pressure spectra at different wall-normal locations in normalized with Kolmogorov scale.

in the spectra premultiplied with $(k\delta)^{7/3}$.

The turbulent boundary layer has been traditionally divided into inner and outer regions. The inner region is largely influenced by the presence of the wall, while the outer region is largely free from the influence of the wall and to a certain degree acts like a free shear layer. In Townsend's [4] model, the inner region of the turbulent boundary layer consists of eddies that are attached to the wall while the outer layer is populated with eddies that are detached from the wall. However, recently it has been shown that the outer layer eddies or the superstructures show an imprint on the velocity fluctuations in the inner region [13]. Hence, the observation of a -7/3 law in the outer region, along with the observation of a -7/3 law in free shear layers [7], and the outer layer imprint on the inner region further suggests that inner-outer interaction might be playing a role in the disappearance of the -7/3 law in pressure spectrum.

From the above discussion, it is plausible that a significant amount of inner-outer interaction or straining of small scales by large-scale structures can cause deviations from the isotropic spectrum. Figure 11 shows spectra of streamwise velocity fluctuations at different wall-normal

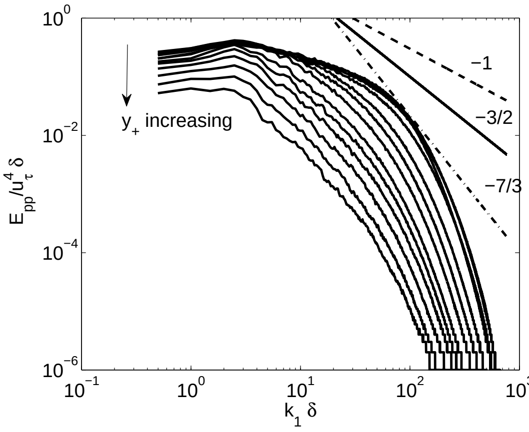

Figure 7. Pressure spectra at different wall-normal locations normalized with outer scales.

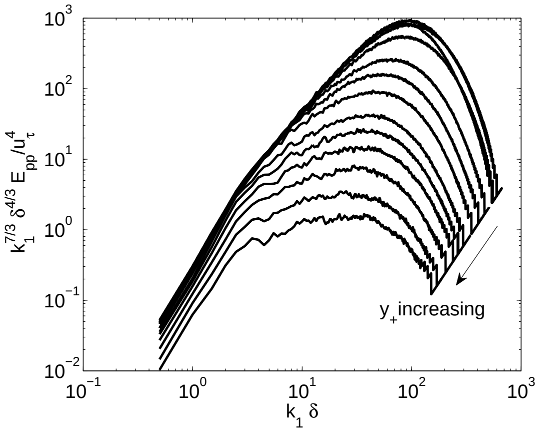

Figure 8. Premultiplied ($k^{7/3}$) pressure spectra at different wall-normal locations normalized with outer scales.

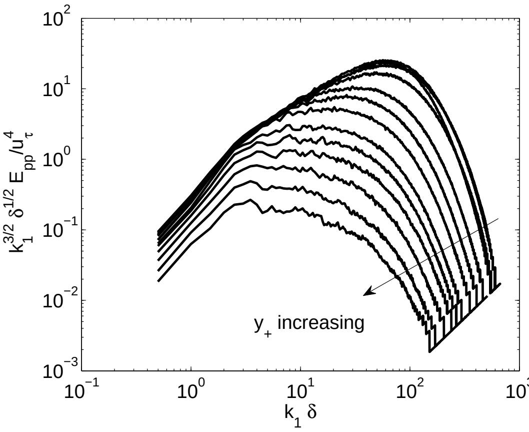

Figure 9. Premultiplied ($k^{3/2}$) pressure spectra at different wall-normal locations normalized with outer scales.

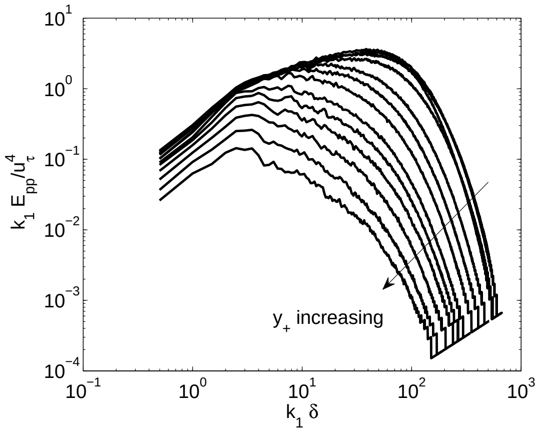

Figure 10. Premultiplied (k) pressure spectra at different wall-normal locations normalized with outer scales.

locations in the outer scaling. The celebrated -5/3 law of Kolmogorov is also shown in the figure as a dashed line. It is interesting to note that similar to the pressure, the velocity spectra does not show resemblance to -5/3 law close to the wall at $y_+ = 15$ to 96. Away from the wall, the slope decreases and it more looks like -5/3 in the outer layer, $y_+ = 846$. Note that the δ_+ value for this boundary layer is 1690. Similar to the pressure spectra, the premultiplied spectra for the streamwise velocity are plotted in figure 12 for clear visibility of the -5/3 slope in the outer region. The velocity and pressure spectra indicate a trend of moving near the isotropic prediction away from the wall, where the shear is less than in the near-wall region. To quantify the plausible inner-outer interaction in pressure, we carried out the amplitude modulation exercise on the pressure and velocity signals from the DNS data. The exercise makes use of space series and not time series as in the original procedure [13].

Here, we briefly describe the procedure for calculating the amplitude modulation which is essentially similar to the one used by Mathis *et. al.* [13] for time series. First we take the Fourier transform of the space series of the fluctuating quantity. The Fourier components are split into two parts: one representing the large-scale motion ($\lambda_x > \delta$) and the other representing small-

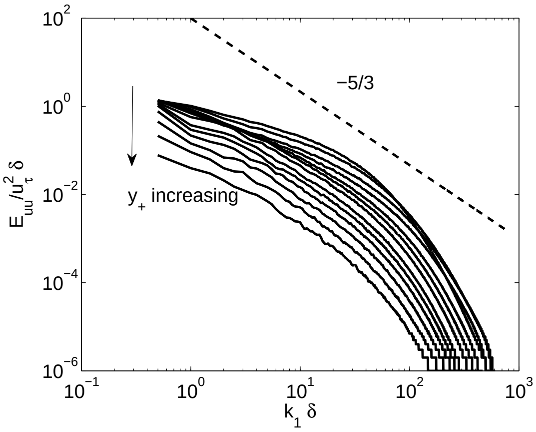

Figure 11. Streamwise velocity spectra at different wall-normal locations, normalized with outer scales.

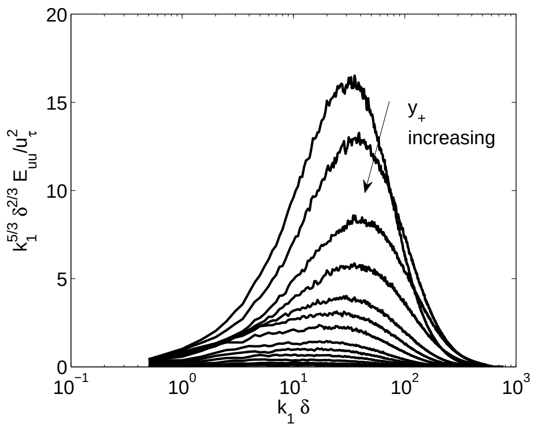

Figure 12. Premultiplied ($k^{5/3}$) streamwise velocity spectra at different wall-normal locations, normalized with outer scales.

scale motion ($\lambda_x < \delta$). The length scale δ is chosen as a threshold consistent with the choice of Mathis *et. al.* [13]. The original signal is then split into two parts by taking the inverse Fourier transform of the split Fourier components. Then, the envelope of the small-scale component is calculated by using a Hilbert transform technique. It is presumed that if there is an amplitude modulation effect of large scales on small scales then the large-scale envelope of the small scales should be well correlated with the large-scale part of the original signal. Hence, the envelope itself is split into small- and large-scale signals, and the correlation between the large-scale part of the envelope with the large-scale part of the original signal is calculated, which represents the amplitude modulation coefficient.

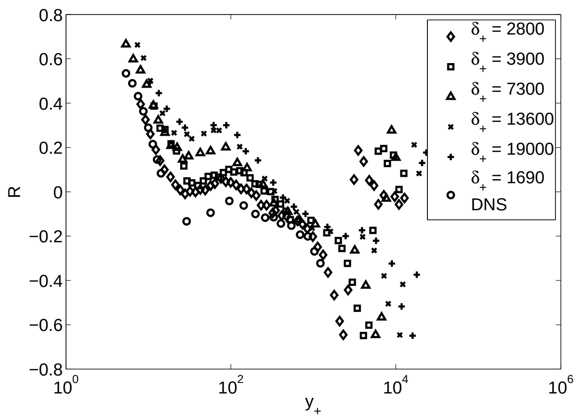

Figure 13. Amplitude modulation in the streamwise velocity signal.

Figure 13 shows the wall-normal variation of the amplitude modulation of the streamwise velocity. This plot represents the correlation between the large-scale part of the signal ($\lambda_x > \delta$) and the large-scale part of the Hilbert transform envelope of the small-scale part of the signal ($\lambda_x < \delta$) at a given wall-normal location. The circles are from DNS data at UPM [9], compared with the experimental data published in Mathis *et. al.* [13]. The experimental data have higher

Reynolds numbers than the DNS, the highest being an order of magnitude larger. The DNS data shows a correct trend both in the wall-normal profile and in the Reynolds-number effect. The velocity fluctuations close to the wall show a very high degree of amplitude modulation. The modulation effect decreases and becomes negative as we go away from the wall. The first minimum in the DNS data occurs below zero. The value of the first minimum is seen to increase with the Reynolds number, showing that the DNS data captures the right trend in regards to the variation of amplitude modulation. The result of the same exercise carried out on pressure fluctuations is shown in figure 14. Unlike the velocity, the pressure signals show no amplitude modulation close to the wall and the value of the correlation is zero. Surprisingly, the nature of the plot is similar to the velocity plot away from the wall, but the remarkable result is that there is no amplitude modulation close to the wall and that they are never positively correlated.

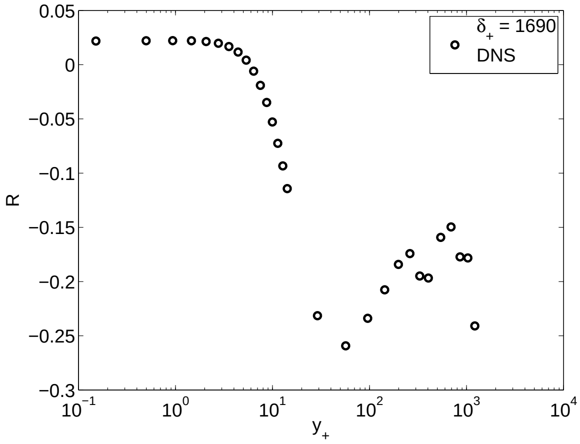

Figure 14. Amplitude modulation in the pressure signal.

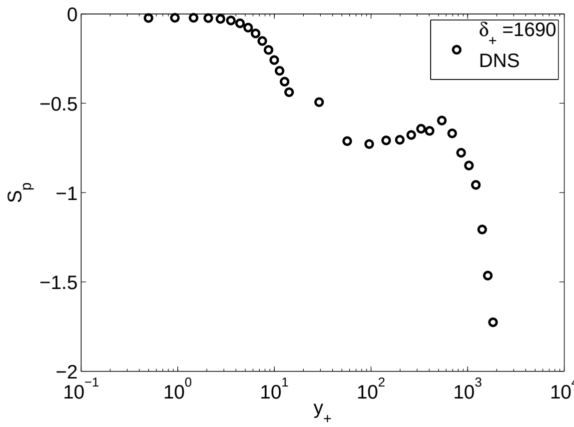

Figure 15. Variation of the skewness factor of pressure fluctuations.

Mathis *et. al.* [13] also observed a similarity between the amplitude modulation and the skewness factor for velocity. The same is evident in the case of pressure fluctuations, as seen in figure 15. The shape of the pressure skewness plot is similar to that of the amplitude modulation, which suggests that there is an inherent relationship between amplitude modulation and skewness for other variables as well.

3.1. Pressure spectra in other scalings

The result in the previous section that there is no modulating effect of the large-scale pressure fluctuations on the small-scale pressure fluctuations is surprising because the velocity shows a strong modulating effect. This raises the question as to how the outer region can influence the inner region in the absence of pressure modulation. It is possible that the modulation is mild (perhaps because of the moderate values of the Reynolds number) and hence velocity is more sensitive than pressure to large-scale influences. In order to understand further the inner-outer effect or "active–inactive" effect, we have plotted pressure spectra using normalizing variables other than u_τ. Figure 16 and 20 show pressure spectra normalized using $q^2 = \overline{u'^2} + \overline{v'^2} + \overline{w'^2}$ and $-\overline{u'v'}$ respectively. Two observations can be made from these plots. The spectra normalized by $\overline{u'v'}$ collapse well in the near wall region compared to the ones normalized by q^2. The region over which this good collapse is observed extends from $y_+ = 15$ to $y_+ = 329$. Also, away from the wall, spectra normalized with $\overline{u'v'}$ are closer to collapsing on a single line than the ones normalized by q^2. Again for better visualization of this, the compensated spectra are presented in figures 17-19 in q^2 scaling and in figures 21-22 in $\overline{u'v'}$ scaling. The original idea by

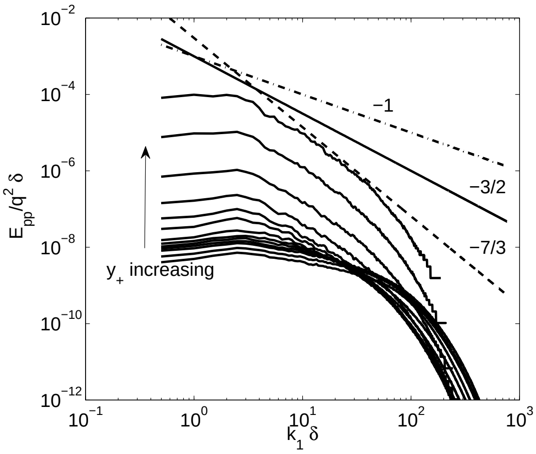

Figure 16. Pressure spectra normalized by q^2.

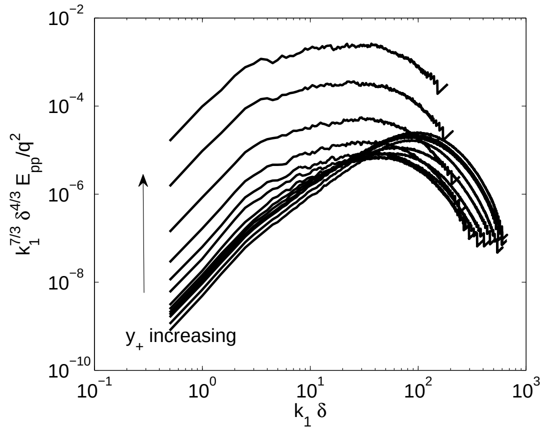

Figure 17. Premultiplied ($k^{7/3}$) pressure spectra normalized by q^2.

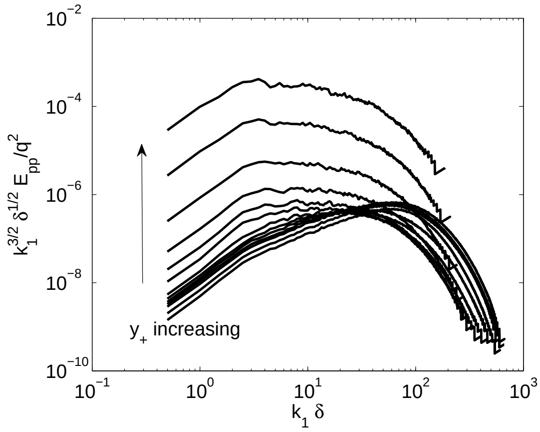

Figure 18. Premultiplied ($k^{3/2}$) pressure spectra normalized by q^2.

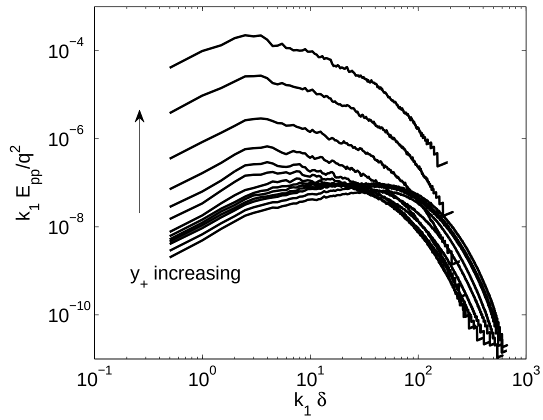

Figure 19. Premultiplied (k) pressure spectra normalized by q^2.

Townsend [4], that the inactive motion does not contribute to the shear stress but contributes to the kinetic energy, is appropriate in this context – a better scaling of pressure spectra with $\overline{u'v'}$ than q^2 is indicative of active motion. Such a better scaling of the pressure spectra with the Reynolds stress was previously observed in channel flows [15].

4. Conclusions

The main objective of this work was to analyze DNS data of pressure in a zero-pressure-gradient turbulent boundary layer in order to understand more about scaling of pressure spectra and the lack of a -7/3 law. Pressure spectra at different wall-normal locations in a boundary layer show that the slope decreases from -1 to lower values as we move away from the wall. Far out in the boundary layer at $y_+ = 1620$, the spectra is close to -7/3 line. The same trend is observed in the case of streamwise velocity spectra, which approach a -5/3 law in the outer region. This observation suggests that the shear near the wall is somehow responsible for the deviation from isotropic prediction. This also supports the possibility of inner-outer interaction, or active-inactive motion in Townsend's [4] words, which could cause large-scale straining of

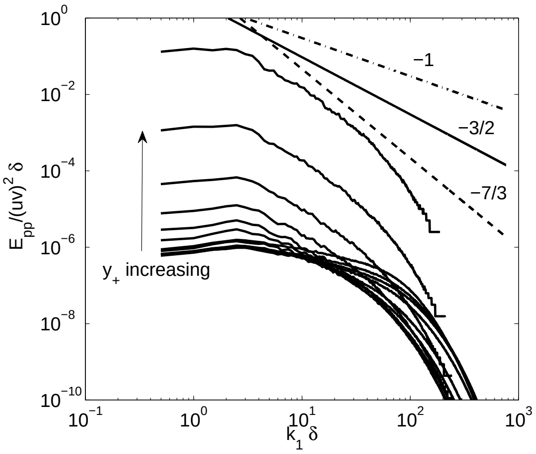

Figure 20. Pressure spectra normalized by $\overline{u'v'}$.

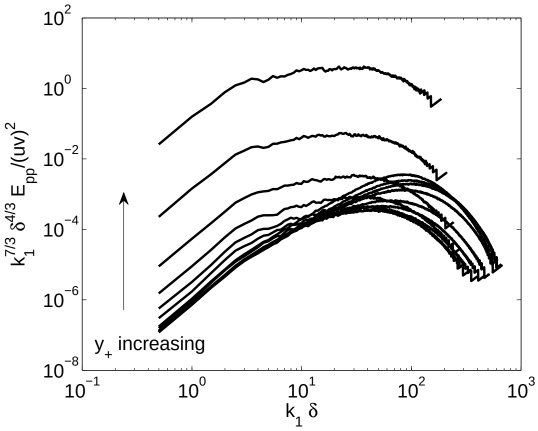

Figure 21. Premultiplied ($k^{7/3}$) pressure spectra normalized by $\overline{u'v'}$.

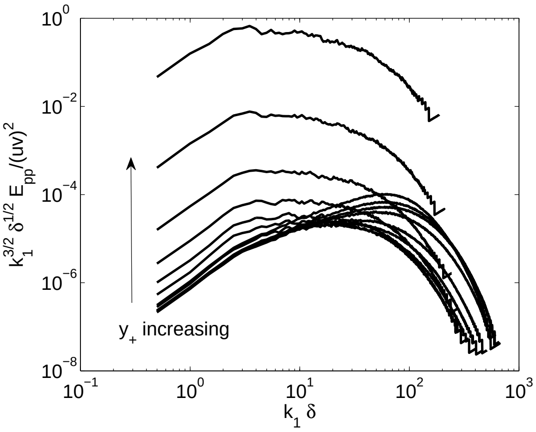

Figure 22. Premultiplied ($k^{3/2}$) pressure spectra normalized by $\overline{u'v'}$.

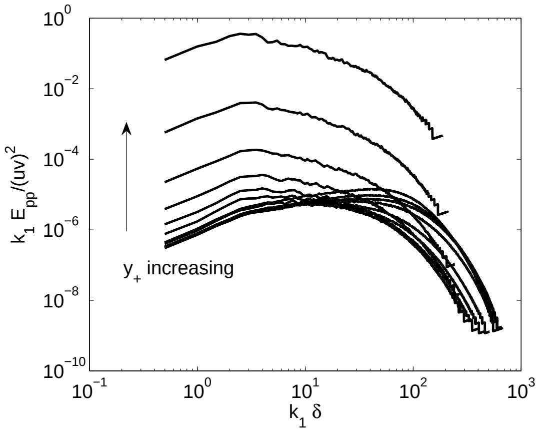

Figure 23. Premultiplied (k) pressure spectra normalized by $\overline{u'v'}$.

small scales in the near-wall region. To study this interaction, we followed the procedure of Mathis *et. al.* [13] to calculate the amplitude modulation effects on small scales by the large scales in the case of pressure fluctuations. However, to our surprise, we found that close to the wall there is no amplitude modulation of the pressure signal, unlike the streamwise velocity. The skewness of pressure also showed a shape similar to the amplitude modulation profile, as in the case of the velocity. Further, we investigated the pressure spectra in other scalings using as normalizing variables $\overline{u'v'}$ and q^2 (kinetic energy). The spectra scaled with $\overline{u'v'}$ show a better scaling compared to the ones scaled with q^2. This is consistent with the classical description by Townsend that inactive motion does not contribute to the Reynolds shear stress, but only to the kinetic energy.

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