

# Physics on the boundary between classical and quantum mechanics

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**Abstract.** Nature's laws in the domain where relativistic effects, gravitational effects and quantum effects are all comparatively strong are far from understood. This domain is called the Planck scale. Conceivably, a theory can be constructed where the quantum nature of phenomena at such scales can be attributed to something fundamentally simpler. However, arguments that quantum mechanics cannot be explained in terms of any classical theory using only classical logic seem to be based on sound mathematical considerations: there can't be physical laws that require "conspiracy". It may therefore be surprising that there are several explicit quantum systems where these considerations apparently do not apply. In the lecture we will show several such counterexamples. These are quantum models that do have a classical origin. The most curious of these models is superstring theory. This theory is often portrayed as to underly the quantum field theory of the subatomic particles, including the "Standard Model". So now the question is asked: how can this model feature "conspiracy", and how bad is that? Is there conspiracy in the vacuum fluctuations?

## 1. Introduction

When the foundations of quantum mechanics are investigated, this is often done from different perspectives. What is agreed about is that the theory contains mysteries which somehow require explanation, but then the question comes what 'clarification' of such mysteries might mean:

- i* One might assume that quantum mechanics as we experience it in today's understanding of the physical laws, is an approximation, an idealization, of some deeper underlying theory. The underlying theory is not quantum mechanical, but obeys classical logic, which in some approximation takes the shape of Schrödinger's equation. It is difficult to imagine a theory that is more basic than quantum mechanics, while it would only roughly mimic pure quantum mechanics, but this is the only choice one has if one takes for granted that theories with classical logic cannot reproduce quantum mechanics exactly.
- ii* Alternatively, one may accept quantum mechanics as it is, but desires to obtain a more precise picture of how exactly quantum mechanics describes physical reality. What is physical reality? Do we have an infinity of different universes that all can be called 'real', as in the Many World Hypothesis? Or should the wave function be interpreted exactly as in the Copenhagen interpretation, as a complex quantity that replaces the standard



probability function  $W$  in classical statistical mechanics?

- iii* Search for an explanation as to how quantum mechanics can emerge as an exact statistical description of an underlying theory, which, nevertheless, is suspected to exhibit classical logical equations anyway. In this case, one must also provide an explanation of how this can be reconciled with the usual arguments that such theories are impossible. Since these arguments are usually considered to be impeccable, such approaches are rare, but what can be done is this: let us simply try and investigate various candidate models that seem to be exactly equivalent to quantum mechanical systems, but will usually suffer from some defects. Then investigate these defects. Maybe we hit upon models that, even if they are classical, must exhibit some unusual features as well. These will be worth-while to investigate for their own sake.

This report is about progress made[1][2] in this last approach, number *iii*. It is clear that theories along these lines, which are deterministic but also appear to obey locality, so that they seem to be “local hidden variable theories”, must be quite different from what we are used to. Quite probably, their elementary mass- and distance scales are in the Planck domain, about which we have very little direct information. Regardless what kind of theories we will try, there are all sorts of difficulties here: not only do we have the difficulties associated with the unrenormalizable divergences, the black hole information paradox, and questions concerning non-trivial topologies in space-time, but one must also not forget that the group of Lorentz transformations is essentially non-compact, and this creates problems when we wish to consider the possibility of a ‘smallest distance’ in space and/or time. This may explain why our progress is slow, but we do wish to report progress anyway, and it is interesting.

To set the scene, we have a number of very illuminating ‘toy models’, which display very well the promising nature of this approach. Three examples will be discussed here. All three are integrable, which makes their discussion very transparent, and all three models share the property that they are classical and quantum mechanical at the same time:

- 1 The quantum harmonic oscillator. It is mathematically equivalent to a *classical point particle moving around the unit circle at constant speed*. More generally, we claim that the quantum harmonic oscillator is represented by any periodic classical system.
- 2 The chiral Dirac fermion. It is mathematically equivalent to *an infinite plane moving with the speed of light in one of the two possible directions orthogonal to the plane*.
- 3 The non-interacting sector of a fully quantized superstring, which is mathematically equivalent to a classical string moving over a target space that is a lattice[4]. The lattice spacing  $a$  obeys  $a = 2\pi\sqrt{\alpha'}$ , where  $\alpha'$  is the string slope parameter.

The interactions of strings with themselves still lead to problems that have not been solved. In the following sections we discuss these models. At the end, we discuss their potentials for the general quantum discussion.

## 2. The harmonic oscillator

Consider the spectrum of the harmonic oscillator. Its energy eigen states  $|n\rangle_E$  have energy eigen values

$$E_n = (n + \frac{1}{2})\hbar\omega . \quad (2.1)$$

First, consider a high-energy cut-off: the spectrum has an upper limit:  $n < N$ . We now also remove the vacuum energy  $\frac{1}{2}\hbar\omega$  as being not very essential for the discussion that follows<sup>1</sup>. Our Hilbert space is  $N$ -dimensional, so a basis has always  $N$  elements. Because of the equal spacings in the energy levels, we have indeed periodicity with period  $T$ :

$$e^{-2\pi i H/\hbar\omega} = 1, \quad T = 2\pi/\omega. \quad (2.2)$$

This suggests that we look at a new basis of states, labeled with an integer  $k = 0, \dots, N-1$ , and defined by the finite Fourier transformation:

$$|k\rangle_{\text{ont}} \equiv \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} e^{-2\pi i kn/N} |n\rangle_E, \quad k = 0, \dots, N-1. \quad (2.3)$$

This turns our system into a model with classical periodicity. At time steps that are integer multiples of  $\delta t = 2\pi T/N$ , the system evolves by permuting these “ontological” states:

$$|0\rangle_{\text{ont}} \rightarrow |1\rangle_{\text{ont}} \rightarrow \dots \rightarrow |N-1\rangle_{\text{ont}} \rightarrow |0\rangle_{\text{ont}}. \quad (2.4)$$

This is as classical as a model can be. The quantum number  $k$  in Eq. (2.3) will be called *beable*, in honor of J.S. Bell[3].

The *continuum limit*  $N \rightarrow \infty$  can be taken, but requires some caution<sup>2</sup>. It is instructive to compute the matrix elements of harmonic oscillator operators such as  $\hat{x}$  and  $\hat{p}$  in the ontological basis. Since we know the explicit wave functions of the energy eigen states in the  $x$  and the  $p$  basis, we can calculate these operators. For simplicity we put  $\omega = 1$ . The period is fixed,  $T = 2\pi$ , while the fundamental time steps  $\delta t$  become infinitesimally short.

Let the continuum limit then be described by the ontological states

$$|\varphi\rangle = \frac{1}{\sqrt{\mathcal{N}}} |k\rangle_{\text{ont}}, \quad \varphi = 2\pi k/N, \quad (2.5)$$

where  $\mathcal{N}$  is the norm that normalizes the  $|\varphi\rangle$  states such that  $\langle\varphi|\varphi'\rangle = \delta(\varphi - \varphi')$ . The matrix transforming these to the  $x$  eigen states and back is found to be

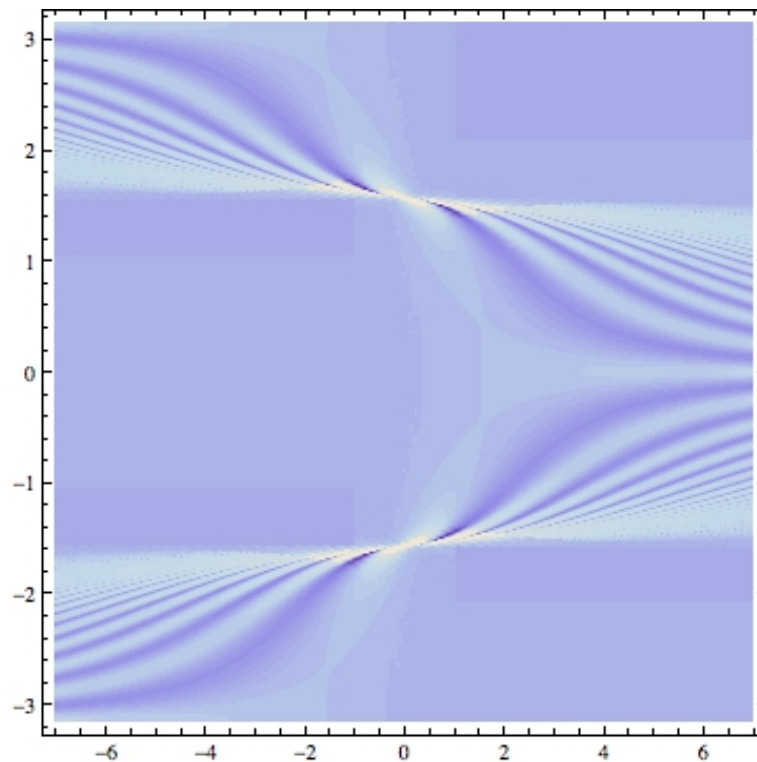
$$\langle\varphi|x\rangle = \sum_{n=0}^{\infty} \frac{H_n(x) e^{-\frac{1}{2}x^2}}{2^{(n+1)/2} \pi^{3/4} \sqrt{n!}} e^{in\varphi} \quad (2.6)$$

where  $H_n(x)$  are the hermite polynomials.

At all fixed values of  $x$  and  $\varphi$ , the sum converges, but very slowly, because we need to include all energy eigen states that contribute at a given  $x$ , until the energies are so high that the bulk of the wave function stretches much further, away from  $x$ . Mathematically working out the sum is laborious, and it does not lead to simple special functions because of the square root. We did find powerful approximation methods, and arrived at functions that can be plotted, see Figure 1. The most salient features of this matrix are visible here: the function oscillates wildly, and eigen values of the operator  $\hat{x}$  jump from predominantly positive to predominantly negative as  $\varphi$  passes the values  $\pm\frac{1}{2}\pi$ .

<sup>1</sup> The term  $\frac{1}{2}\hbar\omega$  makes the system *antiperiodic* with period  $T$ . But here we consider a total shift of the energy that removes the  $\frac{1}{2}\hbar\omega$  as a harmless procedure, so that the minus sign over an odd number of periods is ignored.

<sup>2</sup> In the case of finite  $N$ , the  $\hat{x}$  operator connects the lowest energy state  $|0\rangle_E$  to the highest energy state  $|N-1\rangle_E$ ; one would be tempted to ignore this term in the limit  $N \rightarrow \infty$ , in which case unitarity will be violated. The fact that there is a small subset of states where the mapping classical – quantum mechanical fails may be an important feature to keep in mind; it happens in other examples as well.



**Figure 1.** Plot of the real part of the transformation matrix  $\langle \varphi | x \rangle$ . Horizontally,  $x$  runs from  $-10$  to  $10$ , vertically,  $\varphi$  runs from  $-\pi$  to  $\pi$ . In the busiest parts of the picture, the rapid oscillations are no longer visible.

The most important conclusion of this chapter is that the quantum harmonic oscillator is closely related to completely classical periodic systems in Nature. In elementary particle physics, we encounter many oscillating modes whose energy packets are the fundamental particles. We now see that these may be linked to periodic structures in a classical underlying theory. Of course this leaves the question of interactions wide open, but one may guess where one could try to go from here: seek how interactions can be understood as violations of exact periodicity in the behavior of oscillating modes in whatever the ‘hidden variable’ theory might contain.

### 3. Massless chiral fermions in three spacelike dimensions

In a massless fermion, even when coupled to a gauge field such as the Maxwell or the Yang-Mills field, the helicity, defined as being the spin projected along the direction of motion, is a conserved quantity. This implies that we can postulate that the particle is always left-handed, or only right handed. In that case, the wave function is only a two-component spinor rather than the usual four components.

Let us consider the non-interacting case. We call our particle a “neutrino”, between quotation marks because it is the idealized version: no mass and no interaction. In the Majorana representation, the single, first quantized “neutrino” is described by the Hamiltonian

$$H = \vec{\sigma} \cdot \vec{p}, \quad (3.1)$$

where  $\vec{p}$  is the momentum operator and  $\vec{\sigma}$  are the Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (3.2)$$

Define

$$\hat{p} = \pm \frac{p}{|p|}, \quad s = (\hat{p} \cdot \vec{\sigma}), \quad r = \frac{1}{2}(\hat{p} \cdot \vec{x} + \vec{x} \cdot \hat{p}). \quad (3.3)$$

The sign here is arbitrary; it could be defined such that  $\hat{p}_3 \geq 0$ . The equations of motion for these operators are derived as follows:

$$\frac{d}{dt}\vec{x} = -i[\vec{x}, H] = \vec{\sigma}, \quad \frac{d}{dt}\vec{p} = 0, \quad (3.4)$$

$$\frac{d}{dt}\sigma_i = 2\varepsilon_{ijk}p_j\sigma_k; \quad \frac{d}{dt}\hat{p} = 0; \quad (3.5)$$

$$\frac{d}{dt}(\hat{p} \cdot \vec{\sigma}) = \pm 2\varepsilon_{ijk}(p_i/|p|)p_j\sigma_k = 0, \quad (3.6)$$

$$\frac{d}{dt}(\hat{p} \cdot \vec{x}) = \hat{p} \cdot \vec{\sigma}, \quad (3.7)$$

so that we have

$$\frac{d}{dt}\hat{p} = 0, \quad \frac{d}{dt}s = 0, \quad \text{and} \quad \frac{d}{dt}r = s = \pm 1. \quad (3.8)$$

Furthermore, we have

$$[\hat{p}, s] = 0, \quad [\hat{p}, r] = 0, \quad \text{and} \quad [s, r] = 0. \quad (3.9)$$

The first of these is obvious, and the last follows from the second. The fact that  $[\hat{p}, r] = 0$  can be found by explicit calculation, but also understood as a consequence of the fact that, in momentum space,  $r$  is the dilation operator divided by  $|p|$ , while  $\hat{p}$  is invariant under dilations (its norm stays one).

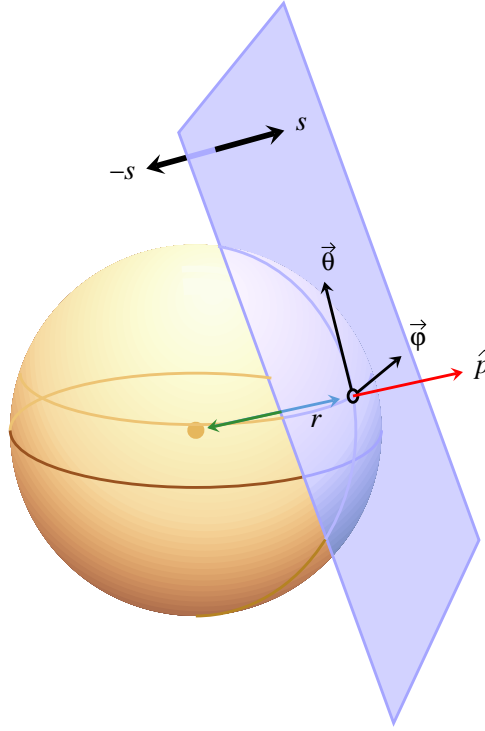
Together, Eqs. (3.8) and (3.9) imply that  $\hat{p}$ ,  $r$  and  $s$  form a set of *beables*[3], which are operators that commute with one another at all times. The best way to describe these observables physically is to say that they represent a *sheet* moving with the speed of light in one of the two possible orthogonal directions, see Fig. 2.

This is why we claim that the ontological theory behind a ‘neutrino’ is an infinite sheet moving with velocity  $v = c$  in one of its two possible orthogonal directions. If interactions or masses would be introduced, we have no obvious ontological theory anymore; one would have to search for the theory replacing the sheet in that case.

The mathematics of the transformation relating infinite sheets to neutrino states is delicate and interesting. The Hilbert space of ‘neutrinos’ can be assumed to be spanned by the states  $|\vec{p}, \alpha\rangle_{\text{neutrino}}$  or the states  $|\vec{x}, \alpha\rangle_{\text{neutrino}}$ , where  $\alpha = \pm 1$  is the spin variable, or, the eigenvalue of the Pauli spin operator  $\sigma_z$ . The sheet states will be denoted as  $|\hat{p}, r, s\rangle_{\text{sheet}}$ . By first Fourier transforming the  $r$  variable to get the states  $|\hat{p}, p_r, s\rangle_{\text{sheet}}$ , the orthogonal matrix relating these two sets of states can be calculated to be

$${}_{\text{neutrino}}\langle \vec{p}, \alpha | \hat{p}, p_r, s \rangle_{\text{sheet}} = p_r \delta^3(\vec{p} - \hat{p} p_r) \chi_\alpha^s(\hat{p}), \quad (3.10)$$

$${}_{\text{neutrino}}\langle \vec{x}, \alpha | \hat{p}, r, s \rangle_{\text{sheet}} = \frac{i}{2\pi} \delta'(r - \hat{p} \cdot \vec{x}) \chi_\alpha^s(\hat{p}), \quad (3.11)$$



**Figure 2.** The “neutrino” as an infinite sheet.  $\hat{p}$  is a unit vector, and therefore is a point of a unit sphere (actually half of a sphere, since its sign can be kept free). It defines the orientation of a sheet. The original coordinate variable  $\vec{x}$  sits in this sheet.  $r$  is the distance of the sheet to the origin. The vectors  $\vec{\theta}$  and  $\vec{\varphi}$  are used in the text.

where  $\chi_{\alpha}^s(\hat{p})$  is the spin eigen state of the operator  $s$  with eigen value  $s$ , in the basis of  $\sigma_z$ . Furthermore,  $\delta'(z) \equiv (\partial/\partial z)\delta(z)$ . The derivative stems from the factor  $p_r$ , which is necessary for a proper normalization of the sheet states.

To express the neutrino operators  $\vec{x}$  and  $\vec{p}$  in terms of sheet operators is not quite so easy. The *phases* of the sheet states still have to be defined. To do that right, we need an orthonormal set of two unit vectors  $\vec{\theta}$  and  $\vec{\varphi}$  on the sheet. They could be defined as in Figure 2. We then define the rotation operators  $L_i^{\text{ont}}$  for the sheet as the generators that rotate the sheet.  $L_i^{\text{ont}}$  do nothing but rotate the unit vectors  $\hat{p}$ .

We have

$$p_r = \pm |p|, \quad \vec{p}^{\text{neutrino}} = \hat{p} p_r, \quad (3.12)$$

but finding an operator  $\vec{x}$  with the proper commutation rules with  $\vec{p}$  is tedious. We first have to define spin flip operators  $s_1$  and  $s_2$ , obeying, just like the Pauli matrices:

$$s_3 = s, \quad s_i s_j = \delta_{ij} + i \varepsilon_{ijk} s_k. \quad (3.13)$$

We found (no derivation is given here)[1]:

$$x_i^{\text{neutrino}} = \hat{p}_i \left( r - \frac{i}{p_r} \right) + \varepsilon_{ijk} \hat{p}_j L_k^{\text{ont}} / p_r + \frac{1}{2p_r} \left( -\varphi_i s_1 + \theta_i s_2 + \frac{\hat{p}_3}{\sqrt{1 - \hat{p}_3^2}} \varphi_i s_3 \right). \quad (3.14)$$

Thus we arrive at a picture where ‘neutrinos’ are identified with flat sheets spanning to infinity, able to move with light speed in one of the two directions in a line orthogonal to the sheet, in a totally classical way. This indicates the potential of our theory but also the difficulties. The potential is that systems normally considered as purely quantum mechanical may yet be completely classical, in disguise. One might object that these sheets may transfer signals with infinite speed inside a sheet, so that indeed the model might be considered non-local in a sense. Perhaps this is the kind of non-locality that will be inevitable in “hidden variable” theories.

The difficulty clearly shows up when one wishes to consider mass terms and/or interactions. What can happen if two sheets interact at a point? Most likely one first has to second-quantize the system: have many neutrinos described by many sheets. But even then it is difficult to see how things can interact; a sheet cannot easily be made to change its direction of motion at one point by means of local interactions. One will have to consider the possibility of having imperfect sheets, but more likely models will be found that are more profoundly different.

#### 4. Real numbers and integers

Imagine that, in contrast to appearances, the real world, at its most fundamental level, were not based on real numbers at all. Real numbers are a man-made invention in response to the apparent continuous nature of our world, such as the positions and sizes of things. Our impression of everything being continuous received its first blow when it became quite evident that matter is made by atoms, which are discrete. Subsequently, it was found that energy is quantized. Now the energy quantum is controlled by the frequency of a wave function, and since time still seems to be continuous as far as we know, frequencies can be varied continuously, and in spite of our discovery of *quantum* mechanics, we still use real numbers in most of our calculations.

On the other hand, however, there are indications that, eventually, real numbers might get to be gradually replaced as our understanding of space and time further improves. A curious indication that this might happen is the theory of the information content of black holes. The microscopic regions near the horizon of a black hole seem to harbor only discrete amounts of information, in the form of *black hole microstates*. Information is fundamentally discrete. It is generally believed that space and time cannot be represented as a continuum of real-valued coordinates at distance scales as tiny as the Planck scale.

This is a motivation for searching for procedures to replace real numbers by integers. As it turns out, standard mathematics used in quantum mechanics provides us with the tools to do this. Again we use mappings, where one basis of Hilbert space is replaced by another. Surprisingly, what we shall derive is the following:

There is a natural way to map a Hilbert space where the basis elements are characterized by a set of two integers, which we shall call  $P$  and  $Q$ , onto a set of states where the basis elements are characterized by one real number.[5]

The mapping is one-to-one, with only one single state escaping at one side: it generates a singularity. The mapping can be trivially extended to the mapping of  $2N$  integers,  $P_i$  and  $Q_i$ ,  $i = 1, 2, \dots, N$ , onto basis elements of *either*  $N$  real numbers  $q_i$ ,  $i = 1, 2, \dots, N$  *or*  $N$  real numbers  $p_i$ . The construction is such that  $p_i$  and  $q_i$  can be handled as positions and momenta in accordance with canonical theories of mechanics. Because  $[q_i, p_j] = i\hbar\delta_{ij}$ , the  $|\vec{p}\rangle$  states and the  $|\vec{q}\rangle$  states are not independent.

The integers, which we shall always write as capitals  $Q_i$  and  $P_i$ , completely commute. For this reason, it will be relatively easy to formulate deterministic theories based on  $Q_i$  and  $P_i$ ,

since these can be the beables of our system.

The mapping is easy to formulate. Consider first just one set of states  $|Q\rangle$ , where  $Q$  is an integer. It may run from  $-\infty$  to  $\infty$ . We can then introduce the unitary step operator  $U$  by

$$U|Q\rangle = |Q+1\rangle. \quad (4.1)$$

Since  $U$  is unitary, and is easily seen to be non-degenerate, we can write it as

$$U = \epsilon^{i\eta}, \quad \epsilon \equiv e^{2\pi} \approx 535.5; \quad \epsilon^{Ni\eta}|Q\rangle = |Q+N\rangle, \quad (4.2)$$

for any  $N$ , where  $-\frac{1}{2} < \eta \leq \frac{1}{2}$ . The quantity  $\eta$  is an operator, and its relation to the  $Q$  operator is easy to derive:

Consider the function  $\eta$  in the domain  $-\frac{1}{2} < \eta < \frac{1}{2}$ . It is easy to Fourier transform this function,

$$\eta = \sum_{N=-\infty}^{\infty} a_N \epsilon^{iN\eta}, \quad a_N = \int_{-\frac{1}{2}}^{\frac{1}{2}} \eta d\eta \epsilon^{-iN\eta} = \frac{i(-1)^N}{2\pi N} \text{ if } N \neq 0, \quad (4.3)$$

and  $a_0 = 0$ . Therefore, we can write

$$\eta = \sum_{N \neq 0} \frac{i(-1)^N}{2\pi N} \epsilon^{iN\eta}, \quad (4.4)$$

so that

$$\langle Q_1 | \eta | Q_2 \rangle = \frac{i}{2\pi} (1 - \delta_{Q_1, Q_2}) \frac{(-1)^{Q_1 - Q_2}}{Q_1 - Q_2}. \quad (4.5)$$

It follows that

$$[\eta, Q] = \frac{i}{2\pi} (\mathbb{I} - |\psi_0\rangle\langle\psi_0|), \text{ with } \langle Q | \psi_0 \rangle \equiv (-1)^Q. \quad (4.6)$$

Here the single state  $|\psi_0\rangle$  describes the one state for which the commutator is not the canonical one. We shall encounter such states more often; it is the exceptional state alluded to earlier, to be referred to as the *edge state*. We observe that this is also the state for which the operator  $\eta$  has the value  $\pm\frac{1}{2}$ , exactly the edge of the Fourier domain in Eq. (4.3).

The mapping that we wish to use is the mapping from the  $Q$  basis to the  $\eta$  basis and back. Just because we prefer to put  $\eta$  on an open interval rather than a closed circle, we encounter the ‘problem’ of the edge state; it will have to be taken proper care of. The transformation matrix is simply  $\epsilon^{iQ\eta}$ . Because of our special choice of  $\epsilon$  as the base of our exponentials, the normalization of the matrix became trivial.

Now, consider a real number  $q$ . Let us write

$$q = Q + \eta_P, \quad (4.7)$$

where  $Q$  is integer and  $\eta_P$  lies between 0 and 1 (as much as possible, we intend to write real numbers as lower case letters, integers as capitals, and numbers *modulo* 1, or fractional numbers between 0 and 1, as Greek letters). Since this splitting is unique, we can write basis elements  $|q\rangle$  as

$$|q\rangle = |Q, \eta_P\rangle. \quad (4.8)$$



Next, we can Fourier transform

$$|\eta_P\rangle = \sum_{P=-\infty}^{\infty} \epsilon^{iP\eta_P} |P\rangle, \quad (4.9)$$

where  $P$  is integer. Any state  $|\psi\rangle$  can thus be written in the real  $q$  basis or in the  $Q, P$  basis:

$$\langle q|\psi\rangle = \langle Q, \eta_P|\psi\rangle = \sum_P \epsilon^{-iP\eta_P} \langle Q, P|\psi\rangle. \quad (4.10)$$

This is what we mean when we state that quantum mechanics on a basis described by a real number  $q$  can be transformed to quantum mechanics with a basis of a pair of integers,  $Q$  and  $P$ .

Subsequently, one may transform to momentum space, using the matrix

$$\langle q|p\rangle = \epsilon^{ipq}. \quad (4.11)$$

Notice however, that this procedure is not quite symmetric under the interchange  $Q \leftrightarrow P$ ,  $q \leftrightarrow p$ , but this symmetry can be restored. At the same time, one can reduce the effects of edge states. This we do by multiplying the coefficients of a wave function  $|\psi\rangle$  in the space of the pairs  $\eta_Q, \eta_P$ , by a complex phase,  $\epsilon^{i\varphi(\eta_Q, \eta_P)}$ . This phase can be chosen such that the coefficients become exactly periodic both in  $\eta_P$  and in  $\eta_Q$ . This would remove the edge state completely, except for the fact that the phase function  $\epsilon^{i\varphi}$  will feature a vortex singularity. This means that, at the very location of this vortex, we still will have an edge state.

The result of this procedure, requiring a calculation that will be published elsewhere[5], is that we obtain a transformation from the basis  $\langle q|$  or  $\langle p|$  to a basis of states  $\langle Q, P|$  that, apart from a few signs, is entirely symmetric. We write for the  $q$  and  $p$  operators

$$q = Q + a_Q, \quad p = P + a_P, \quad (4.12)$$

where the operators  $a_Q$  and  $a_P$  are not exactly restricted to the interval  $(-\frac{1}{2}, \frac{1}{2})$ , but they will stay of order one:

$$\begin{aligned} \langle Q_1, P_1|a_Q|Q_2, P_2\rangle &= \frac{(-1)^{P+Q+1} i P}{2\pi(P^2 + Q^2)}, \\ \langle Q_1, P_1|a_P|Q_2, P_2\rangle &= \frac{(-1)^{P+Q} i Q}{2\pi(P^2 + Q^2)}. \end{aligned} \quad (4.13)$$

From these, one derives that

$$[q, p] = \frac{i}{2\pi}(1 - |\psi_{\text{edge}}\rangle\langle\psi_{\text{edge}}|), \quad \text{with} \quad \langle Q, P|\psi_{\text{edge}}\rangle = (-1)^{Q+P}. \quad (4.14)$$

In  $(\eta_Q, \eta_P)$  space, the edge state  $|\psi_{\text{edge}}\rangle$  is the delta peak on the spot  $(\eta_Q, \eta_P) = (\pm\frac{1}{2}, \pm\frac{1}{2})$ , exactly where we located the vortex of our phase function  $\varphi(\eta_Q, \eta_P)$ . The eigen values of both operators  $q$  and  $p$  in Eq. (4.12) occupy the entire real line without overlappings, but we do have to restrict ourselves to the states that are orthogonal to the edge state  $|\psi_{\text{edge}}\rangle$ .

## 5. Free massless bosons in 1+1 dimensions

The field operators in a quantized field theory also have their eigenvalues on the real line. We would like to apply the results of the previous chapter to such theories, but in general this would not produce viable deterministic models. The reason is, that the splitting (4.12) does not transform in a simple way when two real numbers are added or subtracted.

However, the  $(Q, P) \leftrightarrow (\hat{q}, \hat{p})$  mapping can nevertheless be significant for field theories, if we can arrange the dynamical variables in such a way that the classical field equations involve displacements without additions or subtractions.

The rest of this chapter was written down earlier in Ref.[2]. Take a free, massless quantum field theory in 1+1 dimensions, having field variables  $\phi(x, t)$  and a canonical momentum field  $p(x, t)$  obeying commutation rules

$$[\phi(x, t), p(x', t)] = \frac{i}{2\pi} \delta(x - x') , \quad [\phi(x, t), \phi(x', t)] = [p(x, t), p(x', t)] = 0 . \quad (5.1)$$

The Klein-Gordon equation,

$$(\partial_x + \partial_t)(\partial_x - \partial_t)\phi(x, t) = 0 , \quad (5.2)$$

implies the existence of left-movers  $\phi^L(x + t)$  and right-movers  $\phi^R(x - t)$ :

$$\phi(x, t) = \phi^L(x + t) + \phi^R(x - t) , \quad p(x, t) = \frac{1}{2}a^L(x + t) + \frac{1}{2}a^R(x - t) , \quad (5.3)$$

$$a^L(x + t) = p(x, t) + \partial_x \phi(x, t) , \quad a^R(x - t) = p(x, t) - \partial_x \phi(x, t) . \quad (5.4)$$

In fact, up to some coefficients, these are Fourier transforms of the familiar particle creation and annihilation operators. We have the hamiltonian density

$$\mathcal{H} = \frac{1}{2}(p^2 + (\partial_x \phi)^2) = \frac{1}{4}(a^{L2} + a^{R2}) . \quad (5.5)$$

In terms of the left- and right-movers, the commutation rules are

$$[a^L, a^R] = 0 , \quad [a^L(x), a^L(y)] = \frac{i}{\pi} \partial_x \delta(x - y) , \quad [a^R(x), a^R(y)] = \frac{-i}{\pi} \partial_x \delta(x - y) . \quad (5.6)$$

Replacing the spacetime continuum by a (dense) lattice, one sees that these commutation rules can be replaced by

$$[\phi(x, t), p(y, t)] = \frac{i}{2\pi} \delta_{x,y} , \quad [a^L(x), a^L(y)] = -[a^R(x), a^R(y)] = \pm \frac{i}{2\pi} \quad \text{if } y = x \pm 1 . \quad (5.7)$$

Now, suppose we introduce integer valued operators  $A^{L,R}(x)$  and their associated momentum operators  $\eta^{L,R}(x)$ , obeying

$$[\eta^L(x), A^L(y)] = [\eta^R(x), A^R(y)] = \frac{i}{2\pi} \delta_{x,y} , \quad [A^L, A^R] = [A^L, \eta^R] = 0 , \quad \text{etc.} \quad (5.8)$$

(ignoring the usual edge states), one finds that we can write

$$a^L(x) = A^L(x) + \eta^L(x + 1) , \quad a^R(x) = A^R(x) + \eta^R(x - 1) , \quad (5.9)$$

so that the commutation rule (5.7) is automatically obeyed.

With some more advanced mathematics, one can reduce the effects of the edge states to a minimum (they cannot be ignored completely), as was briefly explained in the previous section.

The importance of this procedure, and the reason why it *only* works in one space, one time dimension, is that the time evolution of the  $a^{L,R}$  fields involves nothing more than shifts in  $x$  space, without further linear transformations such as additions or subtractions. Linear transformations in the variables  $A^{L,R}$  and  $\eta^{L,R}$  do not lead to similar operators where the  $A$  are integer and the  $\eta$  stay in the interval  $(\frac{1}{2}, \frac{1}{2}]$ .

We now observe that, in string theory, the  $\phi$  fields are the left- and right moving coordinates  $X^\mu$  in a  $D$  dimensional space-time. For convenience, we had chosen the world sheet lattice to have lattice length  $a = 1$ , but it is easy to verify that, if we had chosen any other lattice length, the relation between the quantized variables  $A(x)$  and the space-time coordinates  $X^\mu(x)$  would remain the same. In other words, just because we wished to relate the commutation rules (5.1) and (5.6) to the commutation rules (5.8), and have the relations (5.9) such that the real numbers are smoothly covered, we find that the classical system has its spacetime coordinates  $X^\mu$  defined on a grid, and the grid length is fixed. Re-inserting the usual string units, one finds a space-time lattice with lattice length[4]

$$a_{\text{spacetime}} = 2\pi\sqrt{\alpha'}, \quad (5.10)$$

a remarkable result. Note, that the classical theory has its string equations simply formulated in terms of the integer valued operators  $A^{L,R}(x, t)$  (where  $x$  and  $t$  are the world sheet variables usually denoted as  $\sigma$  and  $\tau$ ). From the field equations (5.2) – (5.4), one derives that the integer-valued string coordinates obey simple classical equations,

$$X^\mu(\sigma, \tau + a) + X^\mu(\sigma, \tau - a) = X^\mu(\sigma + a, \tau) + X^\mu(\sigma - a, \tau). \quad (5.11)$$

These equations must be assumed to describe the transverse string coordinates only. The longitudinal and timelike coordinates have to be derived from the usual string constraint equations. This part of the quantum theory remains unaffected. It means that, only in 26 dimensions, or in 10 dimensions if we add the fermionic degrees of freedom (see the next section), the quantum theory has an enhanced symmetry: Lorentz invariance.

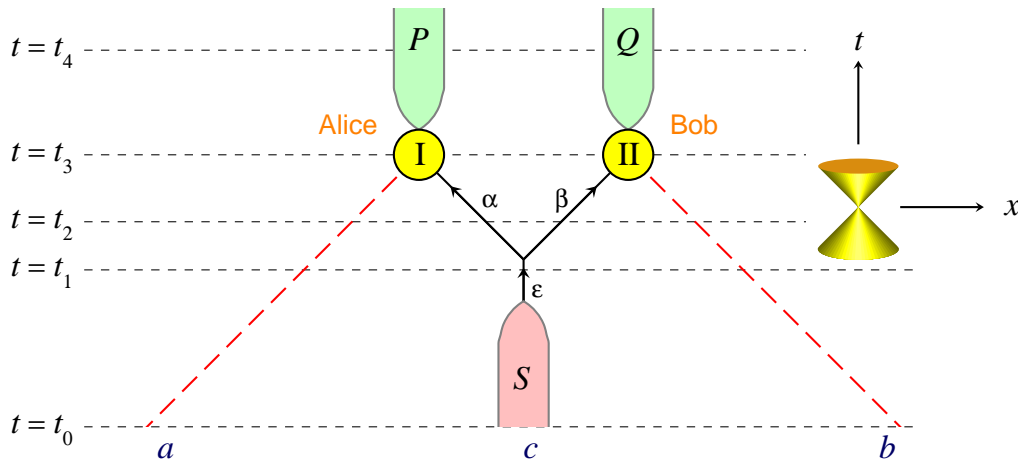
Curiously, Lorentz invariance here is a *quantum* symmetry, which means that its transformations act non-trivially in the quantum  $q, p$  Hilbert space; it is not conspicuously present in the classical  $Q, P$  variables.

The quantized *superstring* contains fermionic degrees of freedom in addition to the bosonic coordinates. Remarkably, these fermionic modes can also be rephrased in classical terms: *Fermions are boolean variables* (variables taking only two values, such as  $\pm 1$ ), existing on the fundamental links of the string. The independent fermionic modes are again the ones in the transverse direction; the longitudinal fermionic modes are constrained by (quantum) equations.

The constraint equations that are needed to guarantee Lorentz invariance have no effect on the bulk behavior of the (super)string, but do appear to cause problems when we consider string interactions. Deterministic interactions can be considered, but it is not obvious that these coincide with the standard quantum interactions of the usual quantum string. This may turn out to be a serious limitation of the present considerations, but even if we would have our mappings only for the non-interacting strings, we found it quite worth-while to report this.

## 6. Bell inequalities and other fundamental quantum features

Not much of this chapter is new, but it had to be added to clarify the author's position concerning J.S. Bells's important observation[6]: local classical theories cannot reproduce the



**Figure 3.** Space-time diagram for a Bell-type experiment, see text.

correlations generated in entangled particles that may typically arise in a quantum experiment. Schematically, we imagine that at  $t = t_1$  (see Fig. 3), two entangled particles  $\alpha$  and  $\beta$  are produced; think of photons whose spins add up to zero. At  $t = t_3$ , two widely separated observers, Alice and Bob, check whether their photon passes through a filter. The polarization angles of the filters are given by the normalized vectors  $\vec{a}$  and  $\vec{b}$ , or angles  $a$  and  $b$ . Alice and Bob each have the freedom to choose their angles.

Using quantum mechanics, we can calculate under all circumstances how Alice's and Bob's measurements are correlated. According to Bell, these correlations can never arise from classical signals travelling with the photons; a direct, "spooky" signal has to be assumed to be transmitted between Alice and Bob *after* they chose the orientations of their filters. Alice and Bob chose their orientation angles both by free will; there was no way these angles could have been known prior to  $t = t_3$ .

At first sight, this argument can easily be countered when deterministic theories are considered: *Alice and Bob do not have such a free will*; their decisions were already decided at the very beginning of the Universe.

But this is not good enough. Suppose that Alice and Bob both base their decisions on measurements of fluctuations in signals coming from distant sources  $a$  and  $b$ . We may imagine that these are distant quasars that emitted their photons at  $t = t_0$ , shortly after the Big Bang. One should be allowed to assume that these signals are uncorrelated. In that case also, the outcome of the experiment then cannot be explained by classical arguments.

The correlations that emerge in simple quantum mechanical descriptions of entangled photons will require correlations in the quasars  $a$  and  $b$  as well. A simple calculation yields the required correlations as follows. The probability distribution  $W$  between the angle  $a$  imposed by Alice in her experiment, the angle  $b$  that Bob decided upon, and the angle  $c$  that one might assume to be carried by both entangled photons (in a classical effective description), is then required to be at least of the form

$$W(\vec{a}, \vec{b}, \vec{c}) \propto |\sin(2a + 2b - 4c)|. \quad (6.1)$$

Notice that, if we randomize either  $a$ , or  $b$  or  $c$ , the two remaining quantities become uncorrelated, so if we drop our information of the photons  $c$ , then Alice and Bob need no exchange of signals. The problem is that in a classical theory, one would have expected a classical observable  $c$ , not correlated to  $a$  or  $b$  at all.

Eq. (6.1) describes what is usually called “conspiracy” [7]. The photons should not “know” what Alice and Bob are going to do. One could imagine exceedingly complex devices that randomize the signals  $a$  and  $b$  on their way from the quasars  $a$  and  $b$  to the filters  $I$  and  $II$  used by Alice and Bob. It does not make sense if the photons could anticipate the outcome of these calculations.

On the other hand there may be reasons to demand caution. Relativistic quantum field theory does explicitly lead one to observe the existence of *vacuum fluctuations*. These are fluctuations that fields undergo in the lowest energy state, the vacuum. Not only do vacuum fluctuations show correlations between data that are spacelike separated (the propagators of the fields do not vanish outside the light cone), but they also have the property that all scattering amplitudes, and therefore also the future behavior, can be obtained from these correlation functions, simply by analytic continuation. In this sense, there is conspiracy, while this is a kind of conspiracy that never worried anyone.

We must conclude that statistical arguments may be invalid in deterministic theories. Also, counterfactual arguments are not acceptable. In fact, Bell’s gedanken experiment was useful to set the scene, but the problem in question already arises for single, unentangled photons. As soon as an observer decides to choose the polarisation angle of a filter with the aim of measuring photons, the wave functions of these photons are diagonalized according to the coordinate frame generated by the filter, and extrapolating to the distant past, the photon *always was* in an ontological state defined by the filter that lies in its future. This conceptual difficulty will have to be accepted or ignored in all deterministic theories. What counts is that we have models that seem to exhibit such behavior. Yet at the classical level there is no trace of any conspiracy.

As an intermezzo, consider an example from number theory. Consider an “arbitrary” large prime number  $P$ . Consider an even large number  $Q$  defined by

$$Q = 2^{P-1} - 1. \quad (6.2)$$

What is the expected distribution of the number

$$P_1 = Q \text{ modulo } P? \quad (6.3)$$

One might have expected a flat distribution, but in fact,  $P_1$  is always zero, a simple result from number theory. This illustrates our statement that statistical arguments in deterministic systems should not be trusted.

Let us emphasize that the peculiarity found by Bell in his considerations for entangled particles indeed leads to “conspiratorial” correlations in the classical theories we describe. We suspect that statistical arguments, and also counterfactual arguments [8] are illegal for our classical models, but how to explain this apparent ‘conspiracy’?

A state considered in some experimental setup may either be a physical state, which we shall call ‘ontological’, or it is a superposition of ontological states. In our analysis of the experiment, such states are indistinguishable. However, if an ‘ontological basis’ exists, which we believe to be the case, then there is a *conservation law*: the ontological nature of a state is conserved in time. If, at some late time, a photon is observed to be in a given polarization state, just because it passed through a filter, then *that* is its ontological state, and the photon has been in that ontological state from the moment it was emitted by its source. It seems to be inevitable to

demand that the ‘ontological basis’ is unobservable, that is, indistinguishable from other bases, as it is for instance in our description of string theory. ‘Conspiracy’ is then unobservable.

Let us also stress the importance of another observation: in quantum field theory, locality means that the Hamiltonian can be written as the integral over space of a Hamiltonian density. The Hamiltonian density is the generator of local time translations, and hence it is part of the algebra of general coordinate transformations. This may mean that general relativity will be an essential ingredient of a complete quantum theory; without general relativity, locality of quantum mechanics may be difficult to understand. Indeed, the same conclusion can be reached by considering the black hole microstates, which quite possibly correspond to the ultimate, classical degrees of freedom of an underlying theory, while they fundamentally arise at the Planck scale only. The problem of “quantizing” curved space-time, quantum gravity for short, is notoriously complex and far from understood. It may well be that a complete understanding of the quantum nature of our world will have to come together with the complete resolution of the quantum gravity problem.

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