

Thermoconvective instability and local thermal non-equilibrium in a porous layer with isoflux-isothermal boundary conditions

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Abstract. The effects of lack of local thermal equilibrium between the solid phase and the fluid phase are taken into account for the convective stability analysis of a horizontal porous layer. The layer is bounded by a pair of plane parallel walls which are impermeable and such that the lower wall is subject to a uniform flux heating, while the upper wall is isothermal. The local thermal non-equilibrium is modelled through a two-temperature formulation of the energy exchange between the phases, resulting in a pair of local energy balance equations: one for each phase. Small-amplitude disturbances of the basic rest state are envisaged to test the stability. Then, the standard normal mode procedure is adopted to detect the onset conditions of convective rolls. Beyond the Darcy-Rayleigh number, playing the role of order parameter for the transition to instability, the relevant dimensionless parameters are the inter-phase heat transfer parameter and the thermal conductivity ratio. The disturbance governing equations, formulated as an eigenvalue problem, are solved numerically by a shooting method. Results are reported for the neutral stability curves and for the critical values for the onset of instability.

1. Introduction

The onset of thermoconvective instability in a horizontal fluid-saturated porous layer heated from below has been extensively studied in the last decades. The attention deserved by this subject is due to the many applications of this interdisciplinary research field, ranging from geophysical research to biophysical applications as well as to petroleum and heat transfer engineering. A detailed discussion of the literature on this subject can be found in the books by Nield and Bejan [1], and by Straughan [2], as well as in the reviews by Tyvand [3], Rees [4], and Barletta [5]. Cornerstone studies of the onset of thermoconvective instability in porous layers are reported by Horton and Rogers [6] and Lapwood [7]. They presented the first linear stability analysis of what is now well-known as either the Horton-Rogers-Lapwood problem, or the Darcy-Bénard problem. The Darcy-Bénard problem is closely linked to the classical Rayleigh-Bénard problem for clear fluids. It consists of a basic motionless state with a uniform temperature drop across a porous layer with a heated isothermal lower wall and a cooled isothermal upper wall.

A variant of the Darcy-Bénard problem will be investigated in this paper. Darcy's law for momentum transfer in the porous medium is assumed to be valid, and a two-temperature model, a temperature for the fluid phase and a temperature for the solid phase, is employed. Our



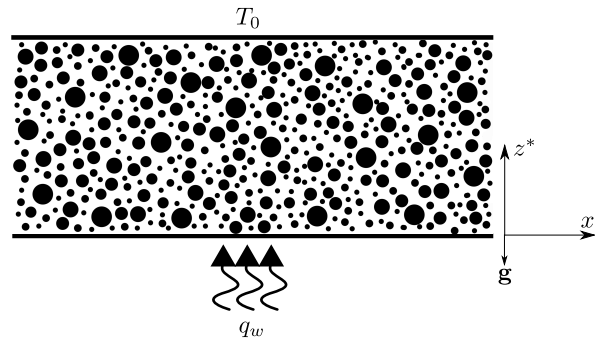


Figure 1. Sketch of the porous layer.

aim is to analyse the effect of the local thermal non-equilibrium model on the onset of the thermoconvective instability. An inter-phase heat transfer coefficient is defined and the local energy balance equations, one for the fluid phase and one for the solid phase, contain heat exchange terms proportional to the local temperature difference between the two phases [1, 8, 9].

We aim to extend two previous analyses [10, 11] of the Darcy-Bénard problem with local thermal non-equilibrium. Banu and Rees [10] considered a porous layer bounded by impermeable isothermal walls, while Barletta and Rees [11] studied the case where both the boundary walls are impermeable and isoflux.

In our paper we will assume, in analogy with [10, 11], that the boundary walls are impermeable. On the other hand, unlike in the studies reported in [10, 11], we will analyse asymmetric temperature conditions, where the upper wall is kept at constant temperature, while a uniform heat flux is prescribed on the lower wall. The uniform heat flux condition will be formulated according to the Amiri-Vafai-Kuzay Model A [11, 12, 13]. The latter model is valid when the boundary wall has a finite thickness and a high thermal conductivity. A linear stability analysis of the motionless basic state, where the fluid is thermally stratified in the vertical direction, is performed. The normal mode method is employed to investigate the onset conditions of the thermoconvective instability. A one-dimensional eigenvalue problem is thus obtained. The analysis is focused on determining the neutral stability curves and the critical values of the Darcy-Rayleigh number. Results are obtained numerically and are compared with those valid for the limiting case of local thermal equilibrium between the phases.

2. Mathematical model

We consider a plane porous layer bounded by impermeable boundaries at $z^* = 0$ and $z^* = L$. A uniform upward heat flux, q_w , is prescribed at $z^* = 0$ and a uniform temperature, T_0 , is prescribed at $z^* = L$, see Figure 1. The porous medium is homogeneous and isotropic. Darcy's law and the Oberbeck-Boussinesq approximation can be employed. The viscous dissipation effect is negligible and no internal heat source exists. The local thermal non-equilibrium between the solid and fluid phases is modeled through an inter-phase heat transfer coefficient h .

2.1. Governing equations

The local thermal non-equilibrium model prescribes an energy balance equation for each phase: one for the solid phase, and the other for the fluid phase. The set of dimensional governing equations is given by a local mass balance equation, a local momentum balance equation and

two local energy balance equations, namely

$$\nabla^* \cdot \mathbf{v}^* = 0, \quad (1a)$$

$$\frac{\mu}{K} \nabla^* \times \mathbf{v}^* = \rho_f g \beta \nabla^* \times [(T_f^* - T_0) \mathbf{e}_z], \quad (1b)$$

$$(1 - \varphi) \frac{\partial T_s^*}{\partial t^*} = (1 - \varphi) \alpha_s \nabla^{*2} T_s^* + \frac{h}{(\rho c)_s} (T_f^* - T_s^*), \quad (1c)$$

$$\varphi \frac{\partial T_f^*}{\partial t^*} + \mathbf{v}^* \cdot \nabla^* T_f^* = \varphi \alpha_f \nabla^{*2} T_f^* - \frac{h}{(\rho c)_f} (T_f^* - T_s^*), \quad (1d)$$

where the subscripts f, s denote the fluid and solid phase, respectively, while the asterisks denote dimensional variables and operators. We denoted as \mathbf{v}^* the velocity field with Cartesian components (u^*, v^*, w^*) , while $T_{f,s}^*$ are the fluid and solid phase temperatures. Moreover, $\alpha_{f,s}$ are the thermal diffusivities of the two phases, *viz.* $\alpha_{f,s} = k_{f,s}/(\rho c)_{f,s}$, t^* is time, μ is the dynamic viscosity, K is permeability, ρ is density, g is the modulus of the gravitational acceleration \mathbf{g} , β is the thermal expansion coefficient, T_0 is the reference temperature, \mathbf{e}_z is the unit vector for the direction z , φ is porosity, and c is the heat capacity per unit mass. The set of boundary conditions can be expressed as [11, 12, 13]

$$\begin{aligned} z^* = 0 : \quad & -\varphi k_f \frac{\partial T_f^*}{\partial z^*} - (1 - \varphi) k_s \frac{\partial T_s^*}{\partial z^*} = q_w, \quad T_f^* = T_s^*, \quad w^* = 0, \\ z^* = L : \quad & T_f^* = T_s^* = T_0, \quad w^* = 0. \end{aligned} \quad (2)$$

A dimensionless formulation of equations (1) and (2) is now introduced by the rescaling

$$\mathbf{x}^* = \mathbf{x} L, \quad t^* = t \frac{L^2}{\alpha_f}, \quad \mathbf{v}^* = \mathbf{v} \frac{\alpha_f}{L} = (u, v, w) \frac{\alpha_f}{L}, \quad T_{s,f}^* = T_0 + T_{s,f} \Delta T. \quad (3)$$

Here, $k_m = (1 - \varphi) k_s + \varphi k_f$ is the average thermal conductivity, and $\Delta T = q_w L / k_m$ is the reference temperature difference. The dimensionless set of governing equations can now be written as

$$\nabla \cdot \mathbf{v} = 0, \quad (4a)$$

$$\nabla \times \mathbf{v} = \varphi \frac{1 + \gamma}{\gamma} R \nabla \times (T_f \mathbf{e}_z), \quad (4b)$$

$$\lambda \frac{\partial T_s}{\partial t} = \nabla^2 T_s + H \gamma (T_f - T_s), \quad (4c)$$

$$\frac{\partial T_f}{\partial t} + \frac{1}{\varphi} \mathbf{v} \cdot \nabla T_f = \nabla^2 T_f - H (T_f - T_s), \quad (4d)$$

where the Darcy-Rayleigh number R and the other dimensionless parameters employed in equations (4) are defined as

$$\lambda = \frac{\alpha_f}{\alpha_s}, \quad H = \frac{h L^2}{\varphi k_f}, \quad \gamma = \frac{\varphi k_f}{(1 - \varphi) k_s}, \quad R = \frac{g \beta \Delta T K L}{\alpha_m \nu}, \quad (5)$$

with $\alpha_m = k_m / (\rho c)_f$. On the other hand, the dimensionless boundary conditions are given by

$$\begin{aligned} z = 0 : \quad & -\gamma \frac{\partial T_f}{\partial z} - \frac{\partial T_s}{\partial z} = 1 + \gamma, \quad T_f = T_s, \quad w = 0, \\ z = 1 : \quad & T_f = T_s = 0, \quad w = 0. \end{aligned} \quad (6)$$

3. The basic state and its stability

A stationary basic state with vanishing velocity field is allowed as a solution of equations (4) and (6),

$$\mathbf{v}_B = 0, \quad T_{s,B} = T_{f,B} = 1 - z, \quad (7)$$

where B stands for basic state. One may note, on account of equation (7), that the temperatures of the fluid phase and of the solid phase are exactly the same and thus, for the basic state, local thermal equilibrium between the two phases occurs.

We now introduce small amplitude disturbances of the basic state defined by (7), such that

$$\mathbf{v} = \mathbf{v}_B + \epsilon \tilde{\mathbf{V}}, \quad T_s = T_{s,B} + \epsilon \tilde{\Phi}, \quad T_f = T_{f,B} + \epsilon \tilde{\Theta}. \quad (8)$$

Here, ϵ is a perturbation parameter, small enough as to neglect the nonlinear terms $O(\epsilon^2)$, while the perturbation velocity $\tilde{\mathbf{V}}$ has Cartesian components $(\tilde{U}, \tilde{V}, \tilde{W})$, $\tilde{\Phi}$ is the perturbation temperature of the solid phase and $\tilde{\Theta}$ is the perturbation temperature of the fluid phase. A linear system of equations governing the disturbances is obtained by substituting equations (7) and (8) into (4), namely

$$\nabla \cdot \tilde{\mathbf{V}} = 0, \quad (9a)$$

$$\nabla \times \tilde{\mathbf{V}} = \varphi \frac{1+\gamma}{\gamma} R \nabla \times (\tilde{\Theta} \mathbf{e}_z), \quad (9b)$$

$$\lambda \frac{\partial \tilde{\Phi}}{\partial t} = \nabla^2 \tilde{\Phi} + H\gamma (\tilde{\Theta} - \tilde{\Phi}), \quad (9c)$$

$$\frac{\partial \tilde{\Theta}}{\partial t} - \frac{1}{\varphi} \tilde{W} = \nabla^2 \tilde{\Theta} - H (\tilde{\Theta} - \tilde{\Phi}). \quad (9d)$$

Since the basic state, equation (7), is invariant by rotations around the z -axis, a two-dimensional formulation of the problem, based on the coordinates (x, z) , is allowed without any loss of generality. Thus, a streamfunction can be defined

$$\tilde{U} = \varphi \frac{\partial \tilde{\Psi}}{\partial z}, \quad \tilde{W} = -\varphi \frac{\partial \tilde{\Psi}}{\partial x}. \quad (10)$$

The governing equations (9), on account of (10), are rewritten as

$$\nabla^2 \tilde{\Psi} + \frac{1+\gamma}{\gamma} R \frac{\partial \tilde{\Theta}}{\partial x} = 0, \quad (11a)$$

$$\lambda \frac{\partial \tilde{\Phi}}{\partial t} = \nabla^2 \tilde{\Phi} + H\gamma (\tilde{\Theta} - \tilde{\Phi}), \quad (11b)$$

$$\frac{\partial \tilde{\Theta}}{\partial t} + \frac{\partial \tilde{\Psi}}{\partial x} = \nabla^2 \tilde{\Theta} - H (\tilde{\Theta} - \tilde{\Phi}). \quad (11c)$$

From equations (6)-(8), the boundary conditions associated with (11) are

$$\begin{aligned} z = 0 : \quad & \gamma \frac{\partial \tilde{\Theta}}{\partial z} + \frac{\partial \tilde{\Phi}}{\partial z} = 0, \quad \tilde{\Phi} = \tilde{\Theta}, \quad \tilde{\Psi} = 0, \\ z = 1 : \quad & \tilde{\Phi} = \tilde{\Theta} = 0, \quad \tilde{\Psi} = 0. \end{aligned} \quad (12)$$

The normal mode method is here employed. The disturbances are expressed in terms of plane waves, namely

$$\begin{Bmatrix} \tilde{\Psi}(x, z, t) \\ \tilde{\Phi}(x, z, t) \\ \tilde{\Theta}(x, z, t) \end{Bmatrix} = \begin{Bmatrix} i \Psi(z) \\ \Phi(z) \\ \Theta(z) \end{Bmatrix} e^{i(ax - \omega t)}. \quad (13)$$

Functions $\Psi(z)$, $\Phi(z)$ and $\Theta(z)$ express the amplitude of the normal modes, a is the real-valued wave number, while ω is a complex parameter such that its real part represents the angular frequency of the wave. A positive imaginary part of ω means a perturbation exponentially growing in time and, hence, an unstable behaviour. A negative imaginary part of ω yields an exponential damping of the perturbation, corresponding to linear stability. Hereafter, we will be interested in the stability/instability threshold, *viz.* in the neutral stability. At neutral stability, the parameter ω is real-valued. Moreover, the principle of exchange of stabilities implies that the neutrally stable modes are non-travelling, *i.e.* that ω is zero. Therefore, on account of (13), equations (11) and (12) yield a system of real ordinary differential equations, namely

$$\Psi'' - a^2\Psi + a \frac{1+\gamma}{\gamma} R\Theta = 0, \quad (14a)$$

$$\Phi'' - a^2\Phi + H\gamma(\Theta - \Phi) = 0, \quad (14b)$$

$$\Theta'' - a^2\Theta - H(\Theta - \Phi) + a\Psi = 0, \quad (14c)$$

with the boundary conditions

$$\begin{aligned} \gamma\Theta'(0) + \Phi'(0) &= 0, & \Phi(0) &= \Theta(0), & \Psi(0) &= 0, \\ \Phi(1) &= \Theta(1) = 0, & \Psi(1) &= 0. \end{aligned} \quad (15)$$

We now define

$$\hat{\Psi} = \gamma\Psi, \quad \Lambda = \gamma\Theta + \Phi, \quad (16)$$

so that the governing equations (14) and the boundary conditions, (15), can be rewritten as

$$\hat{\Psi}'' - a^2\hat{\Psi} + a \frac{1+\gamma}{\gamma} R(\Lambda - \Phi) = 0, \quad (17a)$$

$$\Phi'' - a^2\Phi + H[\Lambda - (1+\gamma)\Phi] = 0, \quad (17b)$$

$$\Lambda'' - a^2\Lambda + a\hat{\Psi} = 0, \quad (17c)$$

$$\begin{aligned} \Lambda'(0) &= 0, & \Lambda(0) &= (1+\gamma)\Phi(0), & \hat{\Psi}(0) &= 0, \\ \Lambda(1) &= \Phi(1) = 0, & \hat{\Psi}(1) &= 0. \end{aligned} \quad (17d)$$

Equations (17) yield an eigenvalue problem, where R is the eigenvalue to be determined for assigned values of (a, H, γ) . This problem can be solved numerically according to the procedure described in the next Section 3.1.

3.1. Numerical solution

Equations (17) can be solved numerically by means of a Runge-Kutta solver combined with a shooting method. More precisely, equations (17) are integrated numerically with the initial conditions

$$\hat{\Psi}(0) = 0, \quad \hat{\Psi}'(0) = \eta, \quad \Lambda(0) = 1, \quad \Lambda'(0) = 0, \quad \Phi(0) = \frac{1}{1+\gamma}, \quad \Phi'(0) = \xi. \quad (18)$$

The additional condition $\Lambda(0) = 1$ serves to break the overall scale invariance of the solution. On the other hand, the unknown parameters η and ξ , are determined, together with the eigenvalue R , by means of a shooting method set up to satisfy the three target conditions at $z = 1$, namely

$$\hat{\Psi}(1) = 0, \quad \Lambda(1) = \Phi(1) = 0. \quad (19)$$

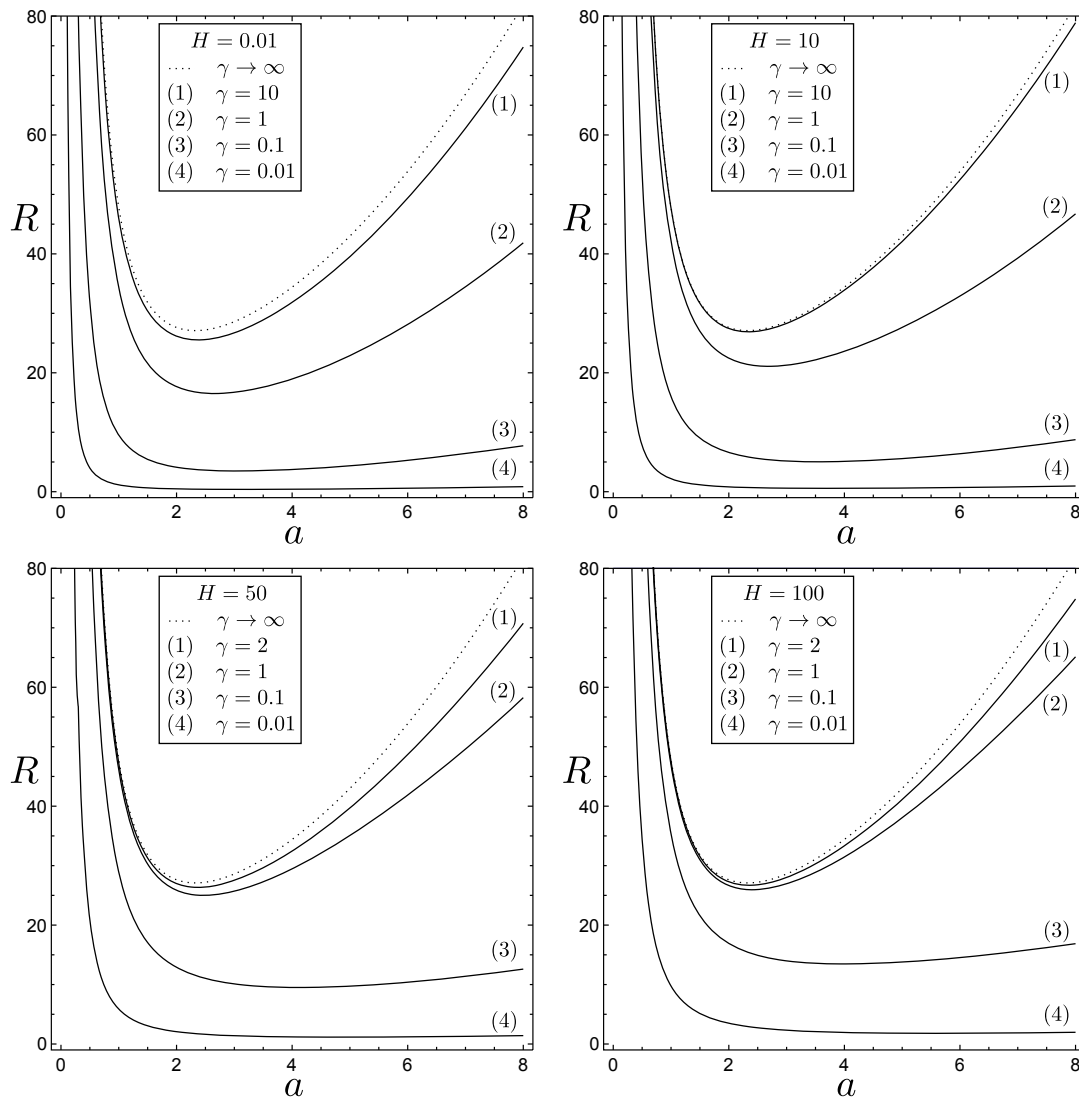


Figure 2. Neutral stability curves.

This numerical technique is accomplished inside the *Mathematica 9* (© Wolfram Research, Inc.) environment [14]. The Runge-Kutta solver is implemented by the built-in function `NDSolve` and the shooting method by the built-in function `FindRoot`. The `FindRoot` function allows one to solve numerically the constraints equation (19). The overall numerical procedure requires the assignment of the input data (a, H, γ) . Once the input data are fixed, a neutral stability curve $R(a)$ for every pair (H, γ) is obtained. Instability occurs when $R > R(a)$, while linear stability is for $R < R(a)$. The absolute minimum of $R(a)$ gives the critical values (a_{cr}, R_{cr}) for the onset of the convective instability.

4. Discussion of the results

We solved the eigenvalue problem defined by equations (17) for a wide range of values of the pair (H, γ) . Figure 2 contains different frames, each frame refers to a different value of H . Starting from the upper left corner, the values of H are: 0.01, 10, 50 and 100. Inside each frame, each curve is relative to a different value of γ . Figure 2 shows that the onset condition of the instability

is very sensitive to the quantity γ . As γ decreases, the system becomes more and more unstable. Very small values of γ , on account of (5), imply a solid phase much more conductive than the fluid phase. This is in fact a condition very close to the behaviour of a gas saturated metallic foam.

As shown by Figure 2, the effect of a decreasing H is also destabilising. We mention that larger and larger values of the product γH lead to the asymptotic condition of local thermal equilibrium. Local thermal equilibrium can be attained by taking the limit $H \rightarrow \infty$ with a finite nonvanishing value of γ . From equation (17b), this limit leads to $\Lambda = (1 + \gamma) \Phi$ and, as a consequence of equations (17), to $\Phi = \Theta$. Thus, in this limit, we can rewrite (17) as

$$\hat{\Psi}'' - a^2 \hat{\Psi} + aR\Lambda = 0, \quad (20a)$$

$$\Lambda'' - a^2 \Lambda + a\hat{\Psi} = 0, \quad (20b)$$

$$\begin{aligned} \Lambda'(0) &= 0, & \hat{\Psi}(0) &= 0, \\ \Lambda(1) &= 0, & \hat{\Psi}(1) &= 0. \end{aligned} \quad (20c)$$

On the other hand, on account of equation (14b), by taking the limit $\gamma \rightarrow \infty$ with a finite nonvanishing value of H , one is lead to $\Phi = \Theta$. Thus, the governing equations (14) reduce to

$$\Psi'' - a^2 \Psi + aR\Theta = 0, \quad (21a)$$

$$\Theta'' - a^2 \Theta + a\Psi = 0, \quad (21b)$$

$$\begin{aligned} \Theta'(0) &= 0, & \Psi(0) &= 0, \\ \Theta(1) &= 0, & \Psi(1) &= 0. \end{aligned} \quad (21c)$$

The eigenvalue problem given by equations (20) is completely equivalent to that given by (21). In other words, the local thermal equilibrium condition is attained either by taking the limit $H \rightarrow \infty$ with a finite γ , or by taking the limit $\gamma \rightarrow \infty$ with a finite H . We mention that the solution of the eigenvalue problem equations (20) or (21) was found by Ribando and Torrance [15], and analysed in detail by Wang [16]. These authors considered a horizontal porous layer with an isoflux lower boundary and an isothermal upper boundary, assuming that both these boundaries are impermeable and that local thermal equilibrium between the solid and the fluid phase holds. Data on the neutral stability condition and on the critical values of R and a for the onset of convection with local thermal equilibrium have been reported by [1, 15, 16]. In particular, we were able to recover the critical values for local thermal equilibrium,

$$R_{cr} = 27.097628, \quad a_{cr} = 2.3262146, \quad (22)$$

by solving numerically equations (20) or (21), with a suitably simplified version of the procedure described in Section 3.1.

The trend of R_{cr} versus γ and that of a_{cr} versus γ , with different finite values of H , are shown in Figures 3 and 4, respectively. Hence, these figures display the effects of a departure from the local thermal equilibrium asymptotic condition ($\gamma H \gg 1$). Interestingly enough, Figure 3 suggests that R_{cr} tends to zero when γ tends to zero, for every finite value of H . We already noted above that this regime may be relevant for the study of heat transfer in gas-saturated metallic foams. In fact, a vanishingly small R_{cr} means that whatever small value of the heat flux q_w prescribed on the lower wall leads to a destabilisation of the basic motionless state through the onset of convective rolls. Another evident feature of Figures 3 and 4 is that the critical values of R and a approach monotonically the local thermal equilibrium asymptotes (dashed lines) when $\gamma \rightarrow \infty$.

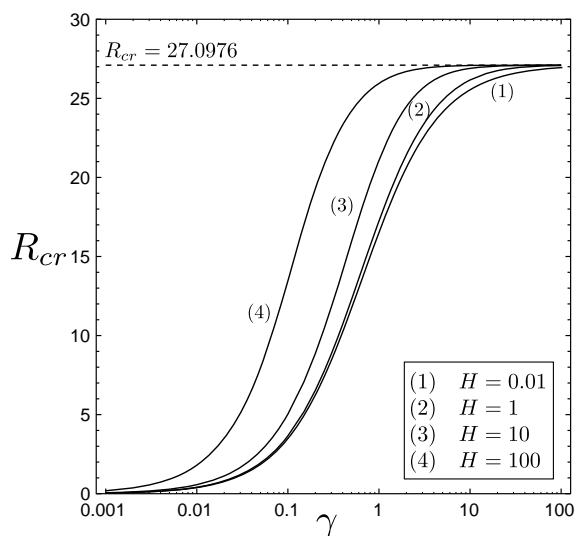


Figure 3. Plots of R_{cr} versus γ .

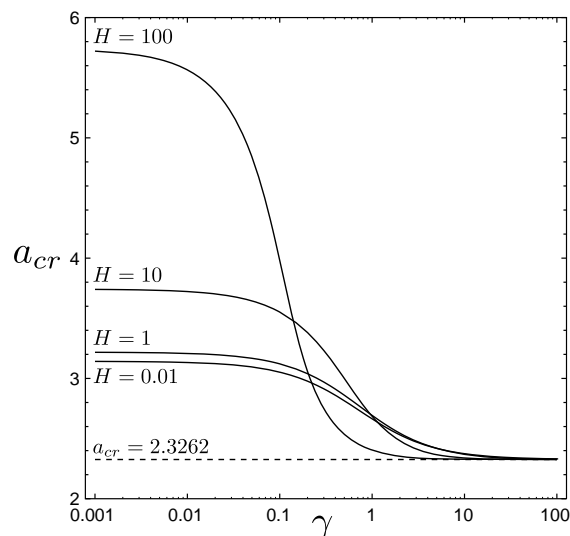


Figure 4. Plots of a_{cr} versus γ .

5. Conclusions

The onset of convection in a horizontal porous layer saturated by a fluid and heated from below by a uniform heat flux has been studied. The assumption of local thermal equilibrium has been relaxed. Different local values of the fluid temperature and of the solid temperature are assumed, with an inter-phase energy exchange effect. The local thermal non-equilibrium thus yields a mathematical model with two energy balance equations. These equations, together with the local mass balance equation and Darcy's law with buoyancy force, have been employed to test the stability of the motionless and local thermal equilibrium basic state of the layer. The disturbance equations, in a streamfunction/temperature formulation, have been linearised and solved for the normal modes. We thus obtained an ordinary differential eigenvalue problem that allowed us to study the neutral stability condition, as well as to determine the critical values of the Darcy-Rayleigh number, R , and of the wave number, a , for the onset of convective rolls.

We pointed out that two dimensionless parameters are crucial in the transition to instability: the thermal conductivity ratio, γ , and the inter-phase heat transfer parameter, H . When the product γH becomes very large, the condition of local thermal equilibrium is attained. On the other hand, small values of γ and H mean significant departures from local thermal equilibrium. In particular, lower and lower values of both γ and H result into a destabilisation of the basic state. A significant effect has been revealed when $\gamma \ll 1$, a condition approached when the solid phase is much more conductive than the fluid phase. In fact, when $\gamma \ll 1$, the critical value of R tends to zero. This feature means that, in this parametric regime, the basic motionless state is always unstable to small-amplitude disturbances. This result is of special interest for all the applications involving a highly-conductive solid, such as the gas-saturated metallic foams. We mention that recently attention in the heat transfer community has been deserved by the properties of the metallic foams with respect to the design of heat exchangers [17].

Acknowledgments

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