A modal formulation for the propagation of guided waves in straight and curved pipes and the scattering at their junction

M El Bakkali[1], A Lhémery[1], V Baronian[1] and S Grondel[2]

[1]CEA, LIST, F-91191 Gif-sur-Yvette cedex
[2] IEMN-DOAE, UMR CNRS 8520, Université de Valenciennes et du Hainaut Cambrésis, Le Mont Houy, F-59313 Valenciennes Cedex 9

E-mail: alain.lhemery@cea.fr

Abstract. The well-established semi-analytic finite element (SAFE) method for predicting guided wave propagation in straight guides is extended to deal with curved pipes using curvilinear coordinates system, following developments by Treyssède and Laguerre [*J. Sound Vib.* **329** 1702-16 (2010)] for helical wires. The modal solution in the elbow can be obtained and the bi-orthogonality relation known for straight guides still holds. The influence of elbow curvature is exemplified by a short parametrical study. The scattering at the junction between a straight guide and an elbow sharing a common section is then considered. Assuming the modal solutions in the straight part and in the curved one are known, the mode-matching method is specifically derived to compute the scattering matrix which deals with waves arising from one part or the other, that is to say, to predict reflection and transmission coefficients at such a junction. Interestingly, the overall formulation requires a single mesh of the section to compute both the modal solutions and the scattering. The scattering matrix computation makes it necessary to take into account evanescent modes locally at the junction of both parts. Again, few examples of numerical results are shown to illustrate specific phenomena arising in this complex scattering case.

1. Introduction

Elastic guided waves (GW) non-destructive techniques are widely used for in-service inspection of pipeline networks in various industries (nuclear, oil, chemical industries…). However, because of the complexity of scattering phenomena involved when guided waves in a straight pipe interact with specific features such as flaws, welds, supports, elbows, junctions, etc., tools for simulating nondestructive examinations must be developed to help the interpretation of measured signals and to improve testing configurations.

In pipe networks, straight lines coexist with elbows. Guided wave propagation in the former parts is well-known and several computational methods as well as the semi-analytic finite element method (SAFE) or Finite Element Method (FEM) can be used to predict it. In the latter parts, the propagation of guided waves is far less documented [1]. The scattering of guided waves at their junction constitutes an even more difficult problem.

The well-known SAFE method is the most commonly used modeling approach to calculate the GW modal solution in components of various shapes. Compared to the FEM, this technique is less computationally expensive and allows dealing with guides of arbitrary section.

In the present paper, the aim is to establish a numerical model based on the SAFE method able, in a first place, to deal with the curvature effect on the elastic guided waves propagation through a pipe elbow and then, to quantitatively predict the scattering phenomena arising when a guided wave is diffracted at the junction between a straight pipe and an elbow.

The paper is organized as follows. The calculation of the modal solution with the SAFE method in a curved guide and a short parametrical study of the curvature effect are first presented. Then, the simulation of the scattering at the junction between a straight guide and a curved one is carried out by using the mode matching method, where modal solutions in both guides are linked. Finally, examples of numerical results are shown to illustrate specific phenomena arising at this junction.

2. SAFE model in curvilinear coordinates system

2.1. Equation of wave motion

The waveguide considered here is a straight-curved-straight pipe (Fig. 1)

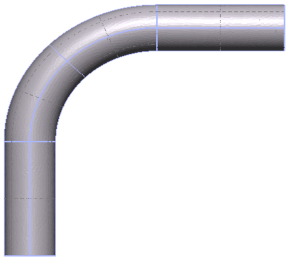

Figure 1. Schematic view of the studied elbow configuration

Our aim is to calculate the scattering matrix at the junction between a straight guide and a curved guide. The first step consists in computing the modal solution in a curved waveguide by means of the SAFE method. The second step consists in computing the scattering matrix at the junction using a mode matching technique.

Guided modes are solutions of the wave equation in an elastic waveguide (pipe in the present case) assumed to have free boundaries and to be homogeneous. The equation of motion is:

$$\begin{cases} -\mathbf{div}\,\sigma(\mathbf{u}) - \omega^2 \rho\,\mathbf{u} = 0 & \text{in } \Omega, \\ \sigma(\mathbf{u}) \cdot \mathbf{n} = 0 & \text{on } \Sigma_1, \end{cases} \tag{1}$$

where $\mathbf{u}$ is the displacement, ρ the mass density, ω the pulsation, Ω the section of the guide and Σ_1 the inner and outer surfaces.

2.2. The SAFE method in curvilinear coordinates system

2.2.1. Overall principle. The Semi-analytical Finite Element (SAFE) method is considered as an efficient and well-adapted numerical method to compute guided waves in different guide geometries made of arbitrary material and section. This technique allows computing a series of guided modes by coupling finite element modeling technique in the guide section to an analytical approach to deal with the guided propagation. In other words, at a given frequency, the FE computation is used to solve the wave equation (1) in the cross section of the guide. Then, the dependency upon the propagation axis of the solution is handled analytically. The SAFE formulation leads to a quadratic eigenvalue system where the eigenvectors are the particle displacements and the eigenvalues are the wavenumbers of the calculated modes.

To use this approach, the following conditions must be satisfied:

- invariant cross section along the propagation axis ,
- invariant material properties,
- equilibrium equation independent of the propagator component.

Then, the displacement components of the elastodynamic field can be decomposed on eigenmodes calculated in the guide section by the SAFE method as follows:

$$\mathbf{u}(x,y,z) = \sum_n A_n \mathbf{u_n}(x,y,z) = \sum_n A_n \tilde{\mathbf{u}}_\mathbf{n}(x,y) e^{j(\omega t - \beta_n z)} \tag{2}$$

where $(\tilde{\mathbf{u}}_\mathbf{n}, \tilde{\boldsymbol{\sigma}}_\mathbf{n}, \beta_n, A_n)$ is the displacement, stress, wave number and amplitude of each eigenmode.

In the case of a curved pipe, the SAFE approach must be derived in curvilinear coordinates in order to deal with the curvature effect. The present derivation is based on that proposed by Treyssède and Laguerre [2] who formulated the SAFE method to study GW propagation in a helical waveguide.

2.2.2. SAFE curvilinear formulation. In the elbow, the pipe centerline is defined by the position vector (see Fig. 2)

$$\mathbf{R}(s) = R_c\left[1-\cos(s/R_c)\right]\mathbf{e_x} + R_c\sin(s/R_c)\,\mathbf{e_z} \tag{3}$$

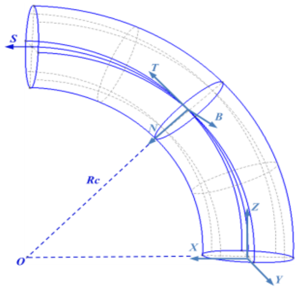

Figure 2. Schematic of an elbow with local curvilinear coordinates system

From this position vector, a new basis (**N**, **B**, **T**) is constructed, which represents the curvilinear basis associated to the configuration. The basis is defined by:

$$\mathbf{T} = \frac{d\mathbf{R}}{ds} \qquad \mathbf{N} = R_c\frac{d\mathbf{T}}{ds} \qquad \mathbf{B} = \mathbf{T}\wedge\mathbf{N} = \mathbf{e_y} \tag{4}$$

The non-orthonormal covariant basis associated to the curvilinear basis can then be defined as:

$$\mathbf{g_1} = \mathbf{N} \qquad \mathbf{g_2} = \mathbf{B} \qquad \mathbf{g_3} = (1-\gamma x)\mathbf{T} \tag{5}$$

For this case, this covariant basis is orthogonal but the axes are no more normalized. The covariant metric tensor is given by:

$$\mathbf{g} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (1-\gamma x)^2 \end{bmatrix} \tag{6}$$

The variational formulation can be expressed in this covariant basis as follows:

$$\int_\Omega \boldsymbol{\delta\varepsilon}_{cov} : \boldsymbol{\sigma}_{cov}\,dV - \omega^2\int_\Omega \rho\boldsymbol{\delta}\mathbf{u}_{cov}\cdot\mathbf{u}_{cov}\,dV = 0 \tag{7}$$

Then, using the relation between the components of the elastodynamic field in the covariant and the curvilinear basis, respectively, each term of the previous formulation can be expressed in the (**N**, **B**, **T**) basis; the variational formulation might be re-written as

$$\int_\Omega \boldsymbol{\delta\varepsilon}^{\mathbf{T}}_{cur}\mathbf{C}\boldsymbol{\varepsilon}_{cur}\sqrt{\det(\mathbf{g})}\,dxdyds - \omega^2\int_\Omega \rho\boldsymbol{\delta}\mathbf{u}^{\mathbf{T}}_{cur}\mathbf{u}_{cur}\sqrt{\det(\mathbf{g})}\,dxdyds = 0 \tag{8}$$

with: $\mathbf{u}_{cur} = \mathbf{u}_{cur}(x,y)e^{i(ks-\omega t)}$ $\quad \boldsymbol{\delta}\mathbf{u}_{cur} = \boldsymbol{\delta}\mathbf{u}_{cur}(x,y)e^{-i(ks-\omega t)}$.

In the equilibrium equation written in the curvilinear coordinate system, the s component does not explicitly appear. Hence, an exponential e^{iks} can be factorized in all elastodynamic field components. From this, we can write the following semi-analytical variational formulation by restricting the integration in the volume to a surface integration over the cross-section:

$$\begin{aligned}&\int_S \boldsymbol{\delta}\mathbf{u}^{\mathbf{T}}_{cur}\mathbf{L}^{\mathbf{T}}_{xy}\mathbf{C}\mathbf{L}_{xy}\mathbf{u}_{cur}\sqrt{\det(\mathbf{g})}dxdy + ik\int_S \boldsymbol{\delta}\mathbf{u}^{\mathbf{T}}_{cur}(\mathbf{L}^{\mathbf{T}}_{xy}\mathbf{C}\mathbf{L}_s - \mathbf{L}^{\mathbf{T}}_s\mathbf{C}\mathbf{L}_{xy})\mathbf{u}_{cur}\sqrt{\det(\mathbf{g})}dxdy\\ &+k^2\int_S \boldsymbol{\delta}\mathbf{u}^{\mathbf{T}}_{cur}\mathbf{L}^{\mathbf{T}}_s\mathbf{C}\mathbf{L}_s\mathbf{u}_{cur}\sqrt{\det(\mathbf{g})}dxdy - \omega^2\int_S \rho\boldsymbol{\delta}\mathbf{u}^{\mathbf{T}}_{cur}\mathbf{u}_{cur}\sqrt{\det(\mathbf{g})}dxdyds = 0\end{aligned} \tag{9}$$

where $\mathbf{L}_{xy}$ and $\mathbf{L}_s$ are two elementary matrices appearing in the strain-displacement relation:

$$\boldsymbol{\varepsilon} = (\mathbf{L}_{xy} + \frac{\partial}{\partial s}\mathbf{L}_s)\mathbf{u}$$

$$\mathbf{L}_{xy} = \begin{pmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ -\gamma(1-\gamma x) & 0 & 0 \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & 0 & 2\gamma + (1-\gamma x)\frac{\partial}{\partial x} \\ 0 & 0 & \frac{\partial}{\partial y} \end{pmatrix} \quad \mathbf{L}_{s} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{(1-\gamma x)} \\ 0 & 0 & 0 \\ \frac{1}{(1-\gamma x)} & 0 & 0 \\ 0 & \frac{1}{(1-\gamma x)} & 0 \end{pmatrix} \tag{10}$$

It must be noticed that the guide curvature appears in several coefficients of operators $\mathbf{L_{xy}}$ and $\mathbf{L_s}$. For a null curvature, these elementary matrices are equal to those of a straight guide.

Similarly to the SAFE method in a Cartesian basis, the numerical discretization (2D elements) of the waveguide cross section leads to the following quadratic system of equations to be solved:

$$(\mathbf{K_1^e} + j\beta\mathbf{K_2^e} + \beta^2\mathbf{K_3^e})\mathbf{U} - \omega^2\mathbf{M^e U} = 0 \tag{11}$$

$\mathbf{K}^e$ and $\mathbf{M}^e$ represent the elementary stiffness and mass matrices calculated at each element of the cross section of the guide. As shown previously, $\mathbf{U}$ and β correspond to the displacement vectors and the wavenumbers of each eigenmode.

2.3. Displacement field for a curved and straight pipe

To exemplify the effect of curvature, the three components of the displacement (radial, orthoradial and axial) of different propagative modes in an elbow are shown for different curvature values. The two modes chosen are the first torsional mode T(0,1) and the second extensional mode L(0,2) which are the modes the most commonly used in GW for non destructive inspections.

The elbows are 5mm-thick with an outer diameter of 10mm; the wave frequency is of 100 kHz. The curvature is varied from 0 (straight guide) to 1/15 mm^{-1}.

Fig. 3 shows the displacement field for the T(0,1) mode.

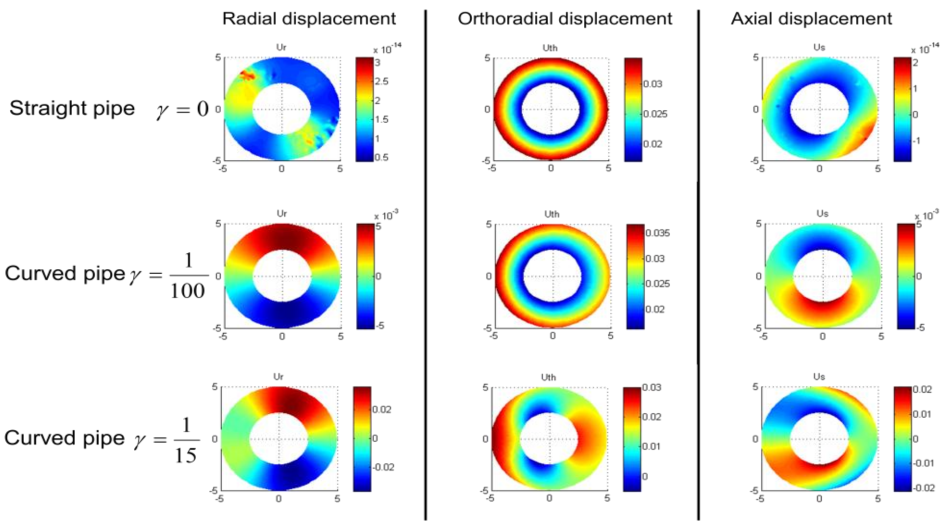

Figure 3. Displacement field for the torsional mode T(0,1)

A non-zero curvature causes distortions of the displacement field that increases for increasing curvature. The axisymmetry of orthoradial displacement the mode in the straight pipe is lost in a curved guide. This component is the only one carrying energy in the straight pipe (the maps for other components are shown at an amplitude scale 10^{-12} lower than that of the orthoradial component, corresponding to numerical errors). In a curved pipe, the two other components of the displacement increase with increasing curvature and are eventually of the same order of magnitude as that of the orthoradial component for the highest curvature. Fig. 4 shows similar results for the L(0,2) mode.

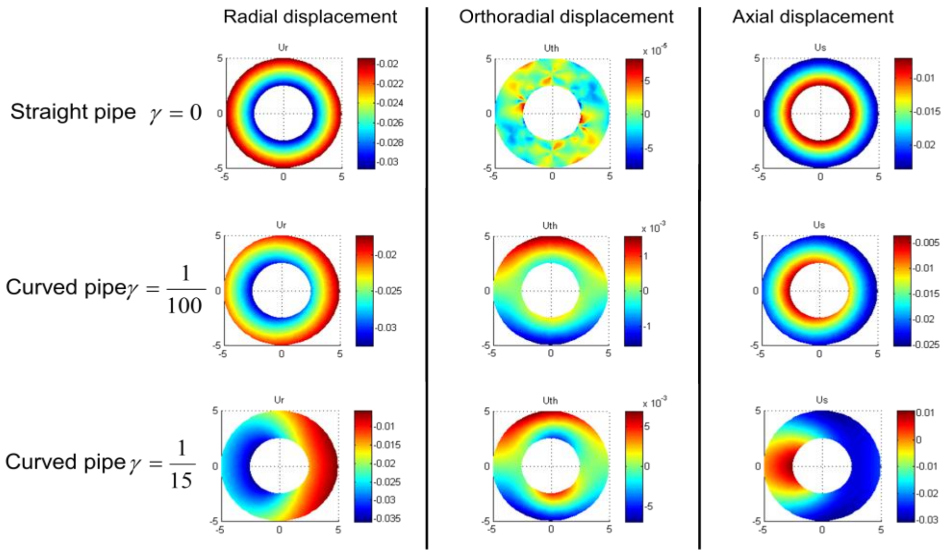

Figure 4. Displacement field for the extensional mode L(0,2)

For this extensional mode, the radial and axial displacements fields dominate in the straight guide. Distortions are observable again as soon as a non-zero curvature is considered.

Similar effects of non-zero curvature can be observed when studying flexural modes.

In addition, for a straight pipe, the flexural modes appear as pairs of modes having the same wavenumber but with displacement components rotated by an angle of $\pi/2$. Conversely, for a curved pipe, these flexural modes are separated, because of the non-axisymmetry of the guide (see fig. 5). It is also observed that the cut-off frequency of each propagative mode in a curved case increases in comparison with the straight case. In Fig. 5, dispersion curves of GW in a straight steel pipe are superimposed with dispersion curves of GW in an elbow.

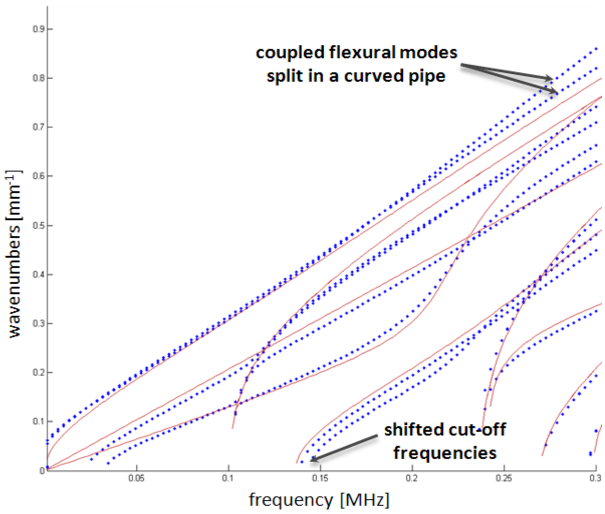

Figure 5. Wavenumbers dispersion curves for a straight (continuous line) and a curved steel pipe (Outer diameter=10mm, Thickness=2.5mm) of 1/25 mm^{-1} curvature (dotted line)

3. The mode matching model at the common junction between two guides

3.1. Analytical mode matching formulation:

In pipeworks, reflection and transmission phenomena arise at each junction of two different parts of the pipe. In the case of an elbow, the scattering phenomena arise at the junction of a straight and a curved part of the guide.

The mode matching technique allows computing the modal decomposition of an incident mode onto reflected and transmitted ones. An interesting mode matching formulation for the computation of Lamb wave scattering matrices has been investigated by Feng *et al*. [3]. It is based on the use of mixed stress-displacement vectors **X** and **Y** introduced by Pagneux and Morel [4] and defined as follows:

$$\mathbf{X}=\begin{pmatrix}\mathbf{t}_S\\ u_z\end{pmatrix}=\begin{pmatrix}\sigma_{xz}\\ \sigma_{yz}\\ u_z\end{pmatrix},\quad \mathbf{Y}=\begin{pmatrix}\mathbf{u}_S\\ t_z\end{pmatrix}=\begin{pmatrix}u_x\\ u_y\\ -\sigma_{zz}\end{pmatrix}. \tag{12}$$

In our case, the mode matching model will be computed at the junction of a straight pipe and a curved one. However, the formal solution is generic and can be easily applied to other cases of junction (curved-curved, straight-straight).

The mode-matching method consists basically in satisfying the continuity between the elastodynamic fields associated to the straight and the curved parts of the guide. Mathematically, it involves the calculation of scalar product coming from the biorthogonality relation linking the displacement and stress components of each elastodynamic fields considered. These scalar products are easily calculated using the bi-orthogonality relation between the hybrid vectors.

Therefore, using the eigenmodes obtained by the SAFE model in the straight and the curved parts, GW displacement and stress fields can be expressed as modal series. From now on, hybrid vectors are explicitly used. The total fields in the straight part (1) and in the curved one (2) can be expressed as the superposition of the calculated eigenmodes:

$$\begin{cases}\begin{pmatrix}\mathbf{X}_{tot1}\\ \mathbf{Y}_{tot1}\end{pmatrix}=\sum_n B_n^1\begin{pmatrix}\mathbf{X}_{n1}^-\\ \mathbf{Y}_{n1}^-\end{pmatrix}\mathrm{e}^{-\mathrm{j}\beta_{n1}^- z}+\sum_n A_n^1\begin{pmatrix}\mathbf{X}_{n1}^+\\ \mathbf{Y}_{n1}^+\end{pmatrix}\mathrm{e}^{-\mathrm{j}\beta_{n1}^+ z}\\ \begin{pmatrix}\mathbf{X}_{tot2}\\ \mathbf{Y}_{tot2}\end{pmatrix}=\sum_n A_n^2\begin{pmatrix}\mathbf{X}_{n2}^+\\ \mathbf{Y}_{n2}^+\end{pmatrix}\mathrm{e}^{-\mathrm{j}\beta_{n2}^+ s}+\sum_n B_n^2\begin{pmatrix}\mathbf{X}_{n2}^-\\ \mathbf{Y}_{n2}^-\end{pmatrix}\mathrm{e}^{-\mathrm{j}\beta_{n2}^- s}\end{cases} \tag{13}$$

where (A_n^1, B_n^1) and (A_n^2, B_n^2) are the amplitude of the incoming and out-coming modes calculated in the straight and the curved parts of the pipe (as shown by Fig. 5)

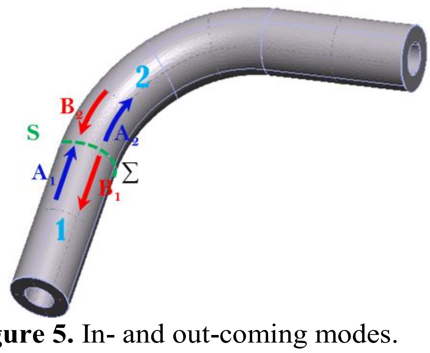

Figure 5. In- and out-coming modes.

$\begin{pmatrix}\mathbf{X}_{n1}\\ \mathbf{Y}_{n1}\end{pmatrix}$ and $\begin{pmatrix}\mathbf{X}_{n2}\\ \mathbf{Y}_{n2}\end{pmatrix}$ are the hybrid vectors associated to the modal solutions obtained by the SAFE calculation in the straight and the curved pipes and which form two biorthogonal bases. The index – (out-coming) and + (incoming) refer to the direction of the propagation, with:

$$\mathbf{X}_{n1}^{-} = -\mathbf{X}_{n1}^{+} \quad \text{and} \quad \mathbf{Y}_{n1}^{-} = \mathbf{Y}_{n1}^{+}$$
$$\mathbf{X}_{n2}^{-} = -\mathbf{X}_{n2}^{+} \quad \text{and} \quad \mathbf{Y}_{n2}^{-} = \mathbf{Y}_{n2}^{+}$$
$$\beta_{n1}^{-} = -\beta_{n1}^{+} \quad \text{and} \quad \beta_{n2}^{-} = -\beta_{n2}^{+}$$

In the mode matching formulation, $(\cdot|\cdot)_S$ denotes the following scalar product:

$$\left(\mathbf{Y}|\,\mathbf{X}\right)_S = \int_S (\mathbf{t}_S \cdot \mathbf{u}_S + t_z u_z)\, dS \text{ ,(14)} \tag{14}$$

where S is the cross-section of the guide.

Then, the continuity condition of elastodynamic fields must be satisfied as follows:

$$\begin{pmatrix} \mathbf{X}_{tot1} \\ \mathbf{Y}_{tot1} \end{pmatrix} = \begin{pmatrix} \mathbf{X}_{tot2} \\ \mathbf{Y}_{tot2} \end{pmatrix} \quad \text{at } z = s = z_0, \tag{15}$$

re-written as:

$$\sum_n B_n^1 \begin{pmatrix} -\mathbf{X}_{n1} \\ \mathbf{Y}_{n1} \end{pmatrix} \mathrm{e}^{-\mathrm{j}\beta_{n1}^{-}\mathrm{z}} + \sum_n A_n^1 \begin{pmatrix} \mathbf{X}_{n1} \\ \mathbf{Y}_{n1} \end{pmatrix} \mathrm{e}^{-\mathrm{j}\beta_{n1}^{+}\mathrm{z}} = \sum_n A_n^2 \begin{pmatrix} \mathbf{X}_{n2} \\ \mathbf{Y}_{n2} \end{pmatrix} \mathrm{e}^{-\mathrm{j}\beta_{n2}^{+}\mathrm{s}} + \sum_n B_n^2 \begin{pmatrix} -\mathbf{X}_{n2} \\ \mathbf{Y}_{n2} \end{pmatrix} \mathrm{e}^{-\mathrm{j}\beta_{n2}^{-}\mathrm{s}} \tag{16}$$

Finally, the mode matching will consist in the computation of the scattering matrix **S** that links the amplitudes of the incoming modes to those of the out-coming modes at the shared junction:

$$\begin{pmatrix} \mathbf{B_1} \\ \mathbf{A_2} \end{pmatrix} = \mathbf{S} \begin{pmatrix} \mathbf{A_1} \\ \mathbf{B_2} \end{pmatrix} \tag{17}$$

where

$$\mathbf{S} = \begin{pmatrix} \mathbf{R_{1\to2}} & \mathbf{T_{2\to1}} \\ \mathbf{T_{1\to2}} & \mathbf{R_{2\to1}} \end{pmatrix} \tag{18}$$

$\mathbf{R_{1\to2}}, \mathbf{T_{1\to2}}$ ($\mathbf{R}_{2\to1}, \mathbf{T}_{2\to1}$, respectively) are the reflection and transmission matrices of a wave arising from the straight part (from the curved part, respectively).

3.2. Scattering matrix calculation

According to Feng *et al.* [3], the scattering matrix can be defined as

$$\mathbf{S} = \begin{pmatrix} 2\mathbf{J}_1 & \mathbf{J}_{21}^T - \mathbf{J}_{12} \\ \mathbf{J}_{12}^T - \mathbf{J}_{21} & 2\mathbf{J}_2 \end{pmatrix}^{-1} \begin{pmatrix} [0]_{NxN} & \mathbf{J}_{12} + \mathbf{J}_{21}^T \\ \mathbf{J}_{12}^T + \mathbf{J}_{21} & [0]_{NxN} \end{pmatrix} \tag{19}$$

N denotes the number of modes used for the mode matching, including all the propagative modes at the working frequency and a number of evanescent modes. $\mathbf{J}_1$ and $\mathbf{J}_2$ are projection matrices obtained from the biorthogonality relation used for each field separately and $\mathbf{J}_{12}$ and $\mathbf{J}_{21}$ represent projection matrices which couple the stress and displacement components of both elastodynamic fields.

In the present case, the mode matching formulation must handle fields in both straight and curved guides which are described in two different coordinate systems. Each field involves a specific scalar product adapted to each geometry. The key point of our mode matching model is to compute the appropriate projection matrices $\mathbf{J}_{12}$ and $\mathbf{J}_{21}$ dealing with both two different bases.

$\mathbf{J}_{12}$ *and* $\mathbf{J}_{21}$ *computation*

Assuming the completeness of the eigenmodes obtained by the SAFE calculation, the biorthogonality relation (Fraser's relation) allows expressing the biorthogonality between two propagative modes m and n in the same guide through the bi-linear relation given as

$$(\mathbf{X}_n|\mathbf{Y}_m)_S = \int_S (\mathbf{t}_S \mathbf{u}_S + t_z u_z)\, dx\, dy = \mathbf{J}_n \delta_{nm} \tag{20}$$

From this relation, the projection matrices $\mathbf{J}_1$ and $\mathbf{J}_2$ of both the straight and the curved parts of the guide can be numerically computed and we get:

$$(\mathbf{X}_n|\mathbf{Y}_m)_S = \hat{\mathbf{X}}_n^{\mathbf{T}} \mathbf{M}_S \hat{\mathbf{Y}}_m = \mathbf{J}_n \delta_{nm} \tag{21}$$

$\mathbf{M}_S$ is the projection matrix obtained by combining the interpolation matrices $N_e(x,y)$ used to compute the elastodynamic field at each element of the mesh, $\hat{\mathbf{X}}_n^{\mathbf{T}}$ and $\hat{\mathbf{Y}}_m$ being the nodal hybrid vectors.

Since two different coordinate systems are used (Cartesian and curvilinear) to compute the modal solutions in the two different parts of the pipe involving two different scalar products associated to each basis.

In order to use the mode matching approach presented here, we need to calculate mixed scalar products by combining hybrid vectors stemming from the different elastodynamic fields. Hence, the projection of a hybrid vector of a mode n of the first elastodynamic field denoted $\mathbf{X}_n^1(x,y,s)$ on the hybrid vector of a mode m denoted $\mathbf{Y}_m^2(x,y,z)$ of the second one can be expressed as follows:

$$(\mathbf{X}_n^1 \,|\, \mathbf{Y}_m^2)_S = \int_{V_c} \mathbf{X}_n^1(x,y,s)\mathbf{Y}_m^2(x,y,s)\, dV_1 \tag{22}$$

We can observe that the elementary volume of integration dV_1 is associated to the projected component. For the mode matching method, this volume must be expressed in the orthonomal basis. In the present case, because of the use of two coordinate systems, we must express this elementary volume in both the curvilinear and the Cartesian bases.

For a curved guide, the elementary volume corresponding to the coordinate differentials (dx, dy, ds) is given by:

$$dV_1 = |\mathbf{g}_1.(\mathbf{g}_2 \wedge \mathbf{g}_3)|\, dxdyds = \sqrt{\det(\mathbf{g})}dxdyds \tag{23}$$

with the symbol $\wedge$ denoting the vector cross product and where $(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)$ are the vectors of the covariant basis associated to the curvilinear coordinate system previously showed in section 2.

Interestingly, the straight case is a specific case of the curved one where $\sqrt{\det(\mathbf{g})} = 1$, then, the elementary volume definition still the same in a Cartesian coordinate system.

Hence from (22) and (23), considering the hybrid vectors of an eigenmode n of the elbowed pipe $\mathbf{X}_n^c(x,y,s)$ and an eigenmode m $\hat{\mathbf{Y}}_m^s(x,y)$ of the straight guide, the scalar product is defined as:

$$(\mathbf{X}_n^c \,|\, \mathbf{Y}_m^s)_S = \int_S \mathbf{X}_n^c(x,y,s)\mathbf{Y}_m^s(x,y)\sqrt{\det(\mathbf{g})}dxdyds = \mathbf{X}_n^{c\mathbf{T}}\mathbf{M}_s^c\mathbf{Y}_m^s = \mathbf{J}_{21}\delta_{nm} \tag{25}$$

$\mathbf{M}_s^c$ corresponds to the global mass matrix (without the constant ρ) used in the SAFE formulation in the curvilinear coordinate system.

Similarly, the projection of an elastodynamic hybrid vector in the straight part $\mathbf{X}_n^s(x,y,s)$ in an eigenmode in the curved one $\hat{\mathbf{Y}}_m^c(x,y)$ can be expressed as:

$$(\mathbf{X}_n^s \,|\, \mathbf{Y}_m^c)_S = \int_S \mathbf{X}_n^s(x,y,s)\,\mathbf{Y}_m^c(x,y)\sqrt{\det(\mathbf{g})}\; dxdyds = \mathbf{X}_n^{s^{\mathbf{T}}}\mathbf{M}_s^s\mathbf{Y}_m^c = \mathbf{J}_{12}\delta_{nm} \tag{26}$$

Now, it is possible to calculate $\mathbf{J}_1$ and $\mathbf{J}_2$ using (21) and $\mathbf{J}_{12}$and $\mathbf{J}_{21}$ using (25). Therefore, the scattering matrix $\mathbf{S}$ presented in (18) can be computed.

4. Numerical results

4.1. Configuration

The straight-curved-straight pipe shown on Fig. 6 and describe in the figure caption is studied.

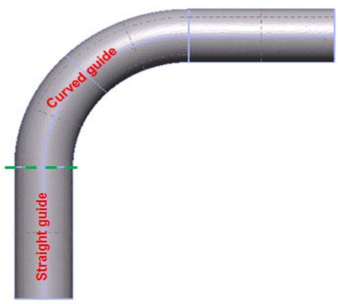

Figure 2. Two straight steel pipes of outer diameter equal to 10 mm and 2.5 mm-thick are linked by a curved pipe of 1/15 mm^{-1} curvature. The examination is made at a frequency of 100 kHz.

4.2. SAFE calculations
The eigenmodes are calculated using the appropriate SAFE model for each part of the guide. At the frequency considered the cut-off frequency of each propagative mode in the curved case increases as compared to the straight case. Therefore, the number of propagative modes in the straight guide and the curved one may differ.

SAFE calculation in the straight guides: at 100 kHz, six propagative modes in the straight parts are predicted: 4 flexural modes F(1,1), F(-1,1), F(1,2), and F(-1,2), 1 torsional mode T(0,1), 1 extensional mode: L(0,1). The flexural modes F(1,1) and F(-1,1) (respectively F(1,2) and F(-1,2)) are calculated in pairs and represent the same flexural mode in the straight case.

SAFE calculation in the curved guide: at 100 kHz, four propagative modes in the curved part are predicted: 2 flexural-like modes: F(1,1) and F(-1,1), 1 torsional-like mode: T(0,1), 1 extensional-like mode: L(0,1)

4.3. Modal decomposition
Scattering phenomena arise at the junctions when propagative modes coming from the straight part reach the curved one. The incident modes give rise to reflected and transmitted modes.

Considering all possible propagative modes coming from the straight guide, the energy partition of the incident energy into transmitted modes in the curved part is detailed in the following table:

Transmitted energy ratio	F(1,1)	F(-1,1)	T(0,1)	L(0,1)	F(1,2)	F(-1,2)
F(1,1)	54,14%	17,65%	23,95%	2,89E-06%	1,00%	2,04%
F(-1,1)	20,98%	64,29%	1,23E-06%	10,80%	2,43%	1,17%
T(0,1)	15,59%	5,08%	73,25%	7,20E-06%	1,55%	3,18%
L(0,1)	2,92%	8,97%	6,33E-06%	82,61%	1,55%	0,74%

Figure 7: Energy partition of transmitted modes in the curved part for the various possible incident modes in the straight part.

This table illustrates the ability of the mode matching model to calculate the modal decomposition of an arbitrary incident mode at the junction of two guides by predicting its modal decomposition. Here, each propagative mode is decomposed onto the modal basis of the curved guide and the transmission is higher into modes in the curved which are of the same nature as that of the incident mode in the straight guide. However, a torsional or extensional incident mode (as typically used in NDT operations) partially converts into transmitted flexural eigenmodes. The modal decomposition of the reflected energy can be obtained using the same model.

In addition, from the same scattering matrix, it is possible to extract the modal decomposition of an incident mode arising from the curved guide into reflected and transmitted modes.

4.4. Curvature effect
In this section, the effect of the curvature is investigated. Here, different curvatures are considered and the mode matching model is used to predict the transmitted and reflected energy ratios for an incoming mode. In the following example, the incident mode is the first extensional L(0,1) in the straight guide.

The total energy (red line on Fig. 8) is conserved, even in the case of the 1/6 mm^{-1} curvature which leads to a radius of the intrados as low as 1 mm (to be compared to the outer diameter of 10 mm). The reflected energy increases for increasing elbow curvature.

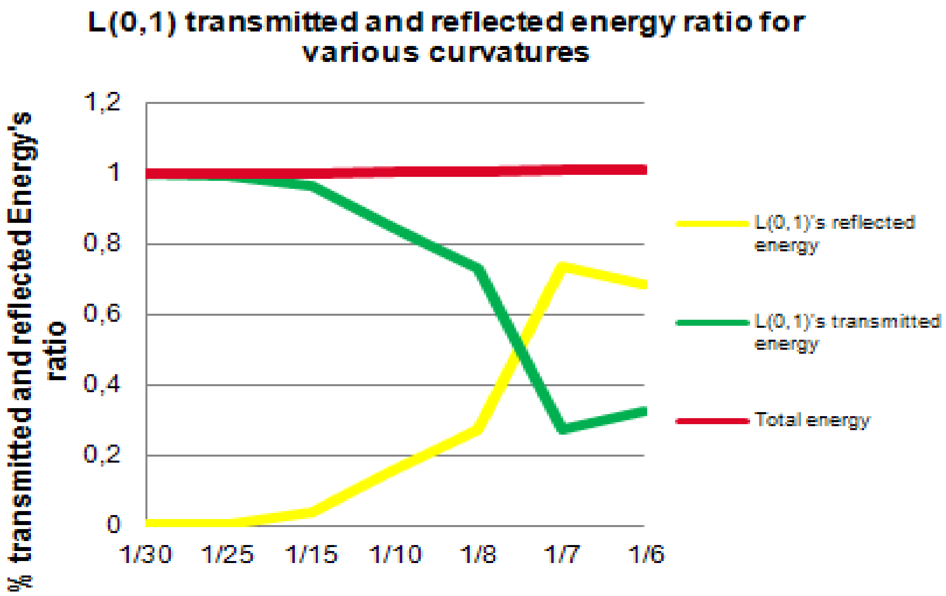

Figure 8. Reflected and transmitted energy ratio at a straight-curved junction for a L(0,1) incident mode in a steel pipe (OD=10mm, Th=2.5mm) as functions of the curvature of the elbow at 100kHz.

Similarly, for the other propagative modes (torsional and flexurals), the transmitted and reflected energies variations are represented as follows:

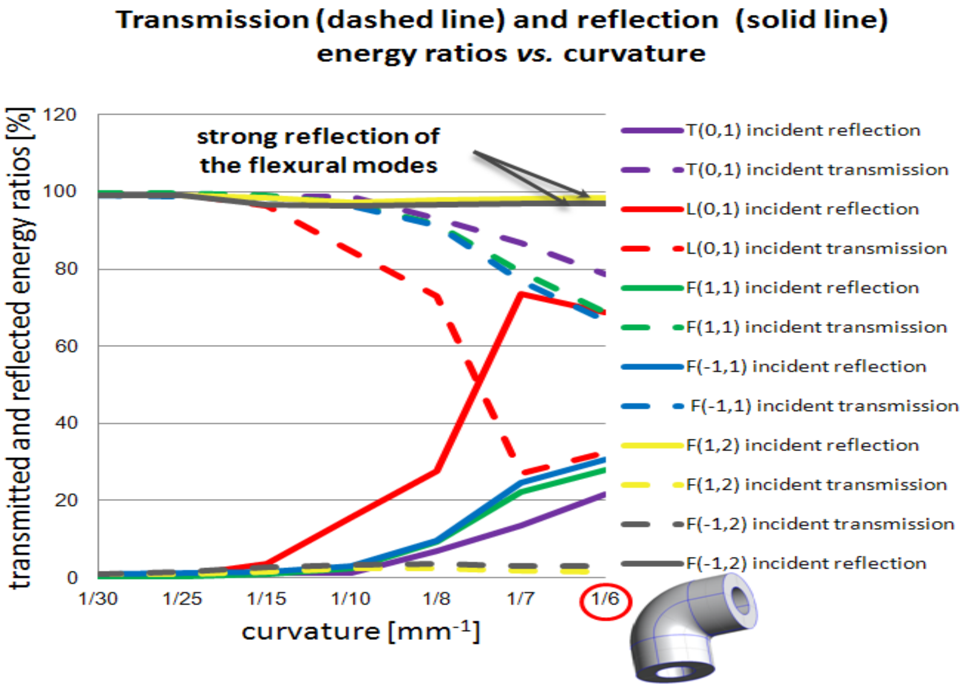

Figure 9. Reflected and transmitted energy ratios at a straight-curved junction for all possible incident modes in a steel pipe (OD=10mm, Th=2.5mm) as functions of the curvature of the elbow at 100kHz.

Globally, the evolution of reflected and transmitted energy ratios has a similar behavior compared to the L(0,1) case, except for the flexural modes F(1,2) and F(-1,2) which are strongly reflected, whatever the curvature. The reason for this behavior is that, at the frequency considered for the examples, these two modes have no equivalent into the curved guide, due to the fact that the cut-off frequencies of these modes in the curved part are higher than those in the straight part and the working frequency is in between cut-off frequency in the straight part and those in the curved part.

Thus, the choice of the frequency used for inspecting a pipework containing curved parts must be considered as an important criterion in the mode transmission through the pipe elbow.

5. Conclusions

In this study, guided waves propagation in a pipework with curved part has been investigated. In a first step, the SAFE method has been derived to compute guided wave propagation in a curved guide. Adapting the formulation proposed in [2] to the case of a curved pipe, eigenmodes expressed in a curvilinear coordinate system have been studied. Parametric studies have shown that the curvilinear SAFE implementation made can accurately deal with arbitrarily curved parts.

Then, the scattering phenomena at the junction of a straight guide and a curved one have been considered. Thanks to the mode matching method, the scattering matrix, that contains transmission and reflection coefficients and links two elastodynamic fields computed in different coordinate systems, has been computed given as modal series. The mode matching model formulation derived herein is generic and can deal with various coordinate systems by using adequate change of basis in regard to the configurations studied. Finally, parametric studies have been presented to illustrate the mode matching model capabilities.

Future work will include the calculation of the global scattering matrix resulting from the combination of several scattering matrices (welded elbow) according to [5], as well as experimental validations.

6. References

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