

On reduction of the general three-body Newtonian problem and the curved geometry

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Abstract. In the framework of an idea of separation of rotational and vibrational motions, we have examined the problem of reducing the general three-body problem. The class of differentiable functions allowing transformation of the $6D$ Euclidean space to the $6D$ conformal-Euclidean space is defined. Using this fact the general classical three-body problem is formulated as a problem of geodesic flows on the energy hypersurface of the bodies system. It is shown that when the total potential depends on relative distances between the bodies, three from six ordinary differential equations of second order describing the non-integrable hamiltonian system are integrated exactly, thus allowing reducing the initial system in the phase space to the autonomous system of the 6th order. In the result of reducing of the initial Newtonian problem the geometry of reduced problem becomes curved. The latter gives us new ideas related to the problem of geometrization of physics as well as new possibilities for study of different physical problems.

1. Introduction

The general three-body classical problem concerns the question of understanding the motions of three arbitrary point masses traveling in space according to Newton's laws of mechanics. Many works on analytical mechanics, celestial mechanics, stellar and molecular dynamics (see for example [1, 2, 3, 4, 5]) are devoted to the study of this problem. For solution of the general problem different approaches based on series expansions methods have been proposed; however, due to the poor convergence of these expansions they are often used and are useful only for solving of particular problems where the system of three-bodies is in a stable bound state [2]. Moreover, the three-body problem is a typical example of a dynamical system where on the large scales of the phase space we observe all features of a complex motion including the bifurcation and chaos. That makes the numerical simulation method a basic way of research of the mentioned problems.

The main aim of this work is to find new opportunities for separation of the internal and external motions in the general classical three-body problem. The latter will have a key importance for reducing the dimensionality of the dynamical problem that will allow to develop an effective algorithm for numerical simulation.



2. The classical three-body system in the laboratory frame

The classical Hamiltonian of the three-body system after Jacobi and mass-scale [6] transformations can be written in the form (see also [7]):

$$H(\mathbf{r}; \mathbf{p}) = \frac{\mathbf{p}^2}{2\mu_0} + V(\mathbf{r}), \quad (1)$$

where $\mathbf{r} = \mathbf{r} \oplus \mathbf{R} \in R^6$ and $\mathbf{p} \in R^6$ are correspondingly the position vector and the momentum of the effective mass (imaginary point) $\mu_0 = [m_1 m_2 m_3 / (m_1 + m_2 + m_3)]^{1/2}$, (m_1, m_2 and m_3 masses of the bodies). Note that \mathbf{r} stands for the distance between the bodies 2 and 3, while \mathbf{R} is the distance between the body 1 and the center of mass of the pair (2,3). In addition, the total potential $V(\mathbf{r})$ depends on the distances between the bodies, that means that the interaction potential in fact depends on three variables.

Let us consider the following system of coordinates:

$$\rho_1 = r = \|\mathbf{r}\|, \quad \rho_2 = R = \|\mathbf{R}\|, \quad \rho_3 = \theta, \quad \rho_4 = \Theta, \quad \rho_5 = \Phi, \quad \rho_6 = \Psi, \quad (2)$$

where the first set of three coordinates; $\{\bar{\rho}\} = (\rho_1, \rho_2, \rho_3)$ describes a position of an imaginary point on the plane formed by bodies triangle, while; $\Theta \in (-\pi, +\pi]$, $\Phi \in (-\pi, +\pi]$ and $\Psi \in [0, \pi]$ are Euler angles describing the rotation of the plane in $3D$ space. The kinetic energy of three-body system in these variables has the form (see also [8]):

$$T = \frac{1}{2\mu_0} \{\dot{\mathbf{r}}^2 + \dot{\mathbf{R}}^2\} = \frac{1}{2\mu_0} \left\{ \dot{r}^2 + r^2 [\boldsymbol{\omega} \times \mathbf{k}]^2 + (\dot{\mathbf{R}} + [\boldsymbol{\omega} \times \mathbf{R}])^2 \right\}, \quad (3)$$

where the direction of unit vector \mathbf{k} in the moving reference frame $\{\varrho\}$ is defined by expression: $\mathbf{R}\mathbf{R}^{-1} = \pm \mathbf{k}$. Below the vector $\mathbf{k} = (0, 0, 1)$ is directed towards a positive direction of the axis OZ , while the angular velocity $\boldsymbol{\omega}$ describes the rotation of the frame $\{\varrho\}$ with respect to laboratory system. By simple calculations in the expression (3) we can find:

$$T = \frac{1}{2\mu_0} \left\{ \dot{r}^2 + \dot{R}^2 + R^2 \dot{\theta}^2 + A r^2 + B R^2 \right\}, \quad (4)$$

where the following notations are introduced:

$$A = (\dot{\Theta}^2 + \dot{\Phi}^2 \sin^2 \Theta) = \omega_X^2 + \omega_Y^2, \quad B = (\omega_X \cos \theta - \omega_Z \sin \theta)^2.$$

Let us note that in deriving the expression (4) we have used the definition of the moving system $\{\varrho\}$ by the requirement that unit vector $\boldsymbol{\gamma}$ lies in the plane OXZ forming angle θ to OZ , i.e. $\boldsymbol{\gamma} = (\sin \theta, 0, \cos \theta)$. As regards of the projections of an angular velocity then they satisfy the equations:

$$\omega_X = \dot{\Phi} \sin \Theta \sin \Psi + \dot{\Theta} \cos \Psi, \quad \omega_Y = \dot{\Phi} \sin \Theta \cos \Psi - \dot{\Theta} \sin \Psi, \quad \omega_Z = \dot{\Phi} \cos \Theta - \dot{\Psi}. \quad (5)$$

Taking into account (4) and (5) we can find the metric tensor:

$$\gamma^{\alpha\beta} = \begin{pmatrix} \gamma^{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma^{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma^{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma^{44} & \gamma^{45} & \gamma^{46} \\ 0 & 0 & 0 & \gamma^{54} & \gamma^{55} & \gamma^{56} \\ 0 & 0 & 0 & \gamma^{64} & \gamma^{65} & \gamma^{66} \end{pmatrix} \quad (6)$$

where the following notations introduced:

$$\begin{aligned}\gamma^{11} = \gamma^{22} = 1, \quad \gamma^{33} = R^2, \quad \gamma^{44} = r^2 + R^2 \cos^2 \Psi \cos^2 \theta, \quad \gamma^{55} = r^2 \sin^2 \Theta + R^2 (\sin^2 \Theta \times \\ \sin^2 \Psi \cos^2 \theta + \cos^2 \Theta \sin^2 \theta - 2^{-1} \sin 2\Theta \sin 2\theta \sin \Psi), \quad \gamma^{66} = R^2 \sin^2 \theta, \quad \gamma^{45} = \gamma^{54} = \\ R^2 (\sin \Theta \sin 2\Psi \cos^2 \theta - 2 \cos \Theta \cos \Psi \sin 2\theta), \quad \gamma^{46} = \gamma^{64} = R^2 \sin 2\theta \cos \Psi, \quad \gamma^{56} = \gamma^{65} = \\ R^2 (\sin \Theta \sin \Psi \sin 2\theta - 2 \cos \Theta \sin^2 \theta).\end{aligned}$$

Without going into details note that the considered problem has 12 integrals of motion. Using them, the initial 18th order system is reduced to the 8th order system [9].

3. The geodesic equations on the hypersurface of energy of three-body system

As it is easy to see, the classical system of three bodies moving in a 3D Euclidean space permanently forms the triangle, and hence Newton's equations describe the dynamical system on the space of such triangles [8]. The latter means that we can formally consider the motion of a system consisting of two parts. The first is the rotational motion of the body-triangle in the 3D Euclidean space and the second is the internal motion of bodies on the plane defined by the triangle. Mathematically, the configuration manifold of solid body R^6 can be represented as a direct product of two subspaces [10]:

$$R^6 : \Leftrightarrow R^3 \times S^3,$$

where R^3 is the manifold which is defined as an orthonormal space of relative distances between bodies while S^3 denotes the space of the rotation group $SO(3)$. However in the problem under consideration the connections between the bodies are not holonomic and consequently we must change the representation for the configuration manifold: $\mathbf{M} : \Leftrightarrow R^6$.

Let us consider the region of localization of the dynamical system (further named *the internal space* \mathcal{M}_t):

$$x^1 = \|\mathbf{r}\| \in [0, \infty), \quad x^2 = \|\mathbf{R}\| \in [0, \infty), \quad x^3 = \|\mathbf{r} + \mathbf{R}\| = \sqrt{(x^1)^2 - 2x^1x^2 \cos \theta + (x^2)^2} \in L, \quad (7)$$

where θ is the angle between the vectors \mathbf{r} and \mathbf{R} , which in the Jacobi coordinates system is the scattering angle, in addition $L = [x^1 - x^2, x^1 + x^2]$. The set of *internal coordinates* is giving by: $\{\bar{x}\} = (x^1, x^2, x^3) \in \mathcal{M}_t$. The rotation of a plane defined by body-triangle will be described by the set of three *external coordinates* $(x^4, x^5, x^6) \in S_t^3$, where S_t^3 is a space of the rotation group $SO(3)$ in a neighborhood of interior points $M_i\{(x^1, x^2, x^3)_i\} \in \mathcal{M}_t$. The subset of all interior points $\check{\mathbf{M}} \subset \mathbf{M}$ is represented as:

$$\check{\mathbf{M}} \cong \mathcal{M}_t \times S_t^3.$$

The set $\mathbf{M} \setminus \check{\mathbf{M}}$ has zero measure. However, in some cases it can be important for the dynamics of the classical three-body system.

So, we can define a local system of coordinates in which further studies will be carried out:

$$\overline{x^1, x^6} = \{x\} \in \check{\mathbf{M}}. \quad (8)$$

Taking into account the well-known work of Krylov [11], we will study the motion of three-body system on the hypersurface of potential energy (HPE) of bodies system. It is obvious that HPE is the curved space the metric tensor of which is defined by the relations:

$$g_{\mu\nu}(\{x\}) = g(\{x\})\delta_{\mu\nu}, \quad g^{\mu\nu} = g^{-1}(\{x\})\delta^{\mu\nu}, \quad g(\{x\}) = [E - U(\{x\})]U_0^{-1} > 0, \quad (9)$$

where E and $U(\{x\}) \equiv V(\mathbf{r})$ are the total energy and the total interaction potential of bodies system respectively, $\delta_{\mu\nu}$ is the Kronecker symbol and $U_0 = \max ||U(\{x\})||$ denotes the maximal depth of the potential. In the case when the potential depends on the relative distances between the particles the metric tensor is equal: $g_{\mu\nu}(\{x\}) = g_{\mu\nu}(\{\bar{x}\})$.

Now, using the variational principle of Maupertuis we can derive the geodesic equations [10, 12]:

$$\ddot{x}^\alpha + \Gamma_{\beta\gamma}^\alpha \dot{x}^\beta \dot{x}^\gamma = 0, \quad \alpha, \beta, \gamma = \overline{1, 6}, \quad (10)$$

where $\dot{x}^\alpha = dx^\alpha/ds$ and $\ddot{x}^\alpha = d^2x^\alpha/ds^2$; in addition, s is a scalar parameter of motion (e.g. the proper time), the Christoffel symbol: $\Gamma_{\beta\gamma}^\alpha(\{x\}) = \frac{1}{2}g^{\alpha\mu}(\partial_\gamma g_{\mu\beta} + \partial_\beta g_{\gamma\mu} - \partial_\mu g_{\beta\gamma})$, where $\partial_\alpha \equiv \partial_{x^\alpha}$.

Taking into account the definition for the metric tensor (9), from (10) we can find the following system of equations describing geodesic flows on the potential energy hypersurface:

$$\begin{aligned} \ddot{x}^1 &= a_1 \left\{ (\dot{x}^1)^2 - \sum_{\mu \neq 1, \mu=2}^6 (\dot{x}^\mu)^2 \right\} + 2\dot{x}^1 \{a_2 \dot{x}^2 + a_3 \dot{x}^3\}, & \ddot{x}^4 &= \dot{x}^4 \{a_1 \dot{x}^1 + a_2 \dot{x}^2 + a_3 \dot{x}^3\}, \\ \ddot{x}^2 &= a_2 \left\{ (\dot{x}^2)^2 - \sum_{\mu=1, \mu \neq 2}^6 (\dot{x}^\mu)^2 \right\} + 2\dot{x}^2 \{a_3 \dot{x}^3 + a_1 \dot{x}^1\}, & \ddot{x}^5 &= \dot{x}^5 \{a_1 \dot{x}^1 + a_2 \dot{x}^2 + a_3 \dot{x}^3\}, \\ \ddot{x}^3 &= a_3 \left\{ (\dot{x}^3)^2 - \sum_{\mu=1, \mu \neq 3}^6 (\dot{x}^\mu)^2 \right\} + 2\dot{x}^3 \{a_1 \dot{x}^1 + a_2 \dot{x}^2\}, & \ddot{x}^6 &= \dot{x}^6 \{a_1 \dot{x}^1 + a_2 \dot{x}^2 + a_3 \dot{x}^3\}, \end{aligned} \quad (11)$$

where $g(\{\bar{x}\}) = g_{11}(\{\bar{x}\}) = \dots = g_{66}(\{\bar{x}\})$, in addition; $a_i(\{\bar{x}\}) = -(1/2)\partial_{x^i} \ln g(\{\bar{x}\})$. In the system (11), the last three equations are integrated exactly:

$$\dot{x}^\mu = J_{\mu-3}/g(\{\bar{x}\}), \quad J_{\mu-3} = \text{const}_{\mu-3}, \quad \mu = \overline{4, 6}. \quad (12)$$

Note that J_1, J_2 and J_3 are integrals of motion. They can be interpreted as projections of the total angular momentum of three-body system $J = \sqrt{J_1^2 + J_2^2 + J_3^2} = \text{const}$ on the corresponding axis.

Finally, substituting (12) into equations (11), we obtain the following system of a non-linear second-order ordinary differential equations:

$$\begin{aligned} \ddot{x}^1 &= a_1(\{\bar{x}\}) \left\{ (\dot{x}^1)^2 - (\dot{x}^2)^2 - (\dot{x}^3)^2 - [J/g(\{\bar{x}\})]^2 \right\} + 2\dot{x}^1 \{a_2(\{\bar{x}\})\dot{x}^2 + a_3(\{\bar{x}\})\dot{x}^3\}, \\ \ddot{x}^2 &= a_2(\{\bar{x}\}) \left\{ (\dot{x}^2)^2 - (\dot{x}^3)^2 - (\dot{x}^1)^2 - [J/g(\{\bar{x}\})]^2 \right\} + 2\dot{x}^2 \{a_3(\{\bar{x}\})\dot{x}^3 + a_1(\{\bar{x}\})\dot{x}^1\}, \\ \ddot{x}^3 &= a_3(\{\bar{x}\}) \left\{ (\dot{x}^3)^2 - (\dot{x}^1)^2 - (\dot{x}^2)^2 - [J/g(\{\bar{x}\})]^2 \right\} + 2\dot{x}^3 \{a_1(\{\bar{x}\})\dot{x}^1 + a_2(\{\bar{x}\})\dot{x}^2\}. \end{aligned} \quad (13)$$

Thus, the system of equations (13) describes the dynamics of an imaginary point with the effective mass μ_0 on the Riemannian manifold: $\mathcal{M} = [\{\bar{x}\} \equiv (x^1, x^2, x^3) \in \mathcal{M}_t; g_{ij} = (E - U(\{\bar{x}\}))U_0^{-1}\delta_{ij} > 0]$, by taking into account the rotation of body-triangle. The system of equations (13) can be represented as a system of six ODEs of the first order.

By taking into account (9) and (12), we can obtain the reduced Hamiltonian:

$$\mathcal{H}(\{\bar{x}\}; \{\dot{\bar{x}}\}) = g_{\mu\nu}(\{\bar{x}\})p^\mu p^\nu = \frac{\mu_0}{2}g(\{\bar{x}\}) \left\{ \sum_{i=1}^3 (\dot{x}^i)^2 + [J/g(\bar{x})]^2 \right\}. \quad (14)$$

Substituting (14) into Hamilton equations:

$$\dot{x}^\mu = \frac{\partial \mathcal{H}}{\partial p^\mu}, \quad \dot{p}^\mu = -\frac{\partial \mathcal{H}}{\partial x^\mu}, \quad (15)$$

and by making simple calculations we can get the system of geodesic equations (13).

4. Transformation of 6D Euclidean space to the 6D conformal-Euclidean space

In deriving the system of equations (13), we have used some physical considerations that, from the mathematical point of view, are insufficiently rigorous, to argue that the dynamical system (13) is equivalent to the Newtonian problem of three-body. For a strict proof of the equivalence of the approaches, we need to prove that there is one-to-one mapping between the two sets of coordinates: $\overline{\rho^1}, \overline{\rho^6} = \{\rho\}$ and $\overline{x^1}, \overline{x^6} = \{x\}$, in addition that is very important in result of transformations 6D Euclidean space must pass into the conformal-Euclidean form.

Let us consider two spaces $\mathbf{E}^6 \cong R^6$ and \mathbf{M} which satisfy the condition of one-to-one mapping; $\mathbf{E}^6 : \Leftrightarrow \mathbf{M}$. We will suppose that the Euclidean space \mathbf{E}^6 is defined by the set of coordinates: $\overline{\rho^1}, \overline{\rho^6} = \{\rho\}$ and by the metric tensor $\gamma_{\mu\nu}(\{\rho\})$, while the configuration space of an imaginary point \mathbf{M} is defined by the local coordinate frame: $\overline{x^1}, \overline{x^6} = \{x\}$ and by the metric tensor $g_{\mu\nu}(\{x\})$, respectively. The linear infinitesimal element in both coordinate systems can be represented as:

$$ds^2 = \gamma^{\alpha\beta}(\{\varrho\})d\varrho_\alpha d\varrho_\beta = g_{\mu\nu}(\{x\})dx^\mu dx^\nu, \quad \alpha, \beta, \mu, \nu = \overline{1, 6}, \quad (16)$$

where the metric tensor $g_{\mu\nu}(\{x\})$ is defined as:

$$g_{\mu\nu}(\{x\}) = \gamma^{\alpha\beta}(\{\varrho\})\varrho_{\alpha;\mu}\varrho_{\beta;\nu}, \quad \varrho_{\alpha;\mu} = \partial\varrho_\alpha/\partial x^\mu. \quad (17)$$

Since the tensor $g_{\mu\nu}(\{x\})$ is still defined in a rather arbitrary way, we can impose an additional conditions on it. In particular, we will require that the metric tensor $g_{\mu\nu}(x)$ describes the conformal-Euclidean space: $g_{\mu\nu}(x) = g(\{x\})\delta_{\mu\nu}$ (see (9)). The latter means that the following algebraic equations take place:

$$\gamma^{\alpha\beta}(\{\varrho\})\varrho_{\alpha;\mu}\varrho_{\beta;\nu} = g(\{x\})\delta_{\mu\nu}. \quad (18)$$

As one can see, the system of algebraic equations (18) is underdetermined since it consists of 21 equations while the number of unknown variables is 36. It is obvious that when these equations are compatible, the system (18) has an infinite number of real and complex solutions. These solutions form two different manifolds of the 15th order. For a classical problem real solutions are important ones, therefore below we will investigate properties of the real manifold. In a similar way, we can obtain the system of algebraic equations for inverse transformations:

$$\gamma_{\alpha\beta}(\{\varrho\})g^{-1}(\{\bar{x}\}) = x^\mu_{;\alpha}x^\nu_{;\beta}\delta_{\mu\nu}, \quad x^\mu_{;\alpha} = \partial x^\mu/\partial \rho^\alpha. \quad (19)$$

It is obvious that if there are direct transformations then there are inverse transformations too.

Let us introduce new notations:

$$x_\mu = \rho_{1;\mu}, \quad y_\mu = \rho_{2;\mu}, \quad z_\mu = \rho_{3;\mu}, \quad u_\mu = \rho_{4;\mu}, \quad v_\mu = \rho_{5;\mu}, \quad w_\mu = \rho_{6;\mu}. \quad (20)$$

Taking into account the fact that the tensor: $g_{\mu\nu}(\{\bar{x}\})$ still is an arbitrary one, we can require fulfillment of the following conditions for its elements:

$$\begin{aligned} x_4 = x_5 = x_6 = 0, & \quad y_4 = y_5 = y_6 = 0, & \quad z_4 = z_5 = z_6 = 0, \\ u_1 = u_2 = u_3 = 0, & \quad v_1 = v_2 = v_3 = 0, & \quad w_1 = w_2 = w_3 = 0. \end{aligned} \quad (21)$$

Using (6), (20) and conditions (21), from the equation (18) we can obtain two independent systems of algebraic equations:

$$\begin{aligned} x_1^2 + y_1^2 + \gamma^{33}z_1^2 &= g(\{\bar{x}\}), & x_1x_2 + y_1y_2 + \gamma^{33}z_1z_2 &= 0, \\ x_2^2 + y_2^2 + \gamma^{33}z_2^2 &= g(\{\bar{x}\}), & x_1x_3 + y_1y_3 + \gamma^{33}z_1z_3 &= 0, \\ x_3^2 + y_3^2 + \gamma^{33}z_3^2 &= g(\{\bar{x}\}), & x_2x_3 + y_2y_3 + \gamma^{33}z_2z_3 &= 0, \end{aligned} \quad (22)$$

and correspondingly:

$$\begin{aligned}\gamma^{44}u_4^2 + \gamma^{55}v_4^2 + \gamma^{66}w_4^2 + 2(\gamma^{45}u_4v_4 + \gamma^{46}u_4w_4 + \gamma^{56}v_4w_4) &= g(\{\bar{x}\}), & a_4u_4 + a_5v_4 + a_6w_4 &= 0, \\ \gamma^{44}u_5^2 + \gamma^{55}v_5^2 + \gamma^{66}w_5^2 + 2(\gamma^{45}u_5v_5 + \gamma^{46}u_5w_5 + \gamma^{56}v_5w_5) &= g(\{\bar{x}\}), & b_4u_5 + b_5v_5 + b_6w_5 &= 0, \\ \gamma^{44}u_6^2 + \gamma^{55}v_6^2 + \gamma^{66}w_6^2 + 2(\gamma^{45}u_6v_6 + \gamma^{46}u_6w_6 + \gamma^{56}v_6w_6) &= g(\{\bar{x}\}), & c_4u_6 + c_5v_6 + c_6w_6 &= 0.\end{aligned}\tag{23}$$

In (23) $a_i = \gamma^{i4}u_5 + \gamma^{i5}v_5 + \gamma^{i6}w_5$, $b_j = \gamma^{j4}u_6 + \gamma^{j5}v_6 + \gamma^{j6}w_6$ and $c_k = \gamma^{k4}u_4 + \gamma^{k5}v_4 + \gamma^{k6}w_4$, where $i = 4, 6$. Note that solutions of systems (22) and (23) form two different manifolds of 3rd order which are in one-to-one mapping with the hypersurface of the potential energy of the three-body system. Recall that similar systems of algebraic equations can be obtained for inverse transformations. It is easy to see that the coordinates defined by formulas (7) and (8) in general case do not satisfy the conditions of transformations (22) and (23) and correspondingly the system of equations (13), generally speaking, is not equivalent to the Newtonian three-body problem. Nevertheless, starting from general considerations, we can prove that there exists a such system of local coordinates. Apparently, the form of these coordinates can be determined explicitly after study of equations properties (22)-(23). Finally, it is important to note that the dynamical equations (13) conserve their previous form and only are changed dependence of the potential from coordinates.

5. Conclusion

As it is shown, with the help of coordinates transformations the general three-body problem can be formulated as a problem of geodesic flows on Riemannian manifold. In the framework of this representation we derive an additional symmetry that allows to integrate an exactly the three non-linear equations of the system containing six equations. In this way, the problem reduced to the autonomous system of the 6th order. Despite the fact that the explicit form of the local coordinate system in general case is not found, it can be proved that there is a class differentiable functions which allow to realize the corresponding coordinate transformations and satisfy the equations (22)-(23), that brings the metric tensor to the conformally-Euclidean form.

Finally it is important to note that the system of equations (13), with definitions of local coordinates (7)-(8), in spite of its inequivalency to the Newtonian problem of three-body, is an important model to study the different dynamical and statistical properties of classical three-body system, and can be interesting for the construction of its quantum analogue.

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