Quantum effects from boundaries in de Sitter and anti-de Sitter spacetimes

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Abstract. We discuss the features of the Casimir effect for two parallel plates in background of de Sitter and anti-de Sitter spacetimes. A massive scalar field with general curvature coupling parameter obeying Robin boundary conditions on the plates is considered. The corresponding results are compared with those in Minkowski spacetime.

1. Introduction

The Casimir effect (for a review see [1]) is now known to be common to systems of very different kinds, involving fluctuating quantities on which external boundary conditions are imposed. It can have important implications on all scales, from subnuclear to cosmological. An interesting topic in the investigations of the Casimir effect is its explicit dependence on the geometry of background spacetime. The relevant information is encoded in the vacuum fluctuations spectrum and analytic solutions can be found for highly symmetric geometries only. In the present paper, we will discuss the Casimir stresses on parallel plates for a scalar field in background of maximally symmetric spaces with zero, positive and negative curvature corresponding to Minkowski, de Sitter (dS) and anti-de Sitter (AdS) spacetimes, respectively.

2. Casimir forces in Minkowski spacetime

Consider a scalar field $\varphi(x)$ with the curvature coupling parameter ξ. The corresponding field equation has the form

$$(\nabla_l \nabla^l + m^2 + \xi\mathcal{R})\varphi = 0, \tag{1}$$

where ∇_l is the covariant derivative operator and $\mathcal{R}$ is the Ricci scalar of the background spacetime. The values of the curvature coupling parameter $\xi = 0$ and $\xi = \xi_D \equiv (D-1)/4D$, with D being the number of spatial dimensions, correspond to the most important special cases of minimally and conformally coupled fields. The background geometry is described by the line element

$$ds^2 = R^2 \eta_{ik} dz^i dz^k = R^2[dt^2 - \sum\nolimits_{i=1}^{D} (dz^i)^2], \tag{2}$$

where η_{ik} is the Minkowskian metric tensor. We will consider special cases $R = 1$ (Minkowski spacetime), $R = \alpha/|t|$ (dS spacetime) and $R = \alpha/z^D$ (AdS spacetime).

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Our main interest is the Casimir forces in the geometry of two infinite, parallel plates located at $z^D = a_j$, $j = 1, 2$. On the plates the field obeys Robin boundary conditions (BCs)

$$(1 + \beta_j n^l_{(j)} \partial_l)\varphi(x) = 0, \quad z^D = a_j, \tag{3}$$

with $n^l_{(j)}$ being the inward-pointing unit normal to the boundary at $z^D = a_j$ (with respect to the region under consideration). Robin type conditions are an extension of Dirichlet ($\beta_j = 0$) and Neumann ($\beta_j = \infty$) BCs and appear in a variety of situations, including the considerations of vacuum effects for a confined charged scalar field in external fields, spinor and gauge field theories, quantum gravity and supergravity. The Robin BCs may, in some geometries, be useful for depicting the finite penetration of the field into the boundary with the 'skin-depth' parameter related to the Robin coefficient. For the region between the plates one has $n^l_{(j)} = (-1)^{j-1}\delta^l_D/R_j$, where $R_j = R|_{z^D=a_j}$. In this case the BCs are written in the form $(1 + (-1)^{j-1} b_j \partial_D)\varphi(x) = 0$, with $b_j = \beta_j/R_j$. In the discussion below we assume that $b_j = \text{const}$. The imposition of BCs leads to a modification of the spectrum for vacuum fluctuations of quantum fields and, as a result, to the changes in the vacuum expectation values (VEVs) for physical quantities. In particular, forces arise acting on constraining boundaries.

First we consider the case of the Minkowski spacetime corresponding to $R = 1$ in (2). The VEV of the energy-momentum tensor, $\langle T^k_i \rangle = \langle 0|T^k_i|0\rangle$, has been investigated in [2]. The normal force acting per unit surface of the plates, p, is determined by the VEV of the normal stress $\langle T^D_D \rangle$: $p = -\langle T^D_D \rangle$. This stress is uniform in the region between the plates and vanishes in the regions $z^D < 0$ and $z^D > 0$ (note that, in general, this is not the case for the energy density and parallel stresses). The Casimir force per unit area (the vacuum effective pressure) is given by

$$p = -\frac{2(4\pi)^{-D/2}}{\Gamma(D/2)} \int_m^\infty dt \, \frac{t^2(t^2 - m^2)^{D/2-1}}{\frac{\beta_1 t-1}{\beta_1 t+1}\frac{\beta_2 t-1}{\beta_2 t+1} e^{2at} - 1}. \tag{4}$$

The force is attractive/repulsive for negative/positive values of p. Note that the Casimir force is the same for both plates and it does not depend on the curvature coupling parameter. The vacuum densities and the Casimir forces for a massive scalar field, subject to Robin boundary conditions on two codimension-one parallel plates, located on a $(D+1)$-dimensional background spacetime with an arbitrary internal space have been investigated in [3]. For Dirichlet and Neumann scalars the forces are attractive: $p_{\rm D} = p_{\rm N} < 0$. For Dirichlet BC on one plate and the Neumann one on the other, the force is repulsive: $p_{\rm DN} < 0$. For a massless field one gets

$$p_{\rm D} = p_0 = -\frac{D\zeta_R(D+1)}{(4\pi)^{(D+1)/2} a^{D+1}} \Gamma\left(\frac{D+1}{2}\right), \; p_{\rm DN} = -(1 - 2^{-D})p_0, \tag{5}$$

with $\zeta_R(x)$ being the Riemann zeta function.

For small separations between the plates, $ma \ll 1$, $a \ll |\beta_j|$, to the leading order one has $p \approx p_0$. The same is the case for Dirichlet BCs on both plates. For Dirichlet BC on one plate and non-Dirichlet BC on the other one has $p \approx p_{\rm DN}$. At large separations, $ma \gg 1$, $a \gg |\beta_j|$, in the leading order we get $p \approx -m^{D/2+1}(4\pi a)^{-D/2} e^{-2ma}$ for a massive field and $p \approx p_0$ for a massless field. The corresponding forces are attractive. For the Neumann BC on one plate and the non-Neumann BC on the other, at large distances we have $p \approx m^{D/2+1}(4\pi a)^{-D/2} e^{-2ma}$ and $p \approx -(1 - 2^{-D})p_0$, for massive and massless fields, respectively. In this case the forces are repulsive. As we see, for Dirichlet BC on one plate and non-Dirichlet BC on the other the forces are repulsive at small separations and attractive at large separations. At some intermediate separation the force vanishes and this equilibrium position is stable. For Neumann BC on one plate and non-Neumann BC on the other the forces are attractive at small separations and repulsive at large separations. In this case the equilibrium position is unstable.

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The results given above can be generalized for non-local BCs on the parallel plates [4]:

$$n^l_{(j)}\partial_l\varphi(x) + \int d\mathbf{z}'_{\parallel}\, f_j(|\mathbf{z}_{\parallel} - \mathbf{z}'_{\parallel}|)\varphi(x') = 0,\ z^D = a_j, \tag{6}$$

where $\mathbf{z}_{\parallel} = (z^1, \ldots, z^{D-1})$. These BCs state that the normal derivative at a given point depends on the values of the field at other points on the boundary. The kernel functions f_j determine the properties of the boundaries. The vacuum force acting per unit surface of the plate:

$$p = -\frac{2^{2-D}\pi^{-(D+1)/2}}{\Gamma((D-1)/2)}\int_0^\infty du \int_{\sqrt{u^2+m^2}}^\infty \frac{u^{D-2}t^2 dt}{\sqrt{t^2-u^2-m^2}}\left[\frac{(t-F_1(u))(t+F_2(u))}{(t+F_1(u))(t-F_2(u))}e^{2at}-1\right]^{-1}, \tag{7}$$

where

$$F_j(x) = \frac{(2\pi)^{(D-1)/2}}{x^{(D-3)/2}}\int_0^\infty du\, u^{(D-1)/2} f_j(u) J_{(D-3)/2}(ux), \tag{8}$$

and $J_\nu(z)$ is the Bessel function. Examples of functions $f_j(z)$ have been considered in [12].

3. Casimir Forces in dS spacetime

In this section we consider the Casimir forces in dS spacetime. This spacetime is among the most popular backgrounds in gravitational physics. It is the maximally symmetric solution of Einstein's equation with a positive cosmological constant and due to this high symmetry numerous physical problems are exactly solvable on this background. A better understanding of physical effects in this background could serve as a handle to deal with more complicated geometries. De Sitter spacetime also plays an important role in most inflationary models. More recently astronomical observations of high redshift supernovae, galaxy clusters and cosmic microwave background indicate that at the present epoch the universe is accelerating and can be well approximated by a world with a positive cosmological constant.

In the case of dS spacetime described in inflationary coordinates, the line element is given by (2) with the function $R = \alpha/|t|$, where t is the conformal time coordinate, $-\infty < t < 0$. The latter is related to the synchronous time t_s by $t = -\alpha e^{-t_s/\alpha}$ and $R = e^{t_s/\alpha}$. For the Ricci scalar one has $\mathcal{R} = D(D+1)/\alpha^2$. In the construction of a quantum field theory in a fixed classical gravitational background, the choice of the vacuum state is among the most important steps. dS spacetime is a maximally symmetric, and it is natural to choose a vacuum state having the same symmetry. In fact, there exists a one-parameter family of maximally symmetric quantum states. Here we will assume that the field is prepared in the dS-invariant Bunch-Davies vacuum state [5]. Among the set of dS-invariant quantum states, the Bunch-Davies vacuum is the only one for which the ultraviolet behavior of the two-point functions is the same as in Minkowski spacetime.

The VEVs of the field squared and the energy-momentum tensor in the geometry of parallel plates in background of dS spacetime have been investigated in [6, 7]. The vacuum force acting per unit surface of the plate at $z^D = a_j$ is determined by the ${}^D_D$-component of the vacuum energy-momentum tensor evaluated at this point. For the region between the plates, the corresponding effective pressures are presented as $p^{(j)} = p_1^{(j)} + p_{(\text{int})}^{(j)}$, $j = 1, 2$. The first term, $p_1^{(j)}$, is the pressure for a single plate at $z^D = a_j$, when the second plate is absent. This term is divergent due to surface divergences in the subtracted VEVs and needs additional renormalization. The term $p_{(\text{int})}^{(j)}$ is the pressure induced by the second plate, and can be termed as an interaction force. This contribution is finite for all nonzero distances between the plates. In the regions $z^D < a_1$ and $z^D > a_2$ we have $p^{(j)} = p_1^{(j)}$. As a result, the contributions to the vacuum force coming from the term $p_1^{(j)}$ are the same from the left and from the right sides of the plate, so there is no net contribution to the effective force.

The interaction force on the plate at $z^D = a_j$ is given by the expression [7]:

$$p_{(\mathrm{int})}^{(j)} = -\frac{8(4\pi)^{-(D+1)/2}}{\Gamma((D-1)/2)\alpha^{D+1}}\int_0^\infty dy\, y^{1-D}\int_y^\infty dx\,\frac{x^2(x^2-y^2)^{(D-3)/2}}{\frac{b_1x/\eta-1}{b_1x/\eta+1}\frac{b_2x/\eta-1}{b_2x/\eta+1}e^{2ax/\eta}-1} \times\left[\frac{2\,(b_j/\eta)^2 G_\nu(y)}{(b_jx/\eta)^2-1}+F_\nu(y)\right], \quad (9)$$

where $\eta = |t|$, and we introduced the notations

$$F_\nu(y) = y^D\left[I_\nu(y)+I_{-\nu}(y)\right]K_\nu(y),\ \nu=\left[D^2/4-D(D+1)\xi-m^2\alpha^2\right]^{1/2},$$
$$G_\nu(y) = \left\{(\xi-1/4)\,y^2\partial_y^2+\left[\xi(2-D)+(D-1)/4\right]y\partial_y-\xi D\right\}F_\nu(y). \quad (10)$$

In (10), $I_\nu(y)$ and $K_\nu(y)$ are the modified Bessel functions. The time dependence of the forces appears in the form a/η and b_j/η. Note that the ratio a/η is the proper distance between the plates measured in units of dS curvature radius α. Unlike to the Minkowskian case, for $\beta_1 \neq \beta_2$ the Casimir forces acting on the left and on the right plates are different. For large values of α, to the leading order, the corresponding result for the geometry of two parallel plates in Minkowski spacetime is obtained.

In the special cases of Dirichlet and of Neumann BCs one finds:

$$p_{(\mathrm{int})}^{(\mathrm{D})} = -\frac{4\alpha^{-D-1}}{(2\pi)^{D/2+1}}\sum_{n=1}^{\infty}\int_0^\infty dy\, yF_\nu(y)\left[(D-1)f_{D/2}(yu_n)+f_{D/2-1}(yu_n)\right], \quad (11)$$

$$p_{(\mathrm{int})}^{(\mathrm{N})} = p_{(\mathrm{int})}^{(\mathrm{D})} - \frac{8\alpha^{-D-1}}{(2\pi)^{D/2+1}}\sum_{n=1}^{\infty}\int_0^\infty dy\,\frac{G_\nu(y)}{y}f_{D/2-1}(yu_n),\ u_n=2na/\eta, \quad (12)$$

where $f_\mu(x) = K_\mu(x)/x^\mu$. For $0 \leqslant \nu < 1$ the integrand in the expression for $p_{(\mathrm{int})}^{(\mathrm{D})}$ is positive which corresponds to an attractive force for all separations.

In the limit of small proper distances between the plates, $a/\eta \ll 1$, the effects induced by the curvature of dS spacetime are subdominant and to leading order we find the result for the Minkowski spacetime, given in the previous section, with the replacement $a \to \alpha a/\eta$.

In considering the large distance asymptotics, corresponding to $a/\eta \gg 1$, the cases of real and imaginary ν must be studied separately. For positive values of ν, one has $p_{(\mathrm{int})}^{(j)} \propto (a/\eta)^{2\nu-D-2}$ for non-Neumann BC on the plate at $z^D = a_j$ ($|b_j| < \infty$) and $p_{(\mathrm{int})}^{(j)} \propto (a/\eta)^{2\nu-D}$ for Neumann BC ($b_j = \infty$). In the case of non-Neumann BC we have assumed that $|b_j|/a \ll 1$. For positive values of ν, at large distances the ratio of the Casimir forces acting on the plate with Neumann and non-Neumann BCs is of the order $(a/\eta)^2$. In neither of these cases does the leading order term in the force depend on the specific value of Robin coefficient in the BC on the second plate. For Dirichlet BC on the plate at $z^D = a_j$ ($b_j = 0$), at large separations the Casimir force acting on that plate is repulsive (attractive) for Neumann (non-Neumann) BCs on the other plate. The nature of the force acting on the plate with Neumann BC can be either repulsive or attractive, in function of the curvature coupling parameter and of the field mass. For minimally and conformally coupled massive scalar fields the corresponding force is attractive (repulsive) for Neumann (non-Neumann) BC on the second plate. For imaginary ν, the leading order terms at large separations between the plates are in the form $p_{(\mathrm{int})}^{(j)} \propto (a/\eta)^{-D-2}\cos[2|\nu|\ln(2a/\eta)+\phi_{(j)}]$ for $|b_j| < \infty$, and $p_{(\mathrm{int})}^{(j)} \propto (a/\eta)^{-D}\cos[2|\nu|\ln(2a/\eta)+\phi_{(j)}^{\mathrm{N}}]$ for $b_j = \infty$ (Neumann BC). In this case the decay of the vacuum forces is oscillatory.

From the discussion given above it follows that for proper distances between the plates larger than the curvature radius of the dS spacetime, $\alpha a/\eta \gtrsim \alpha$, the gravitational field essentially changes the behavior of the Casimir forces compared with the case of the plates in Minkowski spacetime. The forces may become repulsive at large separations between the plates. Recall that, for the geometry of parallel plates on the background of Minkowski spacetime, the only case with repulsive Casimir forces at large distances corresponds to Neumann BC on one plate and non-Neumann BC on the other. A remarkable feature of the influence of the gravitational field is the oscillatory behavior of the Casimir forces at large distances, which appears in the case of imaginary ν. In this case, the values of the plate distance yielding zero Casimir force correspond to equilibrium positions. Among them, the positions with negative derivative of the force with respect to the distance are locally stable. At large separations between the plates the decay of the Casimir forces as functions of the distance is power-law for both cases of massive and massless fields. Recall that, in Minkowski spacetime the corresponding Casimir forces are exponentially suppressed by the factor $\exp(-2ma)$ for a massive filed.

4. AdS spacetime

Now let us turn to the AdS spacetime as a background geometry. Much of the earlier interest in this geometry was motivated by questions of principal nature, mainly related with the quantization of fields on curved backgrounds. Further interest in this subject arose from the discovery that the AdS spacetime generically arises as a ground state in extended supergravity and string theories, what is again potentially most important. In recent developments of the topic, the AdS geometry is an arena for two classes of models. The first is the AdS/CFT correspondence, which relates string theories or supergravity in the AdS bulk with a conformal field theory living on its boundary. The second class of models with the AdS spacetime as background geometry is a realization of the braneworld scenario with large extra dimensions and provides a solution to the hierarchy problem which arises between the gravitational and electroweak mass scales.

For AdS spacetime described in Poincare coordinates the line element is given by (2) with $R = \alpha/z^D$ and for the Ricci scalar one has $\mathcal{R} = -D(D+1)/\alpha^2$. The coordinate z^D is related to the coordinate y, measuring the proper distance from the plates, by $z^D = \alpha e^{y/\alpha}$. The VEVs of the field squared and the energy-momentum tensor in the geometry of two parallel plates have been investigated in [8] (see also [9] for the case of a conformally coupled massless field). The corresponding effective pressures on the plates can be presented as a sum of two terms: $p^{(j)} = p_1^{(j)} + p_{(\mathrm{int})}^{(j)}$, $j = 1, 2$. The first term on the right is the pressure for a single plate at $z^D = a_j$ when the second plate is absent. This term is divergent due to the surface divergences in the VEVs. For the interaction force per unit surface of the plate at $z^D = a_j$ one has:

$$\begin{aligned} p_{(\mathrm{int})}^{(j)} &= \frac{\alpha^{-D-1}a_j^D}{(4\pi)^{D/2}\Gamma(D/2)}\int_0^\infty dx\, x^{D-1}C_{j\mu}(xa_1, xa_2) \\ &\quad \times \frac{(x^2a_j^2 - \mu^2 + 2m^2\alpha^2)B_j^2 - D(4\xi - 1)A_jB_j - A_j^2}{\bar{K}_\mu^{(1)}(xa_1)\bar{I}_\mu^{(2)}(xa_2) - \bar{K}_\mu^{(2)}(xa_2)\bar{I}_\mu^{(1)}(xa_1)}. \end{aligned} \tag{13}$$

where $\mu = \sqrt{D^2/4 - D(D+1)\xi + m^2\alpha^2}$ and we have introduced the functions

$$C_{1\mu}(u,v) = \bar{K}_\mu^{(2)}(v)/\bar{K}_\mu^{(1)}(u),\ C_{2\mu}(u,v) = \bar{I}_\mu^{(1)}(u)/\bar{I}_\mu^{(2)}(v), \tag{14}$$

The barred notations for $j = 1, 2$ are defined as

$$\bar{F}^{(j)}(x) = A_jF(x) + B_jxF'(x),\ A_j = 1 + (-1)^{j-1}(\beta_j/a_j)D/2,\ B_j = (-1)^{j-1}\beta_j/a_j. \tag{15}$$

The parameter μ must be real to ensure stability [10]. For given β_j/a_j, $j=1,2$, the interaction forces depend on a_1 and a_2 in the form of the ratio a_2/a_1, which is related to the proper distance between the plates by $a=\alpha\ln(a_2/a_1)$. As it is seen from (13), the forces acting on the plates are not symmetric under the interchange of the brane indices. It can be seen that $p_{(\mathrm{int})}^{(j)}$ is negative for Dirichlet scalar and for a scalar with $A_1=A_2=0$. The corresponding interaction forces are attractive for all values of the interplate distance.

At small separations compared with the AdS curvature radius, $a\ll\alpha$, the leading terms in the asymptotic expansion of the forces concides with the expressions for the plates in the Minkowski bulk. At large separations, $a\gg\alpha$ (this limit is realized in the Randall-Sundrum model), one has $p_{(\mathrm{int})}^{(1)}\propto e^{-(D+2\mu)a/\alpha}$ and $p_{(\mathrm{int})}^{(2)}\propto e^{-2\mu a/\alpha}$. Note that in AdS spacetime the exponential suppression of the forces with the separation between the plates takes place for a massless fields as well. This is in contrast with the cases of Minkowski and dS spacetimes. In dependence of the values for the coefficients in the boundary conditions, the corresponding forces can be either attractive or repulsive. The expressions of the forces for an untwisted scalar in the $(D+1)$-dimensional Randall-Sundrum braneworld model are obtained from (13) with an additional factor 1/2 and with

$$A_1/B_1=[D(1-4\xi)-\alpha c_1]/2,\quad A_2/B_2=[D(1-4\xi)+\alpha c_2]/2, \tag{16}$$

with c_j being the brane mass term on the brane $z^D=a_j$. For a twisted scalar one has Dirchlet BCs on both branes.

The VEV of the surface energy-momentum tensor in the geometry of two parallel plates on AdS bulk is evaluated in [11]. It has been shown that in the Randall-Sundrum braneworld model, for the interbrane distances solving the hierarchy problem, the cosmological constant generated on the visible brane is of the right order of magnitude with the value suggested by the cosmological observations. The Casimir densities and forces in AdS spacetime with a warped internal compact space have been discussed in [12, 13].

5. Conclusion

From the analysis carried out above, it follows that the curvature of the background spacetime decisively influences the behavior of the Casimir forces at distances larger than the curvature scale. As we have seen, in dS spacetime the decay of the forces at large separations between the plates is power-law. This is quite remarkable and clearly in contrast with the corresponding features of the same problem in Minkowski and AdS spacetimes.

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