

Kerr-like behavior of orbits around rotating Newtonian stars

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Abstract. The innermost stable circular orbit and the splitting of epicyclic and orbital frequencies are among strong-field signatures of general relativity searched for in astronomical objects, particularly in the X-ray data accumulated in observations of neutron stars and black holes. It may come as a surprise that these effects are present in the Newtonian physics of rapidly rotating gravitating bodies, such as the classic Maclaurin spheroids.

1. Introduction

Investigations of orbital motion culminating after some millenia in Kepler's law, $P^2 \propto r^3$, predate the birth of physics. The discovery of extremely short periods in the X-ray emission of neutron stars has renewed interest in the laws governing orbital motion, as it was thought that the phenomenon of these fast quasi-periodic oscillations, as they are called, or kHz QPOs (see [1] for a review) arises in thin accretion disks, where the X-ray emitting fluid follows nearly circular orbits, with the consequence that on the one hand the fluid is nearly in hydrostatic equilibrium, being rotationally supported in the radial direction, while on the other it is in a state of differential rotation leading to vigorous dissipation of energy [2]. Similar effects have been discovered in close binaries harbouring black holes [3]. It was hoped that a close investigation of these quasi-periodic modulations of the X-ray flux may reveal characteristics of motion thought to be a signature of general relativity (GR), such as the innermost stable circular orbit (ISCO, more properly, the marginally stable orbit), or the splitting of orbital and epicyclic frequencies. The question whether such effects are a unique signature of GR is the topic of our contribution.

2. QPOs

Let us start with the case of the 0.11 Hz QPO in the X-ray flux of the cataclysmic variable SS Cygni in outburst [4]. No harmonics or subharmonics have been observed, unlike in X-ray pulsars, so this cannot be a rotating spot, whether on the surface of the white dwarf or in the accretion disk. Further, this 0.11 Hz solitary peak in the power-density spectrum (PDS) is not coherent (hence the name QPO), i.e., it cannot correspond to the (stable) rotation of the white dwarf. In a cataclysmic variable, relativistic corrections to orbital motion are expected to be negligible, with $GM/(rc^2) \sim 10^{-4}$, and this raises the very interesting question why a characteristic frequency should show up at all in the orbital motion of fluid in the accretion disk. After all, Keplerian frequency is scale-free (Fig. 1).



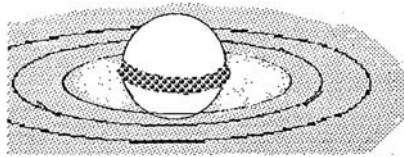
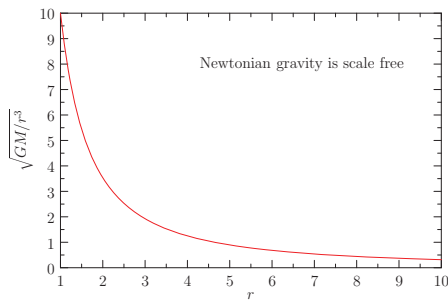


Figure 1. Keplerian frequency is scale-free, but a characteristic radius, e.g., the ISCO, implies a characteristic frequency.

The answer to this puzzle is that the radius of the white dwarf provides a characteristic scale, so the observed frequency should correspond to Keplerian motion close to the stellar surface, as indeed seems to be the case. In GR mass is equivalent to radius, therefore the mass of the gravitating object, be it a BH or neutron star provides another characteristic radius, $r_g = GM/c^2$, and in fact more than one, for there is also the ISCO radius (at $6r_g$ for the Schwarzschild metric). Thus, the maximum orbital frequency in a neutron star may occur at the marginally stable orbit, rather than at the stellar surface [5]. In the cartoon of Fig. 1, the accretion disk terminates close to the marginally stable orbit, well above the surface of a neutron star in a low-mass X-ray binary (LMXB). As remarked earlier, black holes (BH) exhibit similar QPOs, and since a BH has no surface the emission must come from the accretion disk. The BH QPOs occur at frequencies ~ 100 Hz, as would be expected if the frequency is suitably scaled up according to Kepler's law—the ISCO radius of a $6M_\odot$ BH being about 10^2 km, to a white dwarf's radius of 10^4 km.

3. Kepler's law and epicyclic frequencies

As is well known, Kepler's expression for orbital frequency, $\Omega_K(r) = 2\pi/P$, with the proportionality coefficient determined by Newton,

$$\Omega_K(r) = \sqrt{GM/r^3} \quad (1)$$

holds equally for a Newtonian $1/r$ potential and in the Schwarzschild metric, i.e., there is no difference between Newtonian mechanics and Einstein's general relativity in the orbital frequency of a test particle in stable circular motion around a spherically symmetric gravitating mass. However not all circular orbits are stable in GR, even in the spherically symmetric case. While angular momentum l of a test particle in circular orbit is a monotonically increasing function of the radius r in the Newtonian $1/r$ potential, so at all radii the condition for stability of orbits $dl/dr > 0$ is met, in the Schwarzschild metric l has a minimum at $r = 6GM/c^2$, and $dl/dr < 0$ for smaller values of r . Thus stable circular orbits in this metric exist only for $r > 6GM/c^2$.

GR effects become readily apparent for non-circular orbits, and this is the reason why one of the three classic tests of GR concerned the perihelion precession of the orbit of Mercury. The frequency of such a precession can be represented as the difference between the circular orbital frequency of Eq. (1) and radial epicyclic frequency ω_r (of “to and fro” motion), which is always lower (at least in the examples discussed here). Also of interest are slightly inclined, nearly circular, orbits, in which the motion can again be decomposed into circular motion and into vertical epicyclic (“up and down”) motion, occurring at frequency, ω_z . For spherically symmetric bodies, all planes of motion (containing the center of the gravitating mass M) are equivalent, hence $\omega_z = \Omega_K$ for the Schwarzschild metric. However, rotation removes the degeneracy.

In the Kerr metric the expressions for the orbital and epicyclic frequencies [6] are all different:

$$\omega_r^2/\Omega_o^2 = 1 - 6/r + 8j/r^{3/2} - 3j^2/r^2,$$

$$\omega_z^2/\Omega_o^2 = 1 - 4j/r^{3/2} + 3j^2/r^2, \quad (2)$$

where $-1 < j < 1$ is the dimensionless BH spin, i.e., the Kerr parameter, the orbital frequency being $\Omega_o = c^3[GM(r^{3/2} + j)]^{-1}$, with r expressed in units of GM/c^2 . Note that $\omega_r^2 < \omega_z^2 < \Omega_o^2$ for prograde motion ($j > 0$). The location of the marginally stable orbit [7] corresponds to the zero of radial epicyclic frequency. Newton's $1/r$ potential is fully degenerate and therefore bound orbits are closed ($\omega_z = \omega_r = \Omega_K$, as can be checked by taking the limit of $j = 0$ and $r \rightarrow \infty$).

4. Diskoseismology

The presence of the ISCO, occurring as it does at the zero of ω_r , has an interesting consequence: the radial epicyclic frequency necessarily has a maximum in the region of stable circular orbits (at $r = 8GM/c^2$ in the Schwarzschild metric). This fact may have profound astronomical consequences.

Being bodies of fluid close to hydrostatic equilibrium, accretion disks are capable of many modes of oscillation, whose study in GR was pioneered in [8]. As it turns out, the frequency of g -modes must necessarily be less than ω_r . As $\omega_r \rightarrow 0$ on both sides of its maximum (i.e., at the ISCO and at $r \rightarrow \infty$) g -modes of any admissible frequency are trapped between two radii on two sides of the maximum of ω_r . In effect, the radial epicyclic frequency forms a cavity for g -modes, consequently these have a discrete (and rather stable) frequency spectrum [8]. Another interesting mode, the c -mode [6] with frequency $\approx \Omega_o - \omega_z$, corresponds to the Lense-Thirring precession. For a more comprehensive discussion see the reviews [9, 10].

If these theoretical results do indeed correspond to the observed QPOs, the latter are probably a manifestation of GR effects [11]. However, it would be premature to claim that, for instance, the presence of a maximum in the radial epicyclic frequency is unique to GR. Seemingly same phenomena may have very different origins and, as we show below, at least some properties of GR orbits have direct analogues in purely Newtonian gravity.

5. Orbital properties of rapidly rotating stars

While the external metric of non-rotating stars is given by Schwarzschild, the orbits around rotating stars cannot be described by those of the Kerr metric. For slowly rotating stars, the metric is known analytically [12], and through first order in the stellar angular momentum ("spin") the Hartle-Thorne metric coincides with the Kerr one. However, higher order terms carry information about the multipoles of mass distribution, and for high spins these significantly affect the properties of stellar orbits, in particular the epicyclic frequencies. The shape of the $\omega_r(r)$ curve for rapidly rotating massive quark stars implies that rotation-induced oblateness pushes the ISCO out to larger radii [13]. Similar, although less pronounced, effects have been noted in rotating neutron star models [14]. To demonstrate that rotation-induced flattening is indeed responsible for these effects we have investigated models of quark stars, which allow computations both at usual stellar (about Solar) masses and at very low masses (quark matter, if it exists, is self-bound).

Fig. 2 shows the epicyclic and orbital frequencies around two $1.4M_\odot$ quark stars modeled with a simple MIT bag equation of state

$$P = (\rho - \rho_0)c^2/3,$$

with $\rho_0 = 4.2785 \times 10^{14} \text{ g/cm}^3$. The GR calculations were carried out with the RNS code [15]. Note that the frequencies in the plots are reduced by a factor of 2π with respect to the angular frequencies discussed previously ($2\pi\nu = \omega$). The vertical scale is given in kHz, the horizontal in units of gravitational radius. The darker (leftmost) curves correspond to a stellar rotation rate of 600 Hz. This is a modest rotation rate, and qualitatively the behavior is quite similar

to the Kerr case for $j \ll 1$, eq. (2): there is only a very slight splitting of the vertical epicyclic frequency (dashed curve) from the orbital one (solid curve), and the radial epicyclic frequency (dotted curve) goes to zero close to the Schwarzschild ISCO radius of $6M$ (c.f., Kerr metric figures in [6]).

Based on the Kerr behavior, one could expect that doubling the stellar spin would result in an increased splitting of vertical epicyclic and orbital frequencies preserving the $\omega_z^2 < \Omega_0^2$ relation, an increase of the maximum value of ω_r , and a decrease of the ISCO radius (location of zero of ω_r). Instead, when the stellar rotation rate is increased to 1165 Hz, the maximum value of ν_r decreases, the ISCO is pushed out to larger radii (the sparser, rightmost, dotted curve), while the vertical epicyclic frequency (uppermost dashed curve) is now larger than the orbital frequency (the upper of the solid curves) in the radius range depicted in the figure. In fact, this is the behaviour expected for flattened fluid configurations.

We have investigated [16] the frequencies of Maclaurin spheroids, the classic figures of equilibrium of rotating uniform density fluid configurations, and found that $\omega_r^2 + \omega_z^2 = 2\Omega_0^2$, implying that $\omega_z^2 > \Omega_0^2$. Further, at high ellipticities of the spheroids, a marginally stable orbit appears outside the equator, and is pushed out to larger and larger radii as the ellipticity (or equivalently the rotation rate) of the spheroid increases. Having computed quark star models at extremely low masses of $0.001M_\odot$, we are in a position to compare the numerical models with the analytic results for Maclaurin spheroids. The results are shown in Fig. 3, the dots corresponding to the numerical results [17], and the curves to the analytic formulae [16]. Angular frequencies squared have been plotted against the radius in units of the equatorial radius of the fluid configuration/star. Clearly there is excellent agreement between numerical and analytic models in the Newtonian limit, that is to say, between the GR results of the RNS code in the low-mass limit and the fully Newtonian analytic calculations for the Maclaurin spheroids.

Note that although the results presented in Fig. 3 are Newtonian (unlike those of Fig. 2), they include the presence of a marginally stable orbit [18, 19, 20] (at about 1.15 stellar radii, for the selected ellipticity of 0.954), a maximum in the radial epicyclic frequency, and the splitting of orbital and epicyclic frequencies.

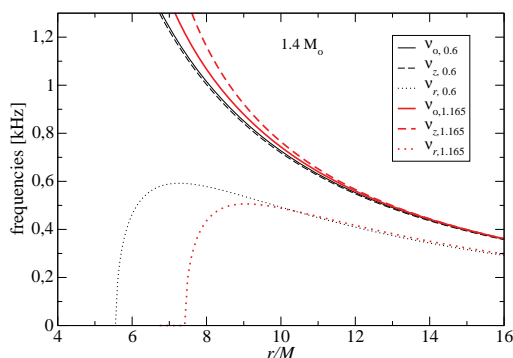


Figure 2. The orbital and epicyclic frequencies for two models of a $1.4M_\odot$ quark star in GR, rotating at 600 Hz (in black), and at 1165 Hz (in red) [17].

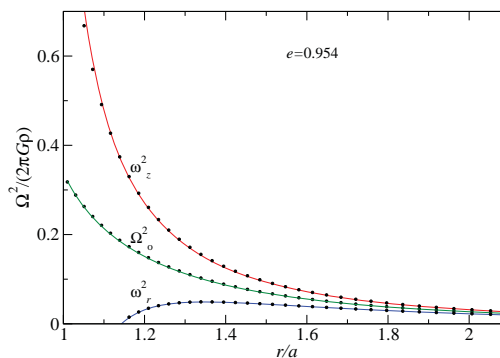


Figure 3. Comparison of analytic results for the Maclaurin spheroid (curves) [16] with numerical results of the RNS code for a $0.001M_\odot$ quark star (dots) [17].

6. Discussion

Astrophysical phenomena, such as X-ray emissions of LMXBs and their modulations (kHz QPOs) can, and should, be understood in terms of our best theory of gravity, Einstein's GR. However, some authors have tried to infer the correct theory of gravity from X-ray observations, and that may be premature, at the current state of observational techniques and theory of accretion.

A case in point is reverberation mapping. It has been suggested that once we have direct evidence of the size of the BH hole accretion disk, comparison with the angular frequencies inferred from observations will provide a quantitative test of GR. In fact, what would be tested in such a situation, is Kepler's law. We note that from the point of view of physics there is no difference between testing the law in the Solar system and in motion around a Schwarzschild black hole, as the problem is scale-free (Fig. 1, left panel). In fact, Kepler's law, with Newton's constant of proportionality, has been tested over some seven orders of magnitude in radius and mass, if one takes into account the motion of the Apollo spacecraft around the moon and that of the Earth around the Sun, and more if one remembers the outer planets. LMXBs are fairly tight binary systems of about Solar mass, with periods of days, hours or even eleven minutes. The "dynamical" masses of BHs in such systems have been derived by extrapolating Kepler's law from the Solar system to smaller distances, by about four orders in magnitude. Describing the fluid orbits in the accretion disk of a BH in a LMXB is a further extrapolation by another four orders in distance. We find it surprising that this last extrapolation should be subjected to attempted experimental verification (although there is no harm in that), while the first extrapolation by as many orders of magnitude is accepted without as much as a second thought.

It has been hoped that some qualitative features of GR, such as the presence of the marginally stable orbit may be tested in LMXBs [21], but we have shown that the same effect is also present in the Newtonian Maclaurin spheroids. The tasks ahead of us are more difficult than expected.

Acknowledgments

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