

Inhomogeneous condensates in dilute nuclear matter and BCS-BEC crossovers

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Abstract. We report on recent progress in understanding pairing phenomena in low-density nuclear matter at small and moderate isospin asymmetry. A rich phase diagram has been found comprising various superfluid phases that include a homogeneous and phase-separated BEC phase of deuterons at low density and a homogeneous BCS phase, an inhomogeneous LOFF phase, and a phase-separated BCS phase at higher densities. The transition from the BEC phases to the BCS phases is characterized in terms of the evolution, from strong to weak coupling, of the condensate wavefunction and the second moment of its density distribution in r -space. We briefly discuss approaches to higher-order clustering in low-density nuclear matter.

1. Introduction

The two-nucleon interactions in nuclear matter at sub-saturation densities (below about half the saturation density) are well constrained by the phase-shift data and analysis of elastic nucleon-nucleon collisions, facilitating a quantitative many-body description of nuclear matter in this regime. However, a significant theoretical challenge is posed by the complexity of behavior arising from the dominant attractive part of the nuclear interaction, notably the formation of nuclear clusters and the emergence of nucleonic pair condensates of the Bardeen-Cooper-Schrieffer (BCS) type. Such a description is of direct relevance for the matter found in supernovae



and neutron stars. These two settings differ somewhat in the regions of parameter space (density, temperature, isospin asymmetry) involved. For example, in supernovae the nuclear matter is at finite isospin asymmetry which, however, is small compared to that of cold β -catalyzed neutron star matter. In this contribution we review some recent progress in establishing the nature of pairing in low-density nuclear matter.

2. Pairing

If we restrict ourselves to only two-body correlations, nuclear matter at extremely low density is a mixture of quasi-free nucleons and deuterons, the only effect of the interaction being to renormalize the mass of the constituents. As bosons, the deuterons may also condense and form a Bose-Einstein condensate (BEC) even without interactions, provided, of course, the temperature is sufficiently low [1, 2, 3, 4, 5, 6]. Increasing the density of the system while keeping its temperature constant serves to increase the degeneracy of the system. As the density ρ approaches the saturation density $\rho_0 = 2.8 \times 10^{14} \text{g cm}^{-3}$ of symmetrical nuclear matter, the abundance of deuterons is reduced by Pauli blocking of the phase space available to nucleons (for a recent discussion see [7]). However, because at high densities the nucleons fill a Fermi sphere and because the interaction between nucleons is attractive, there emerges a BCS-type coherent state – a condensate of paired nucleons.

The dominant pairing interaction comes from the 3S_1 - 3D_1 (SD) partial wave, i.e., the same interaction channel that binds the deuteron. Therefore, we may anticipate that the nuclear matter undergoes a BEC-BCS phase transition as the density increases. This effect has been conjectured to occur in the context of intermediate-energy heavy-ion collisions [1, 2] and more recently in the context of supernovae [8, 9].

The theoretical framework developed by Nozières and Schmitt-Rink [10] for description of the BCS-BEC transition in the condensed-matter context can be transposed to nuclear matter. Importantly, however, isoscalar neutron-proton (np) pairing is disrupted by the isospin asymmetry induced by weak interactions in stellar environments and expected in exotic nuclei, since the mismatch in the Fermi surfaces of protons and neutrons suppresses the pairing correlations [11]. The standard Nozières-Schmitt-Rink theory of the BCS-BEC crossover has been modified to account for this fact [12]. The emergence of two mismatched Fermi surfaces in isospin-asymmetrical nuclear matter introduces a new scale into the problem, namely the shift $\delta\mu = (\mu_n - \mu_p)/2$ in the chemical potentials μ_n and μ_p of neutrons and protons (up or down) from their otherwise common value $\bar{\mu} = (\mu_n + \mu_p)/2$. If sufficiently large, this shift can destroy the condensate.

In the isospin symmetrical case (at $\delta\mu = 0$) the condensate is characterized by a gap Δ_0 in the SD channel of order several MeV. A sequence of unconventional phases has been conjectured to occur with increasing isospin symmetry, i.e., as $\delta\mu$ increases from zero to values of order Δ_0 . One of these is a neutron-proton condensate whose Cooper pairs have non-zero center-of-mass (CM) momentum [13, 14, 15]. This phase is the analogue of the Larkin-Ovchinnikov-Fulde-Ferrell (LOFF) phase in electronic superconductors [?, 16]. Another possibility is phase separation into superconducting and normal components, proposed in the context of cold atomic gases [17]. At large isospin asymmetry, where SD pairing is strongly suppressed, a BCS-BEC crossover may also occur in the isotriplet 1S_0 pairing channel. Finally, it has been conjectured that spontaneous deformation of Fermi surfaces is an alternative to the LOFF phase and can in fact dominate the latter in nuclear systems [14].

Recently, the concepts of unconventional SD pairing and the BCS-BEC crossover were unified in a model of isospin-asymmetric nuclear matter [9] by including some of the phases mentioned above. A phase diagram for superfluid nuclear matter was constructed over wide ranges of density, temperature, and isospin asymmetry. The coupled equations for the gap and the densities of constituents (neutrons and protons) were solved allowing for the ordinary BCS state,

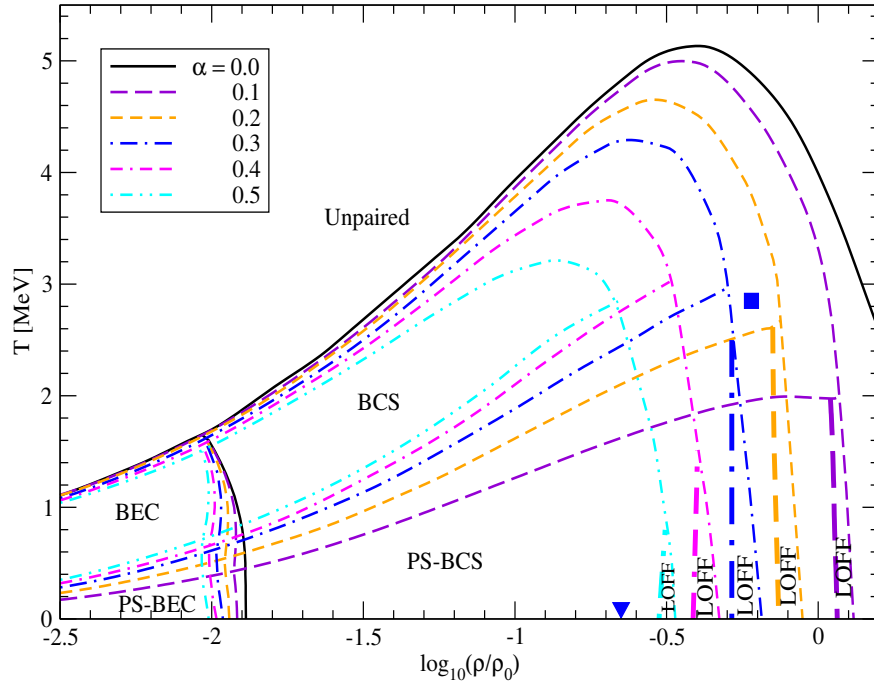


Figure 1. Phase diagram of dilute nuclear matter in the temperature-density plane for several isospin asymmetries α (from [9]). Included are four phases: unpaired phase, BCS (BEC) phase, LOFF phase, and PS-BCS (PS-BEC) phase. For each asymmetry there are two tri-critical points, one of which is always a Lifshitz point [?]. For special values of asymmetry these two points degenerate into single tetra-critical point at $\log(\rho/\rho_0) = -0.22$ and $T = 2.85$ MeV (shown by a square dot) for $\alpha_4 = 0.255$. The LOFF phase disappears at the point $\log(\rho/\rho_0) = -0.65$ and $T = 0$ (shown by a triangle) for $\alpha = 0.62$. The boundaries between BCS and BEC phases are identified by the change of sign of the average chemical potential $\bar{\mu}$.

its low-density asymptotic counterpart BEC state, and two phases that owe their existence to the isospin asymmetry: the phase with a moving condensate (LOFF phase) and the phase in which the normal fluid and superfluid occupy separate spatial domains. The latter phase is referred to as the phase-separated BCS (PS-BCS) phase and, in the strong-coupling regime, the phase-separated BEC (PS-BEC) phase. In this phase the asymmetry is accumulated in the normal domains, whereas the superfluid domain is perfectly isospin symmetric.

The gap equation to be solved has the form

$$\Delta_l(k, Q) = \frac{1}{2} \sum_{l'} \int \frac{d^3 k'}{(2\pi)^3} V_{l,l'}(k, k') \sum_{a,r} \frac{\Delta_{l'}(k', Q)}{2\sqrt{E_S(k')^2 + \Delta_{l'}^2(k', Q)}} [1 - 2f(E_a^r)], \quad (1)$$

where the gap $\Delta(Q)$ in the quasiparticle spectrum is a function of the modulus Q of the total momentum of the pair of particles, the l summation is over the coupled partial waves, $V(\mathbf{k}, \mathbf{k}')$ is the neutron-proton interaction potential, $f(x) = 1/[\exp(x/T) + 1]$, $E_r^a = [E_S^2 + \Delta^2]^{1/2} + r\delta\mu + aE_A$, and $a, r \in \{+, -\}$, with $E_S = (Q^2/4 + k^2)/2m^* - \bar{\mu}$, $E_A = \mathbf{k} \cdot \mathbf{Q}/2m^*$, and $\Delta^2 = \sum_l \Delta_l^2$. Note that the gap in the SD state is angle averaged in the denominator of the gap equation; this approximation can be avoided [15, ?]. Here $\bar{\mu}$ is the average of the chemical potentials μ_n and μ_p of neutrons and protons introduced above, and m^* is the effective mass of the nucleons. (Note that we do not distinguish between the effective masses of neutrons and protons.)

The resulting phase diagram of dilute nuclear matter is shown in Fig 1 for several values of isospin asymmetry $\alpha = (n_n - n_p)/(n_n + n_p)$, where n_n and n_p are the number densities of neutrons and protons. Four different phases of matter are present. (a) The unpaired normal phase is always the ground state at sufficiently high temperatures $T > T_{c0}$, where $T_{c0}(\rho)$ is the critical temperature of the normal/superfluid phase transition at $\alpha = 0$. (b) The LOFF phase is the ground state in a narrow temperature-density strip at low temperatures and high densities. (c) The domains of phase separation (PS) appear at low temperatures and low densities, while the isospin-asymmetric BCS phase is the ground state at intermediate temperatures for densities above the crossover to a BEC phase and below the density where the condensate vanishes in the isospin symmetrical limit.

In the extreme low-density and strong-coupling regime, the BCS superfluid phases have two counterparts: the BCS phase evolves into the BEC phase of deuterons, whereas the PS-BCS phase evolves into the PS-BEC phase, in which the superfluid fraction of matter is a BEC of deuterons. The superfluid/unpaired phase transitions and the phase transitions between the superfluid phases are of second order (thin solid lines in Fig. 1), with the exception of the PS-BCS to LOFF transition, which is of first order (thick solid lines in Fig. 1). The BCS-BEC transition and the PS-BCS to PS-BEC transition are smooth crossovers. At non-zero isospin asymmetry the phase diagram features two tri-critical points where the simpler pairwise phase coexistence terminates and three different phases coexist.

Consistent with the earlier studies of the BCS-BEC crossover, one observes in the phase diagram of Fig. 1 a smooth crossover to an asymptotic state corresponding to a mixture of a Bose condensate of deuterons and a gas of excess neutrons. This however occurs at moderate temperatures, where the unconventional phases do not appear. The new ingredient of the nuclear phase diagram is the crossover seen at very low temperatures, where the heterogeneous superfluid phase is replaced by a heterogeneous mixture of a phase containing a deuteron condensate and a phase containing neutron-rich unpaired nuclear matter.

The transition to the BEC regime of strongly-coupled neutron-proton pairs, which are asymptotically identical with deuterons, occurs at low densities. The criterion for the transition from BCS to BEC is that either the average chemical potential $\bar{\mu}$ changes its sign from positive to negative values, or the coherence length ξ of a Cooper pair becomes comparable to the interparticle distance, i.e., ξ becomes of order $d \sim \rho^{-1/3}$ as it ranges from $\xi \gg d$ to $\xi \ll d$. The coherence length is related to the wave function of the condensate $\Psi(\mathbf{r})$ by

$$\xi = \sqrt{\langle r^2 \rangle}, \quad \langle r^2 \rangle = \int d^3r r^2 |\Psi(r)|^2, \quad (2)$$

where the wave-function is defined in terms of the kernel of the gap equation according to

$$\Psi(\mathbf{r}) = \sqrt{N} \int \frac{d^3p}{(2\pi)^3} [K(\mathbf{p}, \Delta) - K(\mathbf{p}, 0)] e^{i\mathbf{p} \cdot \mathbf{r}}, \quad \int d^3r |\Psi(\mathbf{r})|^2 = N^{-1}. \quad (3)$$

Thus the change in the coherence length is related to the change of the condensate wave-function across the BCS-BEC crossover. In the case of neutron-proton pairing, the criteria for the BCS-BEC transition are fulfilled, i.e., $\bar{\mu}$ changes sign and the mean distance between the pairs becomes larger than the coherence length of the superfluid.

In Fig. 2 we display the integrand $r^2 |\Psi(r)|^2$ of Eq. (2) as a function of radial distance for BCS and PS states at various densities. In its construction, the PS state corresponds to the symmetrical BCS state insofar as we are concerned with the nature of the Cooper pairs; hence there are no effects from the asymmetry in this case; the asymmetry affects the PS condensate indirectly via the number of particles available for pairing. In the weak-coupling (high-density-limit) case, the integrand function has an oscillatory form which extends over

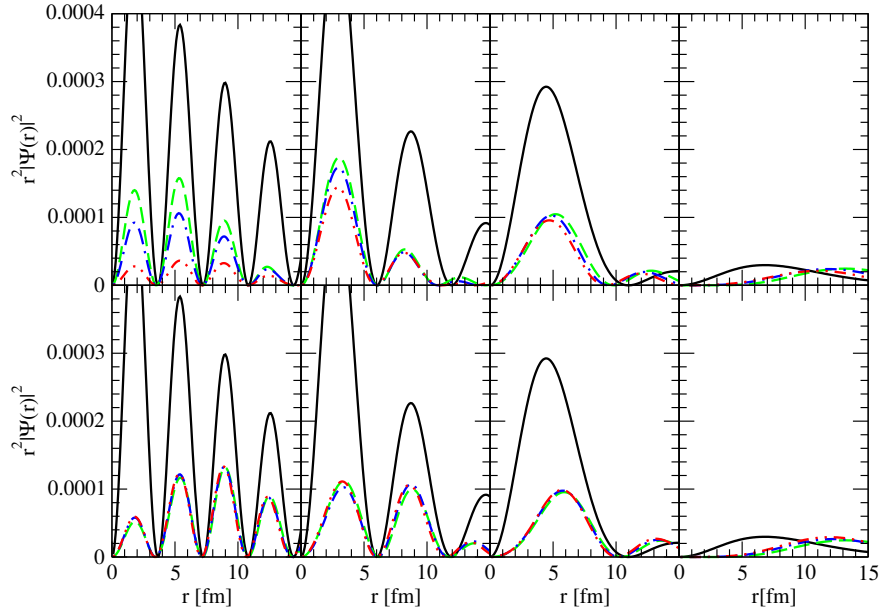


Figure 2. The quantity $r^2|\Psi(r)|^2$ as a function of radial coordinate r for $T = 0.5$ MeV and densities (from left to right) $\log_{10} \rho/\rho_0 = -0.5, -1., -1.5, -2.5$ plotted for the asymmetrical BCS state (upper panel) and the PS phase (lower panel). The values of asymmetry are $\alpha = 0.0$ (solid lines), 0.1 (dashed lines), 0.2 (dash-dotted lines), and 0.3 (dash-double-dotted lines).

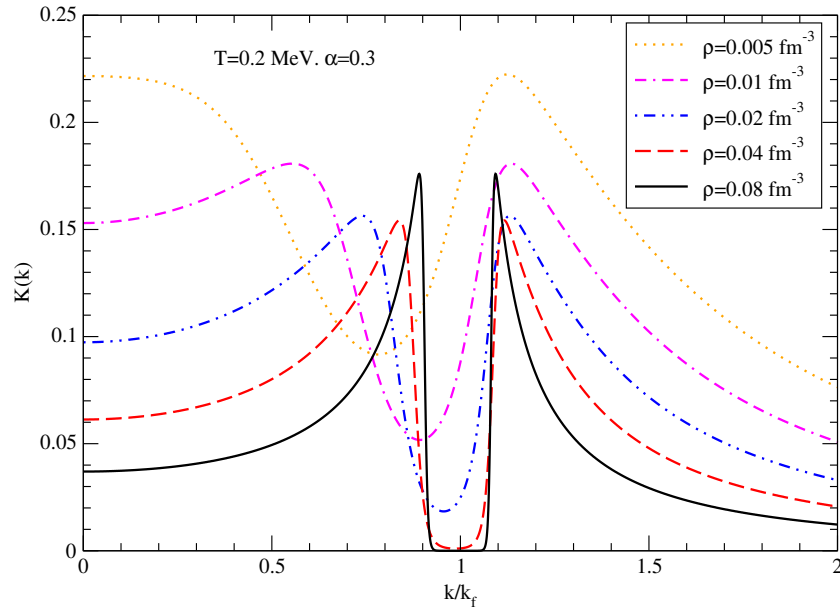


Figure 3. Kernel $K(k)$ of the gap equation (1) as a function of momentum, in units of the relevant Fermi momentum k_F for fixed $\alpha = 0.3$ and $T = 0.2$ MeV at various densities.

many periods of the interparticle distance. This behavior reflects the coherence associated with the condensate, in which the spatial correlations are characterized by scales much larger than the interparticle distance. For smaller densities (larger couplings) the integrand of $\langle r^2 \rangle$ is increasingly concentrated at the origin, with at most a few periods of oscillation. At low density (strong coupling) the pairs are well localized in space within a radius which is small compared to the interparticle distance; there is only one spike in the integrand function. This regime has BEC character, in that the pair correlations just extend over distances comparable to the interparticle distance. In the case of the BCS superfluid it is seen that isospin asymmetry affects only the amplitude of the oscillations, the period being unaffected in the weak-coupling limit and defined by the magnitude of the inverse Fermi momentum.

Further insight into the BCS-BEC crossover can be gained from examining the kernel $K(p)$ of the gap equation (1), which is defined by the sum appearing to the right of the interaction V in this equation. Physically, $K(p)$ can be interpreted as the wave function of the Cooper pairs, since it obeys a Schrödinger-type eigenvalue equation in the limit of extremely strong coupling. The prefactor of the Pauli operator $1 - 2f$ is a smooth function of momentum with a maximum at the Fermi surface, where E_S vanishes. The momentum ranges whose contributions are important in the gap equation in different regimes of the phase diagram can be identified from Fig. 3. In the BCS regime, $K(p)$ has two sharp maxima which are separated by a depression (of width $\sim \delta\mu$) around the Fermi momentum. Because of strong localization in momentum space, the Cooper pairs have an intrinsic structure which is broad in real space, implying a large coherence length. This is characteristic of the BCS regime. The picture is reversed in the strong-coupling (low-density) limit, where $K(p)$ is a broad function of momentum, corresponding to bound states (deuterons) well-localized in the real space. This is characteristic of the BEC regime. In addition, as the density decreases, the lower peak moves toward $k = 0$, due to the fact that $\bar{\mu}$ changes its sign from positive to negative at the transition from the BCS to the BEC regime. As a consequence, the prefactor of the Pauli operator $1 - 2f$ peaks at $k = 0$ in the BEC regime, rather at the Fermi surface as in the BCS regime.

3. Higher-order clustering

In supernovae the matter is at finite isospin asymmetry (but with small values compared to those of cold neutron stars) and at relatively high temperatures. These conditions could allow for a substantial presence of light clusters, notably deuterons, tritons, ^3He nuclei, and alpha particles [8, 18, 19, 20, 21, 22, 23]. The composition of nuclear matter in thermodynamic equilibrium at low density is given by the mass action law, the so-called nuclear statistical equilibrium (NSE). At given temperature T and total baryonic density ρ , the partial densities of the components c (whether nucleons or nuclei) are determined by a phase-space sum (integral) over their Fermi or Bose distribution functions as

$$n_c = g_c \sum_{\mathbf{P}} \left\{ \exp \left[\left(\frac{P^2}{2A_c m} + E_c - Z_c \mu_p - N_c \mu_n \right) / T \right] \pm 1 \right\}^{-1}, \quad (4)$$

where \mathbf{P} is the center-of-mass momentum, Z_c and N_c are the proton and neutron numbers of component c , $A_c = N_c + Z_c$ is its mass number, and g_c is its degeneracy factor (accounting for spin and excitations). A number of studies [8, 18, 19, 20, 21, 22, 23] clearly demonstrate the importance of two- and four-body correlations. The NSE approach is valid at low-densities, where the free-space binding energies of bound states are not affected by the nuclear environment. As the baryon density increases, the energies of bound states are modified due to the strong and electromagnetic interactions. These modifications can be incorporated by solving an “effective”

Schrödinger equations for an A -particle cluster (see [24] and references therein)

$$\left[\sum_{i=1}^A E_i^{\text{mf}} - E \right] \psi(1 \dots A) + \sum_{i < j}^A [1 - f(i) - f(j)] \sum_{1' \dots A'} V_{ij}(1 \dots A, 1' \dots A') \psi(1' \dots A') = 0, \quad (5)$$

where the indices i, j label nucleons, $\psi(1 \dots A)$ is a bound-state wave function, E is its corresponding energy eigenvalue, V_{ij} is the pairwise interaction potential, and $f(i)$ is the distribution function of nucleon i . The mean-field corrections to the quasiparticle energies are included in E_i^{mf} and can be evaluated from the Skyrme or the relativistic mean-field functionals. The modifications of energies of higher-order bound states depend on the center-of-mass momentum \mathbf{P} as well as the thermodynamic parameters of the medium. These have been computed in detail recently in Ref. [24], where fitting formulas are also given. The main physical effect of these modifications is to suppress the abundances of clusters upon asymptotic approach to the nuclear saturation density.

To demonstrate the utility of the “effective” Schrödinger equation (5), we consider the example of a four-nucleon (α -like) cluster and assume for simplicity that both T and α are zero. Bound states can appear below the continuum of scattering states, whose edge is given simply by the sum of the energies E_F of two neutrons and two protons at the tops of their respective Fermi seas. In the zero-density limit, α particles are formed with bound-state energy $E_\alpha^0 = -28.3$ MeV. The energy of the center-of-mass motion vanishes at $\mathbf{P} = 0$. At finite density of the surrounding nuclear matter, the energy of the four-nucleon bound state is shifted due to Pauli blocking, so that $E_4[T, \rho] = E_\alpha^0 + \Delta E_4^{\text{Pauli}}(T, \rho)$. The shift E_4^{Pauli} has been found from Eq. (5) within a variational approach using a Gaussian form for the wave function and a Gaussian separable interaction, adjusted to reproduce the binding energy and the r.m.s. radius of the α particle [25]. The density dependence of the shift (at vanishing T and α) is well approximated by $\Delta E_4^{\text{Pauli}}(\rho) = 4515.9\rho - 100935\rho^2 + 1202538\rho^3$, where the energy is given in units of MeV and the density in units of fm^{-3} . Using this formula, it is easy to deduce that the binding energy of an α particle crosses the continuum $4E_F$ at a density $\rho \sim 0.03 \text{ fm}^{-3}$, above which no α particle bound state can be formed. Analogous fits to in-medium binding energies for other light clusters can be found in [19]. These compare well with exact solutions of the three- and four-body quantum mechanical problems in the medium (see, for example, [3, 7, 26] and references therein). A useful feature of Eq. (4) is that it has the character of a Bose distribution function of a cluster for even A which implies that the physics of BEC of clusters (notably the critical temperature) can be deduced starting from NSE or its modifications that include density- and temperature-dependent binding energies.

4. Outlook

It is apparent from the discussion above that low-density and moderately warm nuclear matter can have a rather complicated phase diagram at finite, but small, isospin asymmetries. It is also apparent that higher-order clusters need to be included in the phase diagram to make the discussion complete. Furthermore, the α particles will form a Bose-Einstein condensate at sufficiently low temperatures (see [27, 28, 29] and references therein). The extent to which these clusters modify the structure of the phase diagram developed here remains to be studied. The diverse microscopic aspects of superfluid, asymmetrical nuclear matter portend diverse ramifications for the astrophysics of supernovae and hot compact stars.

Acknowledgments

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