The Lipkin-Meshkov-Glick model from the perspective of the $SU(1,1)$ Richardson-Gaudin models

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Abstract. Originally introduced in nuclear physics as a numerical laboratory to test different many-body approximation methods, the Lipkin-Meshkov-Glick (LMG) model has received much attention as a simple enough but non-trivial model with many interesting features for areas of physics beyond the nuclear one. In this contribution we look at the LMG model as a particular example of an $SU(1,1)$ Richardson-Gaudin model. The characteristics of the model are analyzed in terms of the behavior of the spectral-parameters or pairons which determine both eigenvalues and eigenfunctions of the model Hamiltonian. The problem of finding these pairons is mathematically equivalent to obtain the equilibrium positions of a set of electric charges moving in a two dimensional space. The electrostatic problems for the different regions of the model parameter space are discussed and linked to the different energy density of states already identified in the LMG spectrum.

1. Introduction

The Lipkin-Meshkov-Glick (LMG) model introduced in the nuclear physics community in a series of three papers in the 60's [1, 2, 3], was originally intended to test different many-body approximation methods and it can be thought as a very schematic model for closed shell nuclei. The model Hamiltonian can be written in terms of the elements of an $SU(2)$ pseudo-spin algebra, and consequently it is an one-quantum-degree of freedom integrable model with the Hamiltonian playing the role of the unique integral of motion. However, simple does not imply trivial and since its introduction different aspects of the model have been studied to gain insights into theoretical concepts such as quantum phase transitions (QPT) [4, 5] and their relations with quantum entanglement properties [6, 7], as well as excited state quantum phase transitions (ESQPT) [8] and quantum decoherence [9]. Moreover, the LMG model has found applications in many other areas of physics like quantum spin systems [10], ion traps [11], Bose-Einstein condensates in double wells [12] or in cavities [13], and in circuit QED [14].

In this contribution we look at the LMG model from the perspective of the Richardson-Gaudin (RG) models for the particular case of a non-compact $SU(1,1)$ algebra. The RG models are Bethe ansatz solvable models which are constructed from a set of L copies of a given Lie algebra. With this set of copies one can construct L independent, quadratic an mutually commuting operators [15]

$$\mathbf{R}_m = \boldsymbol{\xi}_m + \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (1)$$

$$\sum_{n(n\neq m)}^{L} \left(\sum_{\alpha>0} \frac{|\alpha|^2}{2} \left(X(z_n - z_m)\mathbf{E}_m^{\alpha}\mathbf{E}_n^{-\alpha} + X^*(z_n - z_m)\mathbf{E}_m^{-\alpha}\mathbf{E}_n^{\alpha} \right) + Z(z_n - z_m)\sum_{i=1}^{r}\mathbf{H}_m^i\mathbf{H}_n^i \right),$$

where $\mathbf{H}_m^i$, $\mathbf{E}_m^{\alpha}$, and $\mathbf{E}_m^{-\alpha}$ are the Cartan-Weyl weight, raising and lowering operators of the m-th copy of the given Lie algebra, α are its roots, r its rank, and $\boldsymbol{\xi}_m$ is a linear combination of the weight operators $\boldsymbol{\xi}_m = \sum_{i=1}^{r} k^i \mathbf{H}_m^i$. To ensure the commutativity of the operators, the functions $X(u)$ and $Z(u)$ can take three different forms defining, respectively, the rational, trigonometric and hyperbolic families of the RG models: $X(u) = Z(u) = 1/u$; $X(u) = \csc u, Z(u) = \cot u$; and $X(u) = \operatorname{csch}(u)$, $Z(u) = \coth(u)$. The operators depend on $L + r$ free parameters (z_m and k^i), additionally any function of the $\mathbf{R}_m$ operators defines an integrable model. The number of free parameters allows to link the RG model with a great number of different physical models [16, 17]. The eigenfunctions and eigenvalues of the $\mathbf{R}_m$ can be obtained in terms of a set of spectral parameter or pairons which solve a set of non-linear equations, the so-called Richardson equations. Every solution set of these equations defines a common eigenstate of the operators $\mathbf{R}_m$. It is well known [18] that the set of non-linear Richardson equations is mathematically equivalent to find the equilibrium positions, in a two dimensional plane, of a set of electrical charges (the pairons) in presence of a fixed external electric field or fixed charges (whose positions and values are given by the free parameters z_n and k^i).

In reference [17], it was shown that the LMG model can be obtained as a RG model for the particular case of two copies of an $SU(1,1)$ algebra. In reference [19] this relation was exploited to explore the LMG from the numerical solutions obtained in the RG models. In this contribution some aspects of this relation are discussed, in particular the LMG parameter space is classified in terms of the different families (rational, trigonometric and hyperbolic) of the $SU(1,1)$ RG models. This classification explains naturally the different regimes identified in the energy density of states (EDoS) found in the LMG spectrum [20]. This contribution is organized as follows, in section 2 the explicit relation between the LMG and the $SU(1,1)$ RG models is established. In section 3 the different regimes in the parameter space for the EDoSs of the LMG model are presented and they are linked to the different electrostatic problems associated to the families of the RG models. Conclusions are given in the last section.

2. LMG and $SU(1,1)$ Richardson-Gaudin models

The original physical realization of the LMG model consists of N fermions moving in two N-fold degenerated levels separated by an energy ϵ. In terms of fermionic operators, $\mathbf{a}_{p\sigma}^{\dagger}$ with $p = 1, .., N$ and $\sigma = \pm$, the LMG Hamiltonian reads

$$\mathbf{H}_L = \frac{1}{2}\epsilon \sum_{p\sigma} \sigma \mathbf{a}_{p\sigma}^{\dagger}\mathbf{a}_{p\sigma} + \frac{\lambda}{2}\sum_{pp'\sigma} \mathbf{a}_{p\sigma}^{\dagger}\mathbf{a}_{p'\sigma}^{\dagger}\mathbf{a}_{p'-\sigma}\mathbf{a}_{p-\sigma} + \frac{\gamma}{2}\sum_{pp'\sigma} \mathbf{a}_{p\sigma}^{\dagger}\mathbf{a}_{p'-\sigma}^{\dagger}\mathbf{a}_{p'\sigma}\mathbf{a}_{p-\sigma}.$$

The problem is strongly simplified by noting that the LMG Hamiltonian can be written in terms of $SU(2)$ pseudo-spin operators ($\mathbf{S}_+ = \sum_p \mathbf{a}_{p+}^{\dagger}\mathbf{a}_{p-}$, $\mathbf{S}_- = \sum_p \mathbf{a}_{p-}^{\dagger}\mathbf{a}_{p+}$ and $\mathbf{S}_z = \frac{1}{2}\sum_{p\sigma}\sigma\mathbf{a}_{p\sigma}^{\dagger}\mathbf{a}_{p\sigma}$) in the following form

$$\mathbf{H}_L = \epsilon\mathbf{S}_z + \frac{\lambda}{2}\left(\mathbf{S}_+^2 + \mathbf{S}_-^2\right) + \frac{\gamma}{2}\left(\mathbf{S}_+\mathbf{S}_- + \mathbf{S}_-\mathbf{S}_+\right). \qquad (2)$$

From the previous expression it is clear that the invariant subspaces of the LMG Hamiltonian can be classified by the eigenvalues $[j(j + 1)]$ of the Casimir operator S^2. For a given pseudo-spin j, two extra invariant subspaces exist which can be labeled by the eigenvalues ($p = \pm 1$)

of the parity operator $\mathbf{P} = e^{i\pi(\mathbf{S}_z+j)}$. The ground state of the model belongs to the subspace with maximal pseudo-spin $j = N/2$ and positive parity. The connection between the LMG Hamiltonian and the $SU(1,1)$ RG models, can be obtained [17] by using the Schwinger boson representation of the $SU(2)$ algebra: $\mathbf{S}_z = \frac{\mathbf{b}_2^\dagger\mathbf{b}_2 - \mathbf{b}_1^\dagger\mathbf{b}_1}{2}$, $\mathbf{S}_+ = \mathbf{b}_2^\dagger\mathbf{b}_1$, and $\mathbf{S}_- = \mathbf{b}_1^\dagger\mathbf{b}_2$, where $\mathbf{b}_i$ are boson operators. With these boson operators we construct two copies ($i = 1, 2$) of the rank-1 $SU(1,1)$ algebra

$$\mathbf{K}_i^+ = \frac{1}{2}\mathbf{b}_i^\dagger\mathbf{b}_i^\dagger, \quad \mathbf{K}_i^- = \frac{1}{2}\mathbf{b}_i\mathbf{b}_i, \quad \mathbf{K}_i^z = \frac{1}{2}\left(\mathbf{b}_i^\dagger\mathbf{b}_i + \frac{1}{2}\right), \tag{3}$$

where the ladder $(\mathbf{K}^+, \mathbf{K}^-)$ and weight $(\mathbf{K}^z)$ operators satisfy the commutation relations $[\mathbf{K}^z, \mathbf{K}^\pm] = \pm\mathbf{K}^\pm$ and $[\mathbf{K}^+, \mathbf{K}^-] = -2\mathbf{K}^z$. The only root of the algebra is $|\alpha|^2 = 2$ and the Cartan Weyl basis is given by $\mathbf{H}_1 = \sqrt{2}\mathbf{K}^z$, $\mathbf{E}^\alpha = \mathbf{K}^+$ and $\mathbf{E}^{-\alpha} = -\mathbf{K}^-$. With these two copies and choosing $k^1 = 1/(\sqrt{2}g)$ in Eq.(1), we construct two integrals of motion

$$g\mathbf{R}_i = \mathbf{K}_i^z + g\sum_{j\neq i}\left[-X_{ij}\left(\mathbf{K}_i^+\mathbf{K}_j^- + \mathbf{K}_i^-\mathbf{K}_j^+\right) + 2Z_{ij}\mathbf{K}_i^z\mathbf{K}_j^z\right], \tag{4}$$

where $i = 1, 2$, and we have reparametrized [17] the functions $X(z_i - z_j) = X_{ij}$ and $Z(z_i - z_j) = Z_{ij}$ in the way $X_{21} = -X_{12} = \frac{1+st^2}{2t}$ and $Z_{21} = -Z_{12} = \frac{1-st^2}{2t}$, with $s = -1, 0, 1$ corresponding, respectively, to the hyperbolic, rational and trigonometric versions of the $SU(1,1)$ RG models. The parameters t and g are completely free parameters. By taking the difference between the previous integrals of motion, we obtain the LMG Hamiltonian

$$\mathbf{H}_L = \epsilon(g\mathbf{R}_2 - g\mathbf{R}_1) - \frac{\gamma}{4}, \qquad \text{with} \qquad gX_{21} = -\frac{\lambda}{\epsilon}, \quad \text{and} \ \ gZ_{21} = \frac{\gamma}{\epsilon}. \tag{5}$$

The relation between the LMG Hamiltonian parameters $(\lambda, \gamma, \epsilon)$ and those of the $SU(1,1)$ RG models are $\frac{\lambda}{\epsilon} = -g\frac{1+st^2}{2t}$ and $\frac{\gamma}{\epsilon} = g\frac{1-st^2}{2t}$. In order to obtain a simpler parameter space, we introduce the rescaled parameters [4] $(\gamma_x, \gamma_y) \equiv \frac{2j-1}{\epsilon}(\gamma + \lambda, \gamma - \lambda) = (2j-1)g\left(-st, \frac{1}{t}\right)$. This relation allows to classify the quadrants of the LMG parameter space of Fig.1 in terms of the hyperbolic ($s = -1$) and trigonometric ($s = 1$) RG models. The first ($s = -1$, $g > 0$) and third ($s = -1$, $g < 0$) quadrants correspond to the hyperbolic RG model, whereas the second ($s = 1$, $g > 0$) and fourth ($s = 1$, $g < 0$) are associated with the trigonometric model. The eigenvalues of the LMG model in terms of the pairons (e_α) of the RG model, for pseudo-spin j and positive parity, are given by $E_L = gt\epsilon\sum_\alpha^j \frac{1+se_\alpha^2}{t^2-e_\alpha^2}$. The unnormalized eigenvectors common to the two integrals of motion ($\mathbf{R}_i$) and, consequently, to the LMG Hamiltonian are

$$\prod_{\alpha=1}^j\left(\frac{\mathbf{b}_1^\dagger\mathbf{b}_1^\dagger}{e_\alpha+t} + \frac{\mathbf{b}_2^\dagger\mathbf{b}_2^\dagger}{e_\alpha-t}\right)|0\rangle, \tag{6}$$

with $|0\rangle$ the boson vacuum. The equations which determine the pairons and their mapping to a two-dimensional electrostatic problem for the different version of the RG models are discussed in the next section. Likewise the electrostatic systems are linked to the different regimes for the EDoS already found in [20] for the LMG energy spectrum.

3. Energy Density for the LMG states and the RG electrostatic mapping
In figure 1 the parameter space of the LMG model is classified according to the different EDoSs that can be found in the LMG spectrum. Typical profiles of these densities are shown for the
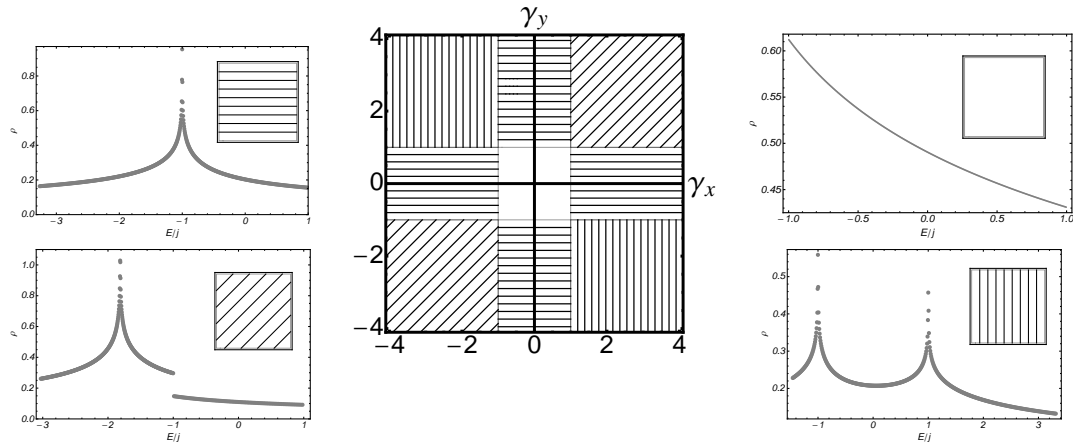
Figure 1. Central plot: parameter space of the LMG model classified according to the different structures that can be found in the Energy Density of States (horizontal, vertical, diagonal and white regions). Typical profiles of these densities (ρ) are shown in the plots surrounding the central one, the corresponding regions are indicated by the squares inside the plots. The density profiles were calculated numerically for the positive sector of finite ($j = 400$) LMG Hamiltonians. The first and third quadrants in the central parameter space correspond to the hyperbolic RG model, whereas the second and fourth are associated with the trigonometric RG family.

different regions. These densities were calculated numerically for the parity positive sector of finite ($j = 400$) LMG systems. The white region around the non-interacting ($\gamma_x = \gamma_y = 0$) case shows monotone EDoSs, with energies in the interval $E \in [-j, j]$. As we move away from the origin a transition to the regions signaled by horizontal lines takes place. These regions are characterized by a sole peak in the EDoS occurring at $E/j = -1$ in the left and bottom horizontal lined regions, and at $E/j = 1$ in the right and top horizontal lined ones. This peak signals an excited state quantum phase transition [21, 8] which transform in an ordinary (Ground-state) quantum phase transition in the left and bottom border of the white region. Away from the axis of the parameter space, two types of regions can be identified. The first type signaled by vertical lines in the second and fourth quadrants of the parameters space, is characterized by two peaks in the EDoS occurring at $E = -j$ and $E = +j$. The double peak structure of the EDoS is related to excited state quantum phase transitions which evolve, respectively, from the ground and from the most excited state. Finally, the second type of regions (signaled by diagonal lines in Figure 1) in the first and third quadrants of the parameter space, presents EDoSs with a peak and a discontinuity. This discontinuity occurs at $E = -j$ in the left bottom diagonal lined region, and at $E = j$ in the other one. The states with energies between the peak and the discontinuity present avoiding crossings, whereas the states beyond the discontinuity show a very small energy density. The qualitative differences between the EDoSs of the first-third and second-fourth quadrants found a natural explanation when we look at the equivalent electrostatic problem which determine the pairons of the RG solution. These pairons determine, in turn, the wave function and energies of the LMG Hamiltonian as it was discussed in the previous section. The equations determining the pairons can be written [19] in the following form, where it is clear their analogy with an electrostatic problem in the complex plane

$$\frac{Q_C}{e_\alpha - P_C} + \frac{Q_D}{e_\alpha - P_D} + \frac{1}{4}\left(\frac{1}{e_\alpha + t} + \frac{1}{e_\alpha - t}\right) + \sum_{\beta \neq \alpha}^{M} \frac{1}{e_\alpha - e_\beta} = 0. \tag{7}$$
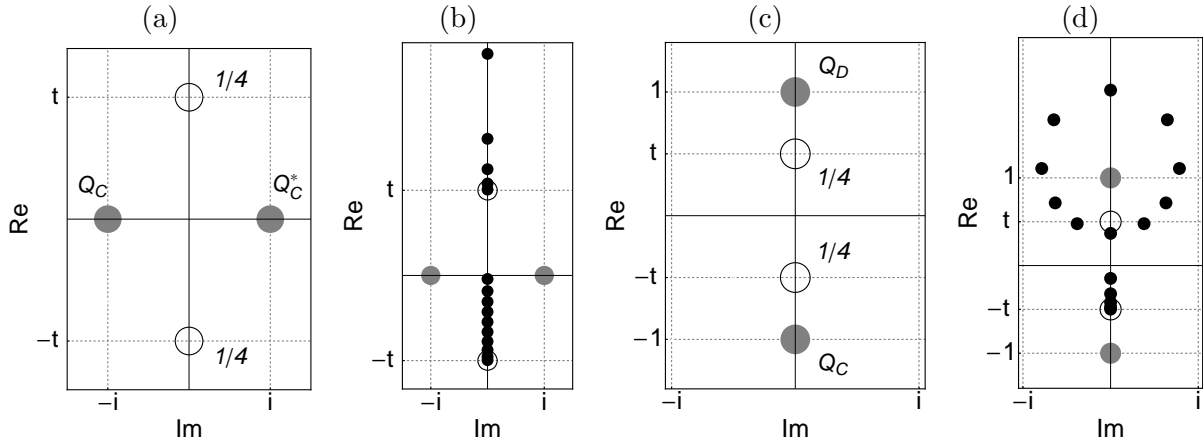
Figure 2. Fixed charges of the equivalent electrostatic problems in the complex plane for the trigonometric (panel a) and hyperbolic (panel c) RG models. The effective charges Q_C and Q_D are complex conjugated for the trigonometric case, whereas in the hyperbolic case they are real. The other two effective charges have a value 1/4 and they are located in the real axis at $\pm t$ for both hyperbolic and trigonometric cases. Two typical equilibrium configurations for two LMG excited states are shown in panel b and c for the trigonometric and hyperbolic cases respectively. A value $j = 15$ was used.

The previous equations describe the electrostatic interaction of a set of j pairons (whose positions are e_α) with positive unit charge in a two dimensional space. The first two terms in (7) describe the electrostatic interaction of the pairons with two effective charges Q_C and Q_D. For the trigonometric case ($s = 1$) the effective charges are complex $Q_C = -\frac{2j-1}{4} - \frac{i}{4g}$, $Q_D = Q_C^*$, and they are located in $P_C = i$, $P_D = -i$. Whereas in the hyperbolic case ($s = -1$) both, the charges and their positions, are real, $Q_C = -\frac{2j-1}{4} + \frac{1}{4g}$ and $Q_D = -\frac{2j-1}{4} - \frac{1}{4g}$, located in $P_C = -1$ and $P_D = 1$ respectively. The third term in (7) represents the interaction of the pairons with two charges $\frac{1}{4}$ at positions $\mp t$. Finally, the fourth term corresponds to the mutual repulsion between pairons.

The panels (a) and (c) of Fig. 2 show the position of the effective fixed charges for the trigonometric and hyperbolic RG families. Each independent solution of the Richardson equations determines the equilibrium position of the pairons in the complex plane. For the trigonometric case, the dynamics of the pairons as a function of the model parameters takes place entirely in the real axis (panel (b) of Fig.2). This result can be understood from the electrostatic mapping as a consequence of the different sign of the imaginary part of the effective charges Q_C and $Q_D = Q_C^*$. These imaginary parts produce effective electric fields whose components parallel to the imaginary axis cancel mutually only in the real axis, constraining the equilibrium positions of the pairons to this axis. Contrarily to this, for the hyperbolic case the effective charges Q_C and Q_D are real and located at the real axis at positions ± 1, moreover the mutual sign of these charges can be different depending on the values of the LMG parameters. When the mutual sign of these charges is negative, the pairons can expand in the complex plane forming arcs around the position of the effective charges Q_C or Q_D. Panel (d) of Fig.2 shows a such configuration with an arc around the Q_D charge. For particular values of the coupling constant [$g = \pm 1/(2j + 1 - 2n)$, with n a positive integer] n pairons can collapse into the position of the effective charge Q_C or Q_D. Similar collapses were found in the $SU(2)$-RG-hyperbolic model for the $p_x + ip_y$ pairing interaction [22]. In that model the collapses of the pairons were associated with a third-order QPT. In the present model the collapses are associated

with other singular behavior: the crossing of different parity states at exactly the same values of the coupling constant where collapses take place. The qualitative differences between the electrostatic systems for the trigonometric (second and third quadrants) and hyperbolic (first and third quadrants) RG models, explain qualitatively the different EDoSs found in the LMG parameter space of Fig. 1.

4. Conclusions

The Lipkin-Meshkov-Glick model was studied from the perspective of a particular realization of the Richardson-Gaudin models, associated with two bosonic copies of an $SU(1,1)$ algebra. In the parameter space of the LMG model different profiles of the energy density of states can be identified. The different profiles were related with the kind of electrostatic problem that determines the pairons or spectral parameters of the RG models. The EDoSs with a peak and a discontinuity are associated with the hyperbolic RG models, for which the pairon dynamics as a function of the model parameters takes place in the whole complex plane. Contrarily, for the trigonometric RG family, where the electrostatic pairon positions are restricted to the real axis, the EDoSs present a structure with two peaks at energies $E = -j$ and $E = j$. As discussed in Ref.[23], the pairon dynamics can help to identify significant physical phenomena. The LMG model is not an exception, for the hyperbolic case the pairon dynamics allows the collapse of pairons into the position of an effective charge. This singular behavior is related with the crossing between different parity states and the avoiding crossing between states of the same parity. Finally, the insights gained in the study of the LMG model, where the set of pairons of every state in the spectrum is easily accessible, can help to identify general mechanisms related with ubiquitous phenomena such as excited state quantum phase transitions.

Acknowledgments

References

[1] Lipkin H J, Meshkov N and Glick A J 1965 *Nucl. Phys.* **62** 188
[2] Lipkin H J, Meshkov N and Glick A J 1965 *Nucl. Phys.* **62** 199
[3] Lipkin H J, Meshkov N and Glick A J 1965 *Nucl. Phys.* **62** 211
[4] Castaños O, López-Peña R, Hirsch J and López-Moreno E 2006 *Phys. Rev.* B **74** 104118
[5] Vidal J, Arias J M, Dukelsky J and García-Ramos J E 2006 *Phys. Rev.* C **73** 054305
[6] Vidal J, Palacios G and Mosseri R 2004 *Phys. Rev.* A **69** 022107
[7] Vidal J, Dukelsky J and Mosseri R 2004 *Phys. Rev.* A **69** 054101
[8] Heiss W D, Scholtz F G and Geyer H B 2005 *J. Phys.* A **38** 1843
[9] Relaño A, Arias J M, Dukelsky J, García-Ramos J E and Pérez-Fernández P 2008 *Phys. Rev.* A **78** 060102
[10] Botet R, Julien R and Pfeuty P 1982 *Phys. Rev. Lett.* **49** 478
[11] Unanyan R G and Fleischhauer M 2003 *Phys. Rev. Lett.* **90** 133601
[12] Links J, Zhou H Q, McKenzie R H and Gould M D 2003 *J. Phys.* A **36** R63
[13] Chen G, Liang J Q and Jia S 2009 *Opt. Express* **17** 19682
[14] Larson J 2010 *Europhys. Lett.* **90** 54001
[15] Lerma H. S and Errea B 2007 *J. Phys.* A **40** 4125
[16] Dukelsky J, Pittel S and Sierra G 2004 *Rev. Mod. Phys.* **76** 643
[17] Ortiz G, Somma R, Dukelsky J and Rombouts S 2005 *Nucl. Phys.* B **707** 421
[18] Dukelsky J, Esebbag C and Pittel S 2002 *Phys. Rev. Lett.* **88** 062501
[19] Lerma H. S and Dukelsky J 2013 *Nucl. Phys.* B **870** 421
[20] Ribeiro P, Vidal J, Mosseri R 2007 *Phys. Rev. Lett.* **99** 050402
[21] Perez-Fernández P, Relaño A, Arias J M, Cejnar P, Dukelsky J and García-Ramos J E 2011 *Phys. Rev.* E **83** 046208
[22] Ibañez M, Links J, Sierra G and Zhao S-Y, 2009 *Phys. Rev.* B **79** 180501
[23] Rubeni D, Foerster A, Mattei E and Roditi I 2012 *Nucl. Phys.* B **856** 698