

Towards Noncommutative Topological Quantum Field Theory - Hodge theory for cyclic cohomology

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Abstract. Some years ago we initiated a program to define Noncommutative Topological Quantum Field Theory (see [1]). The motivation came both from physics and mathematics: On the one hand, as far as physics is concerned, following the well-known holography principle of 't Hooft (which in turn appears essentially as a generalisation of the Hawking formula for black hole entropy), quantum gravity should be a topological quantum field theory. On the other hand as far as mathematics is concerned, the motivation came from the idea to replace the moduli space of flat connections with the Gabai moduli space of codim-1 taut foliations for 3 dim manifolds. In most cases the later is finite and much better behaved and one might use it to define some version of Donaldson-Floer homology which, hopefully, would be easier to compute. The use of foliations brings noncommutative geometry techniques immediately into the game. The basic tools are two: Cyclic cohomology of the corresponding foliation C^* -algebra and the so called "*tangential cohomology*" of the foliation. A necessary step towards this goal is to develop some sort of Hodge theory both for cyclic (and Hochschild) cohomology and for tangential cohomology. Here we present a method to develop a Hodge theory for cyclic and Hochschild cohomology for the corresponding C^* -algebra of a foliation.

Introduction

There have been numerous indications that (pseudo) Riemannian geometry is inadequate to describe space-time at the most fundamental level. These indications come both from 4-dim physical models but also from higher dimensional physical theories: For example in 4 dimensions, Connes et al. [2] exhibited that the full standard model Lagrangian can be geometrically interpreted as the fundamental K-Homology class of a noncommutative manifold arising as the discrete product of a spin 4-dim Riemannian manifold with a discrete space of metric dimension 0 and KO-dimension 6 mod 8. Concerning higher dimensional theories we know that noncommutative spaces (in particular noncommutative tori) appear as admissible compactifications of type IIB superstrings, in non linear sigma models with target space some matrix algebra (matrix models, see [3]) but also in type I superstring theories on D-branes in the presence of an external B-field (see [4]). In addition, holography principle dictates that quantum gravity should be a topological quantum field theory (see [5] and [6]). Recent results coming from the GEO 600 experiment strongly suggest that holography might after all be a valid physical principle (see [7]). Combining all the above evidence it is reasonable

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to look for noncommutative topological quantum field theories as the correct framework for a unified theory of all four known physical interactions (including gravity).

In addition to the motivation coming from physics, there is a good motivation coming from mathematics as well, more specifically from low dimensional geometric topology: Gabai proved that the moduli space of taut codim-1 foliations (modulo coarse isotopy) on practically any closed 3-manifold is a finite set; hence one is tempted to try to construct a noncommutative version of Floer homology using the Gabai moduli space along with a noncommutative topological invariant for foliations defined by the author (see [8]) to decorate or to label each element of the Gabai moduli space instead of the original moduli space of flat connections (modulo gauge invariance) used in the well-known Floer homology (Donaldson-Floer topological quantum field theory). The noncommutative topological invariant in [8] was defined using the Conne-Quillen pairing between K-Theory and cyclic cohomology.

Learning the lesson that the corresponding classical theories teach us, if such an attempt ever succeeds eventually, one needs an important intermediate step: An analogue of Hodge theory for cyclic cohomology and for tangential cohomology which are the basic cohomology theories used in the study of foliations. Here we present a Hodge theory for cyclic cohomology. The construction of the noncommutative bosonic propagator follows automatically since it is the inverse of the noncommutative Laplacian (see below).

Let us briefly recall that the "classical" Hodge Theorem states that on every smooth, compact, Riemannian manifold (also assumed oriented), each de Rham cohomology class has a unique harmonic representative (namely the Laplace operator vanishes). As a consequence, every closed form can be written as the sum of an exact form plus a harmonic form and moreover every form can be written as the sum of a harmonic form plus an exact form plus a coexact form.

Let A be a complex, unital associative algebra (see [9] and [10]) and let

$$\Omega^n A := A \otimes_C \tilde{A}^{\otimes n}$$

for $n > 0$, where C denotes the complex field, $\otimes$ denotes tensor product and the superscript $\otimes n$ denotes the n th tensor power, $\tilde{A} = A/C$ and whereas $\Omega^n A = 0$ for $n < 0$ and $\Omega^0 A = A$. Hence we get an identification

$$a_0 da_1 \dots da_n \sqsubset (a_0, a_1, \dots, a_n).$$

Then we also define

$$\Omega A = \sum_n \Omega^n A$$

which is the graded algebra (GA) of noncommutative differential forms over A , the multiplication being defined via

$$(a_0, a_1, \dots, a_n)(a_{n+1}, a_{n+2}, \dots, a_k) = \sum_{i=0}^k (-1)^{k-i} (a_0, \dots, a_i a_{i+1}, \dots, a_k)$$

for $k > n$. Moreover we define the differential $d: \Omega^n A \rightarrow \Omega^{n+1} A$ as follows:

$$d(a_0 da_1 \dots da_n) = da_0 da_1 \dots da_n$$

or in an equivalent notation

$$d(a_0, a_1, \dots, a_n) = (1, a_0, a_1, \dots, a_n)$$

and hence

$$d\Omega^n A \approx \tilde{A}^{\times(n+1)}$$

for $n \geq 0$. Thus $(\Omega A, d)$ becomes a DGA.

On ΩA , we can also define the Hochschild differential

$$b: \Omega^n A \rightarrow \Omega^{n-1} A$$

given by

$$b(a_0, a_1, \dots, a_n) = \sum_{j=0}^{n-1} (-1)^j (a_0, a_1, \dots, a_j a_{j+1}, \dots, a_n) + (-1)^n (a_n a_1, a_2, \dots, a_{n-1}).$$

Thus one has that

$$b(\omega da) = (-1)^{|\omega|} (\omega a - a \omega) = (-1)^{|\omega|} [\omega, a]$$

and

$$b(a) = 0,$$

where $|\omega|$ denotes the degree of the differential form ω .

One also has the Karoubi operator $k: \Omega^n A \rightarrow \Omega^n A$ which is a degree zero operator on ΩA given by

$$k(\omega da) = (-1)^{|\omega|} (da)\omega$$

(for negative degrees it is given by the identity).

The main result is the following:

Proposition: On ΩA one has the harmonic decomposition

$$\Omega A = \text{Ker}(1 - k)^2 + \text{Im}(1 - k)^2,$$

where the generalised nullspace $\text{Ker}(1 - k)^2$ is analogous to the space of harmonic forms.

Proof (of Proposition): For the proof of this result we need two technical Lemmas:

Lemma 1. The differential d , the Hochschild differential b and the Karoubi operator k are related as follows

$$bd + db = 1 - k.$$

Proof of Lemma 1 (Sketch): This can be proved by direct (yet tedious!) computation.

The above Lemma implies that b and d can be seen as formal adjoints to each other and hence the operator $1 - k$ can be called the *noncommutative Laplacian in cyclic homology*.

Lemma 2. The Karoubi operator k on $\Omega^n A$ satisfies the polynomial relation

$$(k^n - 1)(k^{n+1} - 1) = 0.$$

Proof of Lemma 2 (Basic Idea): This is a technical result and the method to prove it is to use the cyclicity condition to obtain recursive relations between powers of the Karoubi operator k with the differentials b and d . Then by using the definition of the noncommutative Laplacian we get the desired result.

We return to the proof of the Proposition : Since an operator satisfies a polynomial equation, it gives rise to a direct sum decomposition into generalised eigenspaces corresponding to the distinct roots of the polynomial. The roots of $(k^n - 1)(k^{n+1} - 1)$ are the n different n -th roots of unity and the $(n+1)$ different roots of unity of order dividing $(n+1)$. Yet n and $(n+1)$ are relatively prime which means that these two sets of roots have only $k = 1$ in common. Hence 1 is a double root and all other roots are simple.

Consequently $\Omega^n A$ decomposes into the direct sum of the generalised eigenspace $\text{Ker } (1 - k)^2$ corresponding to the eigenvalue $z = 1$ and the ordinary eigenspaces $\text{Ker}(k - z)$ for each root of unity $z \neq 1$ of order dividing n or $(n+1)$.

Combining the above for all n and lumping the eigenvalues $z \neq 1$ together we get the desired decomposition which completes the proof.

References

- [1] Zois I P Noncommutative Topological Quantum Field Theory-Noncommutative Floer Homology, arXiv: hep-th/0510005.
- [2] Chamsedine A H, Connes A and Marcolli M Geometry and the Standard Model with Neutrino Mixing, Adv. Theor. Math. Phys. 11 (2007).
- [3] Connes A, Douglas M R and Schwarz A Noncommutative Geometry and Matrix Theory, Compactifications on Tori, JHEP 9802:003 (1998).
- [4] Seiberg N and Witten E String Theory and Noncommutative Geometry JHEP 9909:032 (1999).
- [5] 't Hooft G Dimensional Reduction in Quantum Gravity gr-qc/9310026.
- [6] Atiyah M F Topological Quantum Field Theories, IHES Publ. Math. 68 (1989).
- [7] Hogan C J Measurement of Quantum Fluctuations in Geometry, Phys. Rev. D 77 (2008).
- [8] Zois I P A new invariant for σ models, Commun. Math. Phys. 209.3 (2004).
- [9] Connes A Noncommutative Geometry, Academic Press (1994).

[10] Quillen D G Chern Simons Forms in Cyclic Cohomology, in The Interface of Mathematics and Particle Physics, editors D G Quillen, G B Segal & S T Tsou, Oxford University Press (1998).