

Minimal subtraction formulation of massless fields for generic competing systems

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Abstract. *We introduce the method of minimal subtraction in the computation of critical exponents of Lifshitz type generic competing systems using massless fields. We first treat the anisotropic cases, when several independent momentum scales define the renormalization group invariance of the scalar fields. In addition, we analyze the isotropic sector. We compute critical exponents using diagrammatic techniques at least up to two-loop level and show their equivalence with other methods presented in the literature.*

1. Introduction

The employment of field theory along with renormalization group arguments are the most appropriate methods in the determination of critical properties of Lifshitz generic competing systems. In particular, critical exponents were determined using normalization conditions either in the massless [1, 2] or in the massive setting [3]. In this work we evaluate critical exponents in the massless field formulation by introducing a minimal subtraction scheme. It provides a new renormalization method to these systems and sheds new light on the interesting mathematical features of the perturbation structure of this higher derivative scalar field theory.

The simplest physical system possessing this sort of critical behavior can be realized using the language of magnetic system. The anisotropic cases can be understood as a d -dimensional Ising model with ferromagnetic interactions between nearest-neighbor spins, with additional antiferromagnetic second neighbors couplings along a m_2 -dimensional subspace as well as ferromagnetic third neighbors exchange forces along m_3 space directions and so on, such that the alternate interactions take place up to the L -th neighbor spins inside an m_L -dimensional subspace, with all (competing) subspaces orthogonal among each other. The isotropic situation occurs whenever there is only one type of subspace, i.e., $d = m_L$. The generic competing systems are described by the following bare Lagrangian density[1]:

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} |\nabla_{(d-\sum_{n=2}^L m_n)} \phi_0|^2 + \sum_{n=2}^L \frac{\sigma_n}{2} |\nabla_{m_n}^n \phi_0|^2 + \sum_{n=2}^L \delta_{0n} \frac{1}{2} |\nabla_{m_n} \phi_0|^2 \\ & + \sum_{n=3}^{L-1} \sum_{n'=2}^{n-1} \frac{1}{2} \tau_{nn'} |\nabla_{m_n}^{n'} \phi_0|^2 + \frac{1}{2} t_0 \phi_0^2 + \frac{1}{4!} \lambda_0 \phi_0^4. \end{aligned} \quad (1)$$



From now on we focus our attention at the Lifshitz point $t_0 = 0$. Furthermore, the critical region we are interested in is characterized by the condition $\delta_{on} = \tau_{nn'} = 0$. Multiplicative renormalizability can be easily stated in terms of one-particle irreducible (1PI) vertex parts, which will be the basic objects needed in our quest to determine the critical exponents. Hereafter we shall restrict our analysis to the vertex parts which display primitive divergences, namely $\Gamma^{(2)}$, $\Gamma^{(4)}$ and the vertex part with an insertion of composite operator $\Gamma^{(2,1)}$. The interested reader should consult the book [4] in order to fix the nomenclature utilized in the remainder.

2. Anisotropic minimal subtraction

The primitively divergent vertex parts in the massless theory can be renormalized solely at nonvanishing momentum scale κ_n of the n th (m_n -dimensional) subspace. This implies a multiscale renormalization group invariance with n independent coupling constants: each vertex part is labeled by the index n specifying its dependence on the appropriate coupling. The flow in momentum space in each subspace is generated by a momentum scale κ_n ($n = 1, \dots, L$). In order to get rid of all dimensionful parameters away from the critical dimension we write the coupling constants in terms of dimensionless couplings u_n and u_{0n} . Hence, the renormalized coupling constants read $g_n = \kappa_n^{n\epsilon_L} u_n$, whereas the bare ones are given by $\lambda_n = \kappa_n^{n\epsilon_L} u_{0n}$. In order to pick out the required subspace labeled by n , we choose $u_n = \delta_{nn'} u_{n'}$ and $\kappa_n = \delta_{nn'} \kappa_{n'}$, i.e., we set to zero all coupling constants and momentum scales with $n \neq n'$ implicitly into the vertex parts on that subspace.

The renormalized vertices can be written in terms of the bare vertices and external momentum configuration as $\Gamma_{R(n)}^{(2)}(k_{(n)}, u_n, \kappa_n) = Z_{\phi(n)} \Gamma_{(n)}^{(2)}(k_{(n)}, u_{0n}, \kappa_n)$, $\Gamma_{R(n)}^{(4)}(k_{i(n)}, u_n, \kappa_n) = Z_{\phi(n)}^2 \Gamma_{(n)}^{(4)}(k_{i(n)}, u_{0n}, \kappa_n)$ and $\Gamma_{R(n)}^{(2,1)}(k_{1(n)}, k_{2(n)}, p_{(n)}, u_n, \kappa_n) = \bar{Z}_{\phi^2(n)} \Gamma_{(n)}^{(2,1)}(k_{1(n)}, k_{2(n)}, p_{(n)}, u_{0n}, \kappa_n)$.

As far as renormalization by minimal subtraction is concerned, these functions should be finite when $\epsilon_L (= 4 + \sum_{n=2}^L \frac{(n-1)}{n} m_n - d) \rightarrow 0$ at every order in the dimensionless renormalized coupling constant u_n . A technical point is that each external momentum is multiplied by κ_n^{-1} . Within this strategy, it is convenient to express the dimensionless bare coupling and the renormalization functions as an expansion in powers of u_n , namely $u_{0n} = u_n \left(1 + \sum_{i=1}^{\infty} a_{in}(\epsilon_L) u_n^i\right)$, $Z_{\phi(n)} = 1 + \sum_{i=1}^{\infty} b_{in}(\epsilon_L) u_n^i$, $\bar{Z}_{\phi^2(n)} = 1 + \sum_{i=1}^{\infty} c_{in}(\epsilon_L) u_n^i$.

The Feynman diagrams of the vertex functions are utilized in the calculation of the coefficients a_{in} , b_{in} and c_{in} . The minimal set of integrals which will be necessary in our undertaking are represented by the following expressions:

$$I_2 = \int \frac{d^{d-\sum_{n=2}^L m_n} q \Pi_{n=2}^L d^{m_n} k_{(n)}}{\left[\left(\sum_{n=2}^L (k_{(n)} + K'_{(n)})^{2n} + (q + P)^2 \right) \left(\sum_{n=2}^L k_{(n)}^{2n} + q^2 \right) \right]}, \quad (2)$$

$$I_3 = \int \frac{d^{d-\sum_{n=2}^L m_n} q_1 d^{d-\sum_{n=2}^L m_n} q_2 \Pi_{n=2}^L d^{m_n} k_{1(n)} \Pi_{n=2}^L d^{m_n} k_{2(n)}}{(q_1^2 + \sum_{n=2}^L k_{1(n)}^{2n})(q_2^2 + \sum_{n=2}^L k_{2(n)}^{2n}) \left[(q_1 + q_2 + P)^2 + \left(\sum_{n=2}^L (k_{1(n)} + k_{2(n)} + K'_{(n)})^{2n} \right) \right]}, \quad (3)$$

$$I_4 = \int \frac{d^{d-\sum_{n=2}^L m_n} q_1 d^{d-\sum_{n=2}^L m_n} q_2 \Pi_{n=2}^L d^{m_n} k_{1(n)} \Pi_{n=2}^L d^{m_n} k_{2(n)}}{\left(q_1^2 + \sum_{n=2}^L k_{1(n)}^{2n} \right) [(P - q_1)^2 + \sum_{n=2}^L (K'_{(n)} - k_{1(n)})^{2n}] (q_2^2 + \sum_{n=2}^L k_{2(n)}^{2n})} \\ \times \frac{1}{\left[(q_1 - q_2 + p_3)^2 + \sum_{n=2}^L (k_{1(n)} - k_{2(n)} + k'_{3(n)})^{2n} \right]}, \quad (4)$$

$$I_5 = \int \frac{d^{d-\sum_{n=2}^L m_n} q_1 d^{d-\sum_{n=2}^L m_n} q_2 d^{d-\sum_{n=2}^L m_n} q_3 \prod_{n=2}^L d^{m_n} k_{1(n)} \prod_{n=2}^L d^{m_n} k_{2(n)} \prod_{n=2}^L d^{m_n} k_{3(n)}}{\left(q_1^2 + \sum_{n=2}^L k_{1(n)}^2\right) \left(q_2^2 + \sum_{n=2}^L k_{2(n)}^2\right) [(q_1 + q_2 - P)^2 + \sum_{n=2}^L (k_{1(n)} + k_{2(n)} - K'_{(n)})^2]} \times \frac{1}{\left(q_3^2 + \sum_{n=2}^L k_{3(n)}^2\right) [(q_1 + q_3 - P)^2 + \sum_{n=2}^L (k_{1(n)} + k_{3(n)} - K'_{(n)})^2]}. \quad (5)$$

In the last expressions P and p_3 are external momenta perpendicular to the m_n competing subspaces ($n = 2, \dots, L$). We are interested only in the singular parts of those integrals. Note that the singular contributions of I_2 and I_4 depend on the external momenta only through the combinations $(P) = p_1 + p_2, p_1 + p_3, p_2 + p_3$ and $(K'_{(n)}) = k'_{1(n)} + k'_{2(n)}, k'_{1(n)} + k'_{3(n)}, k'_{2(n)} + k'_{3(n)}$. Therefore, we can express the poles of the integrals in terms of either the external momenta P along the noncompeting axes or as a function of $K'_{(n)}$ ($n = 2, \dots, L$).

Define $K'_{(1)} \equiv P$ as the momentum associated with the $m_1 (= d - \sum_{n=2}^L m_n)$ -dimensional noncompeting subspace. Now, we require the minimal subtraction of the poles in ϵ_L of the bare vertex parts. In the massless formulation the bare vertex parts can be written in terms of the minimum number of Feynman graphs in a very simple form. Indeed, their perturbative expansions are given by $\Gamma_{(n)}^{(2)}(k_{(n)}, u_{0n}, \kappa_n) = k_{(n)}^2 (1 - B_{2n} u_{0n}^2 + B_{3n} u_{0n}^3)$, $\Gamma_{(n)}^{(4)}(k_{i(n)}, u_{0n}, \kappa_n) = \kappa_n^{\epsilon_L} u_{0n} [1 - A_{2n} u_{0n} + (A_{2n}^{(1)} + A_{2n}^{(2)}) u_{0n}^2]$, $\Gamma_{(n)}^{(2,1)}(k_{1(n)}, k_{2(n)}, p_{(n)}; u_{0n}, \kappa_n) = 1 - C_{1n} u_{0n} + (C_{2n}^{(1)} + C_{2n}^{(2)}) u_{0n}^2$. These coefficients can be expressed as functions of the previous integrals, namely

$$A_{1n} = \frac{(N+8)}{18} [I_2(\frac{k_{1(n)} + k_{2(n)}}{\kappa_n}) + I_2(\frac{k_{1(n)} + k_{3(n)}}{\kappa_n}) + I_2(\frac{k_{2(n)} + k_{3(n)}}{\kappa_n})], \quad (6a)$$

$$A_{2n}^{(1)} = \frac{(N^2 + 6N + 20)}{108} [I_2^2(\frac{k_{1(n)} + k_{2(n)}}{\kappa_n}) + I_2^2(\frac{k_{1(n)} + k_{3(n)}}{\kappa_n}) + I_2^2(\frac{k_{2(n)} + k_{3(n)}}{\kappa_n})], \quad (6b)$$

$$A_{2n}^{(2)} = \frac{(5N+22)}{54} [I_4(\frac{k_{i(n)}}{\kappa_n}) + 5 \text{ permutations}], \quad (6c)$$

$$B_{2n} = \frac{(N+2)}{18} I_3(\frac{k_{(n)}}{\kappa_n}), \quad (6d)$$

$$B_{3n} = \frac{(N+2)(N+8)}{108} I_5(\frac{k_{(n)}}{\kappa_n}), \quad (6e)$$

$$C_{1n} = \frac{N+2}{18} [I_2(\frac{k_{1(n)} + k_{2(n)}}{\kappa_n}) + I_2(\frac{k_{1(n)} + k_{3(n)}}{\kappa_n}) + I_2(\frac{k_{2(n)} + k_{3(n)}}{\kappa_n})], \quad (7a)$$

$$C_{2n}^{(1)} = \frac{(N+2)^2}{108} [I_2^2(\frac{k_{1(n)} + k_{2(n)}}{\kappa_n}) + I_2^2(\frac{k_{1(n)} + k_{3(n)}}{\kappa_n}) + I_2^2(\frac{k_{2(n)} + k_{3(n)}}{\kappa_n})], \quad (7b)$$

$$C_{2n}^{(2)} = \frac{N+2}{36} [I_4(\frac{k_{i(n)}}{\kappa_n}) + 5 \text{ permutations}]. \quad (7c)$$

The singular parts of each integral can not be calculated analytically in the exact form. However, using the orthogonal approximation [1, 2] we can perform the various integrals in each subspace independently. The singular contributions of the integrals are then given by $I_2 = \frac{1}{\epsilon_L} \left[1 + (h_{m_L} - 1) \epsilon_L - \frac{\epsilon_L}{2} L(K'_{(n)}) \right]$, $I_3 = (\sum_{n=1}^L K'_{(n)} \frac{(-1)}{8\epsilon_L}) \left(1 + (2h_{m_L} - \frac{3}{4}) \epsilon_L - 2\epsilon_L L_3(K'_{(n)}) \right)$, $I_4 = \frac{1}{2\epsilon_L^2} \left(1 + (2h_{m_L} - \frac{3}{2}) \epsilon_L - \epsilon_L L(K'_{(n)}) \right)$ and $I_5 = (\sum_{n=1}^L K'_{(n)} \frac{(-1)}{6\epsilon_L^2}) \left(1 + (3h_{m_L} - \epsilon_L - 3\epsilon_L L_3(K'_{(n)})) \right)$,

where $h_{m_L} = 1 + \frac{[\psi(1) - \psi(2 - \sum_{n=2}^L (\frac{m_n}{2n}))]}{2}$, $L(K'_{(n)}) = \int_0^1 dv \ln [v(1-v) \sum_{n=1}^L K'_{(n)}{}^{2n}]$ and $L_3(K'_{(n)}) = \int_0^1 dv v \ln [v(1-v) \sum_{n=1}^L K'_{(n)}{}^{2n}]$.

We have at hand all elements to determine the normalization functions Z_ϕ , \bar{Z}_{ϕ^2} and the bare coupling constants $u_{0n}(u_n)$. Minimal subtraction of dimensional poles in ϵ_L has two consequences. First, the singular terms proportional to the logarithmic integrals Eq. (??) are eliminated in the process. Second, the expansions of u_{0n} , Z_ϕ and \bar{Z}_{ϕ^2} turn out to be given by the expressions:

$$u_{0n} = u_n \left(1 + \frac{(N+8)}{6\epsilon_L} u_n + \left[\frac{(N+8)^2}{36\epsilon_L^2} - \frac{(3N+14)}{24\epsilon_L} \right] u_n^2 \right), \quad (8a)$$

$$Z_{\phi(n)} = 1 - \frac{N+2}{144\epsilon_L} u_n^2 + u_n^3 \left[-\frac{(N+2)(N+8)}{1296\epsilon_L^2} + \frac{(N+2)(N+8)}{5184\epsilon_L} \right], \quad (8b)$$

$$\bar{Z}_{\phi^2(n)} = 1 + \frac{N+2}{6\epsilon_L} u_n + \left[\frac{(N+2)(N+5)}{36\epsilon_L^2} - \frac{(N+2)}{24\epsilon_L} \right] u_n^2. \quad (8c)$$

The Wilson functions are defined by $\beta_n = -n\epsilon_L (\frac{\partial \ln u_{0n}}{\partial u_n})^{-1}$, $\gamma_{\phi(n)}(u_n) = \beta_n \frac{\partial \ln Z_{\phi(n)}}{\partial u_n}$ and $\bar{\gamma}_{\phi^2(n)}(u_n) = -\beta_n \frac{\partial \ln \bar{Z}_{\phi^2(n)}}{\partial u_n}$. Alternatively, they can be expressed as functions of the coefficients from $Z_{\phi(n)}$, $\bar{Z}_{\phi^2(n)}$ and u_{0n} rather simply as $\beta_n = -n\epsilon_L u_n [1 - a_{1n} u_n + 2(a_{1n}^2 - a_{2n}) u_n^2]$, $\gamma_{\phi(n)} = -n\epsilon_L u_n [2b_{2n} u_n + (3b_{3n} - 2b_{2n} a_{1n}) u_n^2]$ and $\bar{\gamma}_{\phi^2(n)} = n\epsilon_L u_n [c_{1n} + (2c_{2n} - c_{1n}^2 - a_{1n} c_{1n}) u_n]$. The fixed points are determined by the conditions $\beta_n(u_n^*) = 0$ which furnish the same value irrespective of the subspace chosen. After replacing all the coefficients just figured out, we obtain the fixed point value $u_n^* = \frac{6}{8+N} \epsilon_L \left\{ 1 + \epsilon_L \left[\frac{(9N+42)}{(8+N)^2} \right] \right\}$. Now $\gamma_{\phi(n)}(u_n^*) = \eta_n$, whereas $\nu_n^{-1} = 2n - \eta_n - \bar{\gamma}_{\phi^2(n)}(u_n^*)$. Putting together all this information, we find

$$\eta_n = \frac{n}{2} \left(\epsilon_L^2 \frac{(N+2)}{(N+8)^2} \left[1 + \epsilon_L \left(\frac{6(3N+14)}{(N+8)^2} - \frac{1}{4} \right) \right] \right), \quad (9a)$$

$$\nu_n = \frac{1}{n} \left(\frac{1}{2} + \frac{(N+2)}{4(N+8)} \epsilon_L + \frac{1}{8} \frac{(N+2)(N^2 + 23N + 60)}{(N+8)^3} \epsilon_L^2 \right). \quad (9b)$$

They are the same from those obtained using normalization conditions and the remaining exponents can be obtained from these two by the scaling relations derived in [1, 2].

3. Isotropic minimal subtraction

The algorithm just described in the anisotropic cases is identical with that in the isotropic case. We are going to highlight only the main differences with what has been explained so far. First, there is only one type of subspace involved since the bare propagator is equal to p^{-2n} . In that case, the Feynman integrals to be evaluated in our procedure are written in the form:

$$I_2 = \int \frac{d^{m_n} k}{(k + K')^{2n} k^{2n}}, \quad (10a)$$

$$I_3 = \int \frac{d^{m_n} k_1 d^{m_n} k_2}{(k_1 + k_2 + K')^{2n} k_1^{2n} k_2^{2n}}, \quad (10b)$$

$$I_4 = \int \frac{d^{m_n} k_1 d^{m_n} k_2}{k_1^{2n} (K' - k_1)^{2n} k_2^{2n} (k_1 - k_2 + k'_3)^{2n}}, \quad (10c)$$

$$I_5 = \int \frac{d^{m_n} k_1 d^{m_n} k_2 d^{m_n} k_3}{(k_1 + k_2 + K')^{2n} (k_1 + k_3 + K')^{2n} k_1^{2n} k_2^{2n} k_3^{2n}}. \quad (10d)$$

The isotropic integrals are simpler: they can be calculated either using the orthogonal approximation or exactly. In what follows we prefer to tackle solely the exact computation. Note that the expansion parameter is now $\epsilon_n = 4n - d$ ($d = m_n$). The ϵ_n -expansion of those integrals are given by

$$I_2 = \frac{1}{\epsilon_n} \left[1 - \frac{\epsilon_n}{2} \left(\psi(2n) - \psi(1) + \frac{\Gamma(2n)}{\Gamma(n)\Gamma(n)} L_n(K') \right) \right], \quad (11a)$$

$$I_3 = \frac{(-1)^n K'^{2n} \Gamma^2(2n)}{4\Gamma(3n)\Gamma(n+1)\epsilon_n} \left[1 + \epsilon_n \left(B_n - \frac{L_{3n}(K')}{A_n} \right) \right], \quad (11b)$$

$$I_4 = \frac{1}{2\epsilon_n^2} \left[1 + \left(D(n) - \frac{\Gamma(2n)L_n(K')}{\Gamma(n)^2} - \sum_{p=1}^{2n-1} \frac{1}{p} \right) \epsilon_n \right], \quad (11c)$$

$$I_5 = \frac{(-1)^n K'^{2n} \Gamma^2(2n)}{3\Gamma(3n)\Gamma(n+1)\epsilon_n^2} \left[1 + \epsilon_n \left(C_n - \frac{3L_{3n}(K')}{2A_n} \right) \right], \quad (11d)$$

where $D(n) = \frac{1}{2}\psi(2n) - \psi(n) + \frac{1}{2}\psi(1)$, $A_n = \frac{\Gamma(2n)\Gamma(n)}{\Gamma(3n)}$, $B_n = D(n) - \frac{1}{2} \sum_{p=1}^{2n-1} \frac{1}{p} - \frac{1}{2} \sum_{p=0}^{2n-1} \frac{1}{n+p}$ and $C_n = 2D(n) - \sum_{p=1}^{2n-1} \frac{1}{2p} + \sum_{p=1}^n \frac{3}{2p} - \sum_{p=0}^{2n-1} \frac{1}{n+p}$. The parametric integrals $L_n(K') = \int_0^1 dx x^{n-1} (1-x)^{n-1} \ln[x(1-x)K'^2]$ and $L_{3n}(K') = \int_0^1 dx x^{2n-1} (1-x)^{n-1} \ln[x(1-x)K'^2]$ generalize those from the case $n = 1$ corresponding to ordinary quadratic field theory.

The Wilson functions can be defined in the same way as before, except that now $\beta_n = -\epsilon_n \left(\frac{\partial \ln u_{0n}}{\partial u_n} \right)^{-1}$. Using the same procedure as before but keeping in mind the relevant differences already pointed out, the coupling constant and normalization functions can be shown to be given by

$$u_{0n} = u_n \left(1 + \frac{(N+8)}{6\epsilon_n} u_n + u_n^2 \left[\frac{(N+8)^2}{36\epsilon_n^2} + \frac{1}{18\epsilon_n} \left(\frac{(-1)^{n+1}(N+2)\Gamma^2(2n)}{2\Gamma(3n)\Gamma(n+1)} - (5N+22)D(n) \right) \right] \right), \quad (12a)$$

$$Z_{\phi(n)} = 1 + \left[\frac{(-1)^n (N+2)\Gamma^2(2n)}{72\Gamma(3n)\Gamma(n+1)\epsilon_n} u_n^2 + u_n^3 \left((-1)^n \frac{(N+8)(N+2)\Gamma^2(2n)}{648\Gamma(3n)\Gamma(n+1)\epsilon_n^2} \left[1 - \epsilon_n \left(D(n) + \frac{1}{2} \sum_{p=1}^{2n-1} \frac{1}{p} - \frac{1}{2} \sum_{p=1}^{2n-1} \frac{1}{n+p} \right) \right] \right) \right], \quad (12b)$$

$$\bar{Z}_{\phi^2(n)} = 1 + \frac{N+2}{6\epsilon_n} u_n + \left[\frac{(N+2)}{12\epsilon_n} \left(\frac{(N+5)}{3\epsilon_n} - D(n) \right) \right] u_n^2. \quad (12c)$$

The β_n function turns out to be

$$\beta_n = -u_n \left\{ \epsilon_n - \frac{(N+8)}{6} u_n + \frac{1}{9} \left[(5N+22)D(n) + \frac{(-1)^n (N+2)\Gamma^2(2n)}{2\Gamma(3n)\Gamma(n+1)} \right] u_n^3 \right\}. \quad (13)$$

The condition $\beta_n(u_n^*) = 0$ originates the fixed point value of the coupling constant, namely

$$u_n^* = \frac{6}{N+8} \epsilon_n \left\{ 1 + \frac{2}{(N+8)^2} \left[(10N+44)D(n) + \frac{(-1)^n (N+2)\Gamma^2(2n)}{\Gamma(3n)\Gamma(n+1)} \right] \epsilon_n \right\}. \quad (14)$$

Substitution of this value into the Wilson functions and recalling that $\gamma_{\phi(n)}(u_n^*) = \eta_n$, whereas

$\nu_n^{-1} = 2n - \eta_n - \bar{\gamma}_{\phi^2(n)}(u_n^*)$, we obtain the following critical exponents

$$\eta_n = \frac{(-1)^{n+1}(N+2)\Gamma^2(2n)}{(N+8)^2\Gamma(3n)\Gamma(n+1)}\epsilon_n^2[1 + \epsilon_n\mathcal{F}(n, N)], \quad (15a)$$

$$\begin{aligned} \nu_n = \frac{1}{2n} + \frac{(N+2)}{4n^2(N+8)}\epsilon_n + \frac{(N+2)}{2n^2(N+8)^2} \left[\frac{(-1)^{n+1}\Gamma^2(2n)}{2\Gamma(3n)\Gamma(n+1)} + \frac{(N+2)}{4n} - 3D(n) \right. \\ \left. + \mathcal{I}(n, N) \right] \epsilon_n^2, \end{aligned} \quad (15b)$$

where

$$\begin{aligned} \mathcal{F} = \frac{4}{(N+8)^2} \left[(10N+44)D(n) + (-1)^n \frac{(N+2)\Gamma^2(2n)}{\Gamma(3n)\Gamma(n+1)} \right] - \left(D(n) + \frac{1}{2} \sum_{p=1}^{2n-1} \frac{1}{p} \right. \\ \left. - \frac{1}{2} \sum_{p=1}^{2n-1} \frac{1}{n+p} \right), \end{aligned} \quad (16a)$$

$$\mathcal{I} = \frac{(10N+44)D(n)}{(N+8)} + \frac{(-1)^n(N+2)\Gamma^2(2n)}{(N+8)\Gamma(3n)\Gamma(n+1)}. \quad (16b)$$

4. Conclusion

The minimal subtraction method presented herein produce critical exponents identical to those computed using normalization conditions using either massless [2] or massive fields [3], albeit the results are represented in different manners. In conjunction with the minimal subtraction using massive fields to be developed in the future, this study can unveil further issues in the problem of Lifshitz quantum field theories [5, 6, 7, 8, 9].

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