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Quantum multiresolution: tower of scales

Antonina N. Fedorova, Michael G. Zeitlin
IPME RAS, St. Petersburg, V.O. Bolshoj pr., 61, 199178, Russia
http://www.ipme.ru/zeitlin.html, http://mp.ipme.ru/zeitlin.html

E-mail: `zeitlin@math.ipme.ru, anton@math.ipme.ru`

Abstract.
We demonstrate the creation of nontrivial (meta) stable states (patterns), localized, chaotic, entangled or decoherent, from the basic localized modes in various collective models arising from the quantum hierarchy described by Wigner-like equations. The numerical simulation demonstrates the formation of various (meta) stable patterns or orbits generated by internal hidden symmetry from generic high-localized fundamental modes. In addition, we can control the type of behaviour on the pure algebraic level by means of properly reduced algebraic systems (generalized dispersion relations).

1. Introduction. New localized modes and patterns: why need we them?

It is widely known that the currently available experimental techniques in the area of quantum physics as well as the present level of the understanding of phenomenological models, outstrips the actual level of mathematical description. Considering the problem of describing the really existing and/or realizable states, one should not expect that (gaussian) coherent states would be enough to characterize complex quantum phenomena. The complexity of a set of relevant states, including entangled (chaotic) ones is still far from being clearly understood and moreover from being realizable [1]. Our motivations arise from the following general questions [2]: how can we represent a well localized and reasonable state in mathematically correct form? is it possible to create entangled and other relevant states by means of these new localized building blocks? The general idea is rather simple: it is well known that the generating symmetry is the key ingredient of any modern reasonable physical theory. Roughly speaking, the representation theory of the underlying (internal/hidden) symmetry (classical or quantum, finite or infinite dimensional, continuous or discrete) is the useful instrument for the description of (orbital) dynamics. The proper representation theory is well known as "local nonlinear harmonic analysis", in particular case of the simple underlying symmetry, affine group, aka wavelet analysis. From our point of view the advantages of such approach are as follows: **i)** the natural realization of localized states in any proper functional realization of (Hilbert) space of states, **ii)** the hidden symmetry of a chosen realization of the functional model describes the (whole) spectrum of possible states via the so-called multiresolution decomposition. Effects we are interested in are as follows: **1).** a hierarchy of internal/hidden scales (time, space, phase space); **2).** non-perturbative multiscales: from slow to fast contributions, from the coarser to the finer level of resolution/decomposition; **3)**.the coexistence of the levels of hierarchy of multiscale dynamics with transitions between scales; **4).** the realization of the key features of the complex quantum world such as the existence of chaotic and/or entangled states with possible destruction in "open/dissipative" regimes due to interactions with quantum/classical environment and transition to decoherent states.

N-particle Wigner functions allow to consider them as some quasiprobabilities. The full description for quantum ensemble can be done by the hierarchy of functions (symbols): $W = \{W_s(x_1,\dots,x_s), s = 0,1,2\dots\}$ which are solutions of Wigner equations:

$$\frac{\partial W_n}{\partial t} = -\frac{p}{m}\frac{\partial W_n}{\partial q} + \sum_{\ell=0}^{\infty}\frac{(-1)^\ell(\hbar/2)^{2\ell}}{(2\ell+1)!}\frac{\partial^{2\ell+1}U_n(q)}{\partial q^{2\ell+1}}\frac{\partial^{2\ell+1}W_n}{\partial p^{2\ell+1}}. \tag{1}$$

The similar equations describe the important decoherence processes.

2. Variational multiresolution representation

We obtain our multiscale/multiresolution representations for solutions of Wigner-like equations (1) via the variational-wavelet approach [2] and represent the solutions as decomposition into localized eigenmodes related to the hidden underlying set of scales:

$$W_n(t,q,p) = \bigoplus_{i=i_c}^{\infty} W_n^i(t,q,p),$$

where value i_c corresponds to the coarsest level of resolution c in the full multiresolution decomposition (MRA) [3] of the underlying functional space:

$$V_c \subset V_{c+1} \subset V_{c+2} \subset \dots$$

and $p = (p_1, p_2, \dots)$, $q = (q_1, q_2, \dots)$, $x_i = (p_1, q_1, \dots, p_i, q_i)$ are coordinates in phase space. We introduce the Fock-like space structure on the whole space of internal hidden scales

$$H = \bigoplus_i \bigotimes_n H_i^n$$

for the set of n-partial Wigner functions (states):

$$W^i = \{W_0^i, W_1^i(x_1;t), \dots, W_N^i(x_1,\dots,x_N;t),\dots\},$$

where $W_p(x_1,\dots,x_p;t) \in H^p$, $H^0 = C$, $H^p = L^2(R^{6p})$ (or any different proper functional space), with the natural Fock space like norm:

$$(W,W) = W_0^2 + \sum_i \int W_i^2(x_1,\dots,x_i;t)\prod_{\ell=1}^{i}\mu_\ell.$$

First of all, we consider $W = W(t)$ as a function of time only, $W \in L^2(R)$, via multiresolution decomposition which naturally and efficiently introduces an infinite sequence of the underlying hidden scales. We have the contribution to the final result from each scale of resolution from the whole infinite scale of spaces. The closed subspace $V_j (j \in \mathbf{Z})$ corresponds to the level j of resolution and satisfies the following properties: let D_j be the orthonormal complement of V_j with respect to V_{j+1}: $V_{j+1} = V_j \bigoplus D_j$. Then we have the following decomposition:

$$\{W(t)\} = \bigoplus_{-\infty<j<\infty} D_j = \overline{V_c \bigoplus_{j=0}^{\infty} D_j},$$

in case when V_c is the coarsest scale of resolution. The subgroup of translations generates a basis for the fixed scale number: $\mathrm{span}_{k\in Z}\{2^{j/2}\Psi(2^j t-k)\}=D_j$. The whole basis is generated by the action of the full affine group [2], [3]:

$$\mathrm{span}_{k\in Z, j\in Z}\{2^{j/2}\Psi(2^j t-k)\}=\mathrm{span}_{k,j\in Z}\{\Psi_{j,k}\}=\{W(t)\}.$$

After the construction of the multidimensional tensor product bases, the next key point is the so-called Fast Wavelet Transform (FWT) [3], demonstrating that for a large class of operators the wavelet functions are a good approximation for true eigenvectors; and the corresponding matrices are almost diagonal. We have the simple linear parametrization of the matrix representation of our operators in the localized wavelet bases and of the action of these operators on arbitrary vectors/states in the proper functional space. FWT provides the maximum sparse and useful form for the wide classes of operators. After that, we can obtain our multiscale/multiresolution representations for observables (symbols), states, partitions via the variational approaches. Let L be an arbitrary (non)linear differential/integral operator with matrix dimension d (finite or infinite), which acts on some set of functions from $L^2(\Omega^{\otimes^n})$: $\Psi\equiv\Psi(t,x_1,x_2,\dots)=\Big(\Psi^1(t,x_1,x_2,\dots),\dots,\ \Psi^d(t,x_1,x_2,\dots)\Big)$, $\quad x_i\in\Omega\subset\mathbf{R}^6$, n is a number of particles:

$$\begin{aligned}
L\Psi &\equiv L(Q,t,x_i)\Psi(t,x_i)=0,\\
Q &\equiv Q_{d_0,d_1,d_2,\dots}(t,x_1,x_2,\dots,\partial/\partial t,\partial/\partial x_1,\partial/\partial x_2,\dots,\int\mu_k)\\
&= \sum_{i_0,i_1,i_2,\dots=1}^{d_0,d_1,d_2,\dots} q_{i_0i_1i_2\dots}(t,x_1,x_2,\dots)\Big(\frac{\partial}{\partial t}\Big)^{i_0}\Big(\frac{\partial}{\partial x_1}\Big)^{i_1}\Big(\frac{\partial}{\partial x_2}\Big)^{i_2}\dots\int\mu_k.
\end{aligned}$$

Let us consider the N mode approximation:

$$\Psi^N(t,x_1,x_2,\dots)=\sum_{i_0,i_1,i_2,\dots=1}^{N} a_{i_0i_1i_2\dots}A_{i_0}\otimes B_{i_1}\otimes C_{i_2}\dots(t,x_1,x_2,\dots).$$

We will determine the expansion coefficients from the following conditions, Generalized Dispersion Relation, related to the proper choosing of variational approach:

$$\ell^N_{k_0,k_1,k_2,\dots}\equiv\int(L\Psi^N)A_{k_0}(t)B_{k_1}(x_1)C_{k_2}(x_2)\mathrm{d}t\mathrm{d}x_1\mathrm{d}x_2\dots=0. \tag{2}$$

Thus, we have exactly dN^n algebraical equations for dN^n unknowns $a_{i_0,i_1,\dots}$. This variational approach reduces the initial problem to the problem of solution of functional equations at the first stage and some algebraical problems at the second one. It allows to unify the multiresolution expansion with variational construction. As a result, the solution is parametrized by the solutions of two sets of reduced algebraical problems, one is linear or nonlinear (depending on the structure of the generic operator L) and the rest are linear problems related to the computation of the coefficients of reduced algebraic equations. It is also related to the choice of exact measure of localization (including the class of smoothness), which is proper for our set-up. These coefficients can be found via functional/algebraic methods by using the compactly supported wavelet basis or any other wavelet families. As a result, the solution of the hierarchies as in c- as in q-region, has the following multiscale or multiresolution decomposition via nonlinear localized eigenmodes

$$\begin{aligned}
W(t,x_1,x_2,\dots) &= \sum_{(i,j)\in Z^2} a_{ij}U^i\otimes V^j(t,x_1,\dots),\\
V^j(t) &= V_N^{j,slow}(t)+\sum_{l\geq N}V^j_l(\omega_l t),\ \omega_l\sim 2^l,\\
U^i(x_s) &= U_M^{i,slow}(x_s)+\sum_{m\geq M}U^i_m(k^s_m x_s),\ k^s_m\sim 2^m,
\end{aligned} \tag{3}$$

which corresponds to the full multiresolution expansion in all underlying time/space scales. The formulas (3) give the expansion into a slow part and fast oscillating parts for arbitrary N, M. So, we may move from the coarse scales of resolution to the finest ones for obtaining more detailed information about the dynamical process. In this way, one obtains contributions to the full solution from each scale of resolution or each time/space scale or from each nonlinear eigenmode. Formulas (3) do not use perturbation techniques or linearization procedures. Numerical calculations are based on compactly supported wavelets and wavelet packets and on the evaluation of accuracy on the level N of the corresponding cut-off of the full system regarding Fock-like norm: $\|W^{N+1} - W^N\| \leq \varepsilon$.

3. Conclusions

By using proper high-localized bases on orbits generated by actions of internal hidden symmetries of underlying functional spaces, we can describe and classify the full zoo of patterns with non-trivial behaviour including localized (coherent) structures in quantum systems with complicated behaviour (Figs. 1, 2). The numerical simulation demonstrates the formation of various (meta) stable patterns or orbits generated by internal hidden symmetry from generic high-localized fundamental modes. These (nonlinear) eigenmodes are more realistic for the modeling of classical/quantum dynamical process than the (linear) gaussian-like coherent states. Here we mention only the best convergence properties of the expansions based on wavelet packets, which realize the minimal Shannon entropy property and the exponential control of the convergence of expansions like (3). Figs. 1, 2 demonstrate the steps of (hidden) multiscale resolution, starting from coarse–graining, during the full quantum interaction/evolution of entangled states leading to the growth of the degree of complexity (entanglement) of the quantum state. It should be noted that we can control the type of behaviour on the level of the reduced algebraic system (Generalized Dispersion Relation) (2) [2].

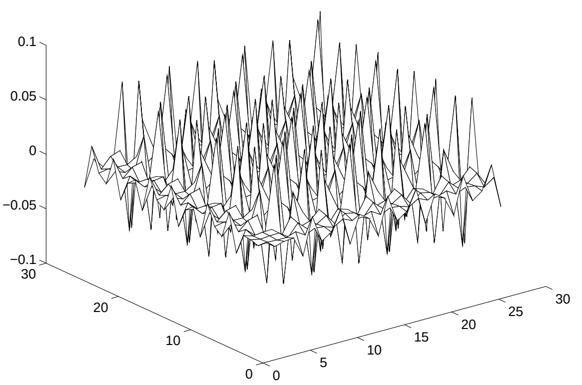

Figure 1. Entangled Wigner function.

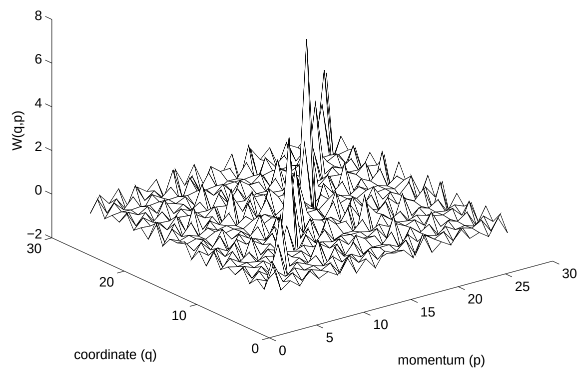

Figure 2. Localized (decoherent) pattern: (waveleton) Wigner function.

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