Vector Finite Elements and the Step by Step Eigenvalue Perturbation Methods

Serkan Günel, Yeşim Zoral

Dokuz Eylül University, Department of Electrical and Electronics Engineering, 35160, Buca, Izmir, Turkey.

E-mail: `serkan.gunel@deu.edu.tr`, `yesim.zoral@deu.edu.tr`

Abstract. The electromagnetic resonant structures and their perturbations have been found their way in many applications in microwave engineering. Analysis of these structures using finite element method results in a generalized eigenvalue problem, where the eigenvalues correspond to the resonant frequencies, and the eigenvectors correspond to the resonant modes. The perturbations of resonant structures yield perturbed eigenvalue problem and can be solved by eigenvalue perturbation methods, effectively. Combining finite element method with step by step eigenvalue perturbation method yields parametric history with respect to perturbation parameter. In this study perturbation of a microwave ring resonator placed in a metalic enclosure has been examined, combining the vector finite element and the step–by–step generalized eigenvalue perturbation methods.

1. Introduction

Material or geometrical perturbations of microwave resonant structures results in a generalized eigenvalue-eigenvector perturbation problem. For small perturbations of these structures, the well established classical perturbation technique can be enough [1]. However, this technique may not be sufficient for large perturbations. Numerical techniques are preferred for analysis of such perturbation problems for which an analitical solution rarely exist. In this respect, the Finite Element Method (FEM) can be used, effectively [2]. If such resonant structures are perturbed by changing a parameter, then analysis reduces to a generalized eigenvalue-eigenvector perturbation problem [3]. When optimization of resonant structures with respect to a geometrical or material parameter is required, FEM can be used to solve the problem by analyzing the entire structure for all parametric values. Meanwhile, if a parametric history data of this process is desired, the approach requires the repetitive solution of the geometry for each parameter value. For large meshes and highly granular parameter variations, the process can be time consuming and computational cost is high. For such cases, the step–by–step generalized eigenvalue perturbation method is an efficient technique [4, 5]. In this study, the step-by-step perturbation technique combined with vector FEM had been investigated. Results are obtained by three dimensional vector FEM code specifically developed for this purpose. Vector FEM formulation is chosen since it is easier to enforce the boundary conditions [2].

In previous studies, it is shown that step by step perturbation technique is effective in conjunction with the moment method and the nodal FEM [5, 6]. Particular advantage of eigenvalue perturbation technique is due to the fact that the mass and the stiffness matrices produced by FEM are inherently symmetric positive definite. Therefore, the problem can be

reduced to tracking of few eigenvalues of interest with simple similarity transformations. In this study, the vector FEM is used in conjunction with the generalized step–by–step eigenvalue-eigenvector perturbation method for parametric history analysis of a perturbed ring resonator. At first, the basic formulation about the vector FEM has been summarized. Then the combination of the vector FEM with the step-by-step eigenvalue perturbation problem has been given. In the last section, a sample geometry and obtained results by three dimensional vector FEM code have been given. The results have been compared to the results of commercial Ansoft HFSS ^{TM} electromagnetic FEM software.

2. Vector FEM for the solution of electromagnetic resonance problems

Let the electric field vector defined in a region Ω be $\boldsymbol{E} = [E_x(x,y,z)\ E_y(x,y,z)\ E_z(x,y,z)]^T$ in phasor form. Then, $\boldsymbol{E}$ satisfies the Helmholtz equations. The solution of the Helmholtz equations minimizes the following functional,

$$F(\boldsymbol{E}) = \frac{1}{2} \int_{\Omega} \frac{1}{\mu_r} \left(\nabla \times \boldsymbol{E}\right) \cdot \left(\nabla \times \boldsymbol{E}\right) - k_o^2 \varepsilon_r \boldsymbol{E} \cdot \boldsymbol{E} d\Omega. \tag{1}$$

In the FEM analysis, the problem domain Ω is divided non-overlapping sub-domains or elements such that $\Omega_e \in \Omega\ e = 1, 2, \ldots, M$. In vector FEM the field components in each region are expressed in terms of the fields along the edges of Ω_e. For tetrahedral elements the electric field in each element can be approximated as $\tilde{\boldsymbol{E}}^e = \sum_{i=1}^{6} \boldsymbol{N}_i^e E_i^e$ where $\boldsymbol{N}_i^e$ are the vector basis functions. Defining elemental matrices, that can be easily evaluated by well known formulation given in [2], as

$$\boldsymbol{A}^e \triangleq \int_{\Omega_e} \frac{1}{\mu_r^e} \left(\nabla \times \boldsymbol{N}^e\right)^T \left(\nabla \times \boldsymbol{N}^e\right) d\Omega \qquad\qquad \boldsymbol{B}^e \triangleq \int_{\Omega_e} \varepsilon_r^e \boldsymbol{N}^{eT} \boldsymbol{N}^e d\Omega \tag{2}$$

and substituting these expressions in (3) gives,

$$F(\tilde{\boldsymbol{E}}) = \frac{1}{2} \sum_{e=1}^{M} \tilde{\boldsymbol{E}}^{eT} \boldsymbol{A}^e \tilde{\boldsymbol{E}}^e - k_o^2 \tilde{\boldsymbol{E}}^{eT} \boldsymbol{B}^e \tilde{\boldsymbol{E}}^e. \tag{3}$$

Applying the standart assembly procedure in FEM analysis, the variational problem of minimizing (3) with respect to E_i^e finally reduces the solution of following eigenvalue/eigenvector problem

$$\boldsymbol{A}\tilde{\boldsymbol{E}} = k_o^2 \boldsymbol{B}\tilde{\boldsymbol{E}} \tag{4}$$

where k_o is the wave number. The stiffness matrix $\boldsymbol{A}$ and the mass matrix $\boldsymbol{B}$ are symmetric and (semi)-positive definite matrices.
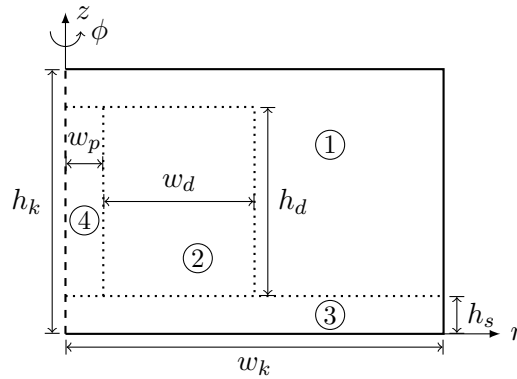
3. Step-by-step perturbation

We can recast (4) in the following form

$$\boldsymbol{L}_0 \phi = \lambda \boldsymbol{M}_0 \phi \qquad\qquad \lambda = \omega^2 \tag{5}$$

where $\boldsymbol{L}_0$ and $\boldsymbol{M}_0$ are the stiffness and mass matrices of the *base* system, ω is the resonant frequency, λ's and ϕ's are the eignvalues and eigenvectors of the associated problem, respectively. Assume that the solution of (5) is obtained and the base system is modified in such a way that the matrices of the perturbed system can be written using unperturbed matrices as:

$$[\boldsymbol{L}_0 + f(p)\boldsymbol{L}_p] \phi(p) = \lambda(p) [\boldsymbol{M}_0 + g(p)\boldsymbol{M}_p] \phi(p) \tag{6}$$

Figure 1. Cylindrical cavity perturbed by a dielectric ring resonator

It has been shown that even large perturbations can be handled by step-by-step perturbation method [5, 4]. The solution of (6) using step by step perturbation technique in conjunction with FEM is particularly advantageous. The resulting symmetric positive (semi)-definite matrices due to FEM formulation, leads to perturbation of diagonal matrix pencils and the formulation can exploit orthogonality of eigenvectors using similarity transformations instead of heavy QZ-factorizations suggested in [5].

Assume that the dielectric permittivity in some sub-region of $\Omega_\ell \subset \Omega$ is to be varied in $[\varepsilon_\ell, \varepsilon_{\max}]$. The parametric history with respect to ε_ℓ can be obtained using the procedure explained in [5]. If K denotes the number of steps, the relative dielectric constant and the perturbation parameter at the kth step can be written as

$$\varepsilon_\ell^{(k)} = \varepsilon_\ell + p^{(k)}\left(\varepsilon_{max} - \varepsilon_\ell\right) \tag{7}$$

$$p^{(k)} = p^{(k-1)} + \Delta p^{(k)}. \tag{8}$$

The functions $f(.)$ and $g(.)$ in (6) are expressed as

$$f^{(k)} = \left(\frac{1}{\varepsilon_\ell^{(k)}} - \frac{1}{\varepsilon_\ell}\right) \qquad\qquad \Delta f^{(k)} = -\frac{\Delta p^{(k)}(\varepsilon_{max} - \varepsilon_\ell)}{\varepsilon_\ell^{(k)}\varepsilon_\ell^{(k-1)}} \tag{9a}$$

$$g^{(k)} = p^{(k)}(\varepsilon_{max} - \varepsilon_\ell) \qquad\qquad \Delta g^{(k)} = \Delta p^{(k)}(\varepsilon_{max} - \varepsilon_\ell) \tag{9b}$$

The problem (6) can be reduced to

$$\left[\mathbf{\Lambda}^{(k-1)} + \Delta f^{(k)} \boldsymbol{L}_p^{(k)}\right] \boldsymbol{Y}^{(k)} = \left[\boldsymbol{I} + \Delta g^{(k)} \boldsymbol{M}_p^{(k)}\right] \boldsymbol{Y}^{(k)} \mathbf{\Lambda}^{(k)} \tag{10}$$

$$\boldsymbol{L}_p^{(k)} = \mathbf{\Phi}^{(k-1)^T} \boldsymbol{L}_p \mathbf{\Phi}^{(k-1)}, \quad \boldsymbol{M}_p^{(k)} = \mathbf{\Phi}^{(k-1)^T} \boldsymbol{M}_p \mathbf{\Phi}^{(k-1)} \tag{11}$$

where $\mathbf{\Lambda}^{(k)}$ is the diagonal matrix of the eigenvalues to be tracked and $\mathbf{\Phi}^{(k)}$ are the eigenvectors at step k, using the transformation $\mathbf{\Phi}^{(k)} \triangleq \bar{\mathbf{\Phi}}^{(k-1)} \boldsymbol{Y}^{(k)}$. This is just perturbation of $(\mathbf{\Lambda}, \boldsymbol{I})$ diagonal matrix pencil, and $\boldsymbol{L}_p$ and $\boldsymbol{M}_p$ can easily be obtained using $\boldsymbol{L}_o$ and $\boldsymbol{M}_o$. Hence, the perturbed solutions can be calculated quite efficiently.

4. Sample problem
The proposed approach has been tested on the structure given in Figure 1 which represents a dielectric ring (region 2) placed in a perfectly conducting enclosure. The setup is useful for

measurement of dielectric constants of fluid substances placed in region 4, particularly. For a sample problem, relative dielectric constant of the region 2 is fixed to 10 and the dielectric constant of the region 4 is changed from value 1 to 10 with no substrate region 3 ($h_s = 0$). The results are compared by the HFSS simulation results, preliminarily. Figure 2 shows the parametric history calculated with both methods on two different but comparable meshes. The relative error between to simulations is about 1.15% at the worst case when $\varepsilon_\ell = 10$, although the step by step perturbation approach is much faster.
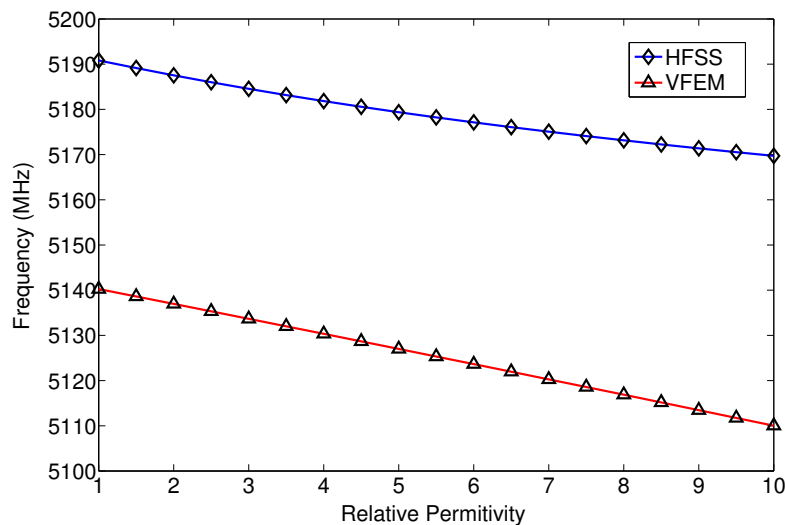
Figure 2. The parametric history results for dielectric ring resonator perturbation.

5. Conclusion

The vector FEM and step by step eigenvalue perturbation for the analysis of microwave resonant structures has been inspected. Analysis using the eigenvalue perturbations is practical, since it does not require the solution of eigenvalue problem for each perturbation step. For large perturbation cases, the step-by-step perturbation method is is particularly advantageous combined with FEM, since FEM formulation results in symmetric positive definite matrix pencils. The proposed method is tested on the perturbation problem of a dielectric ring. The parametric history analysis with respect to dielectric permittivity of the perturber is presented. The results are also compared with the Ansort HFSS ^{TM}results for the sample problem. Preliminary results suggest that the method is promising.

References
[1] van Bladel J 1985 *Electromagnetic Fields* (New York, U.S.A.: Hemisphere Publishing Corporation)
[2] Jin J 2002 *The finite element method in electromagnetics* 2nd ed (John Wiley and Sons, Inc.)
[3] Baumgärtel H 1984 *Analytic Perturbation Theory for Matrices and Operators* (*Operator Theory: Advances and Aplications* vol 15) (Berlin, Germany: Birkhäuser Verlag)
[4] Lü Z H, Shao C and Feng Z D 1993 *Journal of Sound and Vibration* **164** 459–469
[5] Günel S and Zoral E Y 2010 *IET Microwave, Antennas & Propagation* **4** 466–476
[6] Günel S and Zoral Y 2013 *Proc. of XVI International Symposium on Electromagnetic Fields in Mechatronics, Electrical and Electronic Engineering ISEF 2013, September 2013, Ohrid, Machedonia*