

Geometry of a Quantized Spacetime: The Quantum Potential Approach

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Abstract. Abstract: Quantum dynamics in a curved spacetime can be studied using a modified Lagrangian approach directly in terms of the spacetime variables [Mirza, B.M., Quantum Dynamics in Black Hole Spacetimes, IC-MSQUARE 2012]. Here we investigate the converse problem of determining the nature of the background spacetime when quantum dynamics of a test particle is known. We employ the quantum potential formalism here to obtain the modifications introduced by the quantum effects to the background spacetime. This leads to a novel geometry for the spacetime in which a test particle modifies the spacetime via interaction through the quantum potential. We present here the case of a Gaussian wave packet, and a localized quantum soliton, representing the test particle, and determine the corresponding geometries that emerge.

1. Introduction

Quantum effects in curved spacetimes are important features of physical processes that involve extremely high energies such as those occurring in vicinity of a black hole, or a highly dense star, and have been the subject of a wide study (see reference [1]-[5], and for further references). Quantum potential approach, however, involves another important feature that can significantly affect the physics of such systems. The quantum potential is a nonlinear function that can attain very high potential magnitude in some regions of space, hence can significantly affect the background spacetime itself.

The quantum potential approach is based on the general assumption that the Schrödinger equation,

$$\left[-\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}) \right] \psi(\mathbf{r}, t) = i\hbar \frac{\partial \psi(\mathbf{r}, t)}{\partial t}, \quad (1)$$

with the substitution $\psi(\mathbf{r}, t) = R(\mathbf{r}, t) \exp iS(\mathbf{r}, t)/\hbar$, can be separated into the real and imaginary parts:

$$\frac{\partial \rho(\mathbf{r}, t)}{\partial t} + \nabla \cdot \rho(\mathbf{r}, t) \mathbf{v}(\mathbf{r}, t) = 0, \quad (2)$$

$$\frac{\partial S(\mathbf{r}, t)}{\partial t} = \frac{1}{2m} (\nabla S(\mathbf{r}, t))^2 - V(\mathbf{r}) - \frac{\hbar^2}{2m} \frac{\nabla^2 R(\mathbf{r}, t)}{R(\mathbf{r}, t)}. \quad (3)$$

In analogy with the usual Hamilton-Jacobi theory, the phase gradient corresponds to the particle momentum so that equation (3) is regarded as the Hamilton-Jacobi equation with a new potential function involving the Planck's constant. This potential is called the quantum potential. Here we



use the quantum potential approach to investigate modification introduced by this potential in the background Schwarzschild spacetime[7,8], for the case of massive particles (such as electrons) represented by a Gaussian wave packet [4] or a localized soliton wave [6]. We employ here the gravitational units $G = 1 = c$.

2. Modified Schwarzschild Spacetime Metric

In analogy with the Hamilton-Jacobi theory, we have in equation (3), $\partial S/\partial t = E$, and $\nabla S = p$, which gives the conservation of energy law $E = T - V + Q$, where T the kinetic energy, V the potential energy function, and $Q = -\hbar^2 \nabla^2 R(\mathbf{r}, t)/2mR(\mathbf{r}, t)$ is the new quantum potential. This addition in the total energy indicates that there is a modification in the spacetime metric also. First let us recall that the Schwarzschild metric can be deduced from the following dynamical considerations. The kinetic energy of a test particle in the field of a gravitating object has the general covariant expression $T = \frac{m}{2}(ds/dt)^2$, where ds is the arc length, gives

$$ds^2 = \frac{2}{m}(E - V)dt^2 \quad (4)$$

whereas for a locally fixed (that is co-moving) observer, the Schwarzschild metric reduces to

$$ds^2 = -(1 - \frac{2M}{r})dt^2 \quad (5)$$

where M is the mass of the gravitational source. The above two expressions for the metric correspond under the rule

$$E = 0, V(r) = 1 - \frac{2M}{r}, \quad (6)$$

which is the usual choice for the gravitational potential energy also. Thus for the choice $E = 0$, we have in general:

$$ds^2 = -V(r)dt^2 \quad (7)$$

In the above case when there is quantum force present, we thus have for $V(r)$ modified by the quantum potential,

$$ds^2 = -\frac{2}{m}(T + Q)dt^2 \quad (8)$$

Using the general invariant expression

$$T = \frac{1}{2}g_{\alpha\beta} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} \quad (9)$$

we have for the modified metric

$$ds^2 = -g_{\alpha\beta} dx^\alpha dx^\beta - \frac{2Q}{m} dt^2, \quad (10)$$

where as usual the $1/m$ factor has been absorbed into the definition of the metric coefficients. Thus in full, the Schwarzschild metric modifies to

$$ds^2 = -(1 - \frac{2M}{r} + \frac{2Q}{m})dt^2 + \left(1 - \frac{2GM}{r^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \quad (11)$$

2.1. Cases of a Gaussian Wave Packet and a Localized Soliton Wave

The most common representation of a free particle in quantum theory is given by the Gaussian wave packet [4]:

$$\psi(r, t) = \frac{1}{(2\pi s)^{3/4}} \exp \left(ik(r - u_0 t/2) - (r - u_0 t)^2 / 4s\sigma_0 \right), \quad (12)$$

where u is the uniform constant speed of the wave packet, moving along the radial direction. It gives for the quantum potential,

$$Q(r, t) = \frac{\hbar^2}{4m^2\sigma^2} \left(3 - \frac{(r - u_0 t)^2}{2\sigma^2} \right). \quad (13)$$

hence the Schwarzschild spacetime modifies to,

$$ds^2 = - \left(1 - \frac{2M}{r} + \frac{\hbar^2}{2m^3\sigma^2} \left(3 - \frac{(r - u_0 t)^2}{2\sigma^2} \right) \right) dt^2 + \left(1 - \frac{2GM}{r^2} \right)^{-1} dr^2 + \quad (14)$$

$$r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \quad (15)$$

Another representation of a particle in quantum theory that satisfies the set of equations (2) and (3) is the localized soliton wave solution given by [6]:

$$\rho(\xi) = \rho \cos^2 \frac{\sqrt{2mQ_0}}{\hbar} \xi, \quad (16)$$

where $Q_0 = Q(r_0, t)$ is the profile of the quantum potential at any time t , $\xi = r - ut$, and $\rho = \sqrt{R(\mathbf{r}, t)}$. It follows that the quantum potential is a constant for the localized soliton case identically which can be chosen zero here, and it moves radially in free space, with a constant uniform speed $u = c$. Hence the Schwarzschild, in this case, spacetime remains unmodified.

3. Conclusions

We have derived here the general form of the spacetime metric modified by the quantum effects. To this end we have taken a different route and have used the quantum potential approach to the quantum theory. This has the advantage that one can talk directly in terms of dynamical principles, such as the energy conservation, to arrive at the spacetime geometry. The arguments are heuristic in the sense that they are not based on Einstein field equations, however in the absence of a complete quantum theory of the gravitational fields, this approach is useful because of its rather general character. We thus have adopted the route from dynamics to geometry, rather than the opposite one from geometry to dynamics.

Another conclusion that follows from the above study is that whereas a Gaussian wave packet modifies the spacetime metric, a localized soliton does not show any such modification. Its dynamics, therefore, is more correctly representative of a test particle than that of a Gaussian wave packet.

References

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