2nd International Conference on Mathematical Modeling in Physical Sciences 2013 IOP Publishing
Journal of Physics: Conference Series **490** (2014) 012181 doi:10.1088/1742-6596/490/1/012181

# Hierarchical Bass model

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**Abstract.** We propose a new model about diffusion of a product which includes a memory of how many *adopters* or advertisements a *non-adopter* met, where (non-)adopters mean people (not) possessing the product. This effect is lacking in the Bass model. As an application, we utilize the model to fit the iPod sales data, and so the better agreement is obtained than the Bass model.

## 1. Introduction

It is so interesting how fashion diffuses in a society. It has been fascinating scientists and scholars how a new product or innovation spreads in a society. "No other field of behavior science research represents more effort by more scholars in more disciplines in more nations." [1] We shall employ a penetration rate of a product in studying a fashion diffusion scientifically because it is an objective and available statistical data. Generally, a change of the penetration rate in time is slow at first, and then, it peaks, followed again by slow change. The logistic model (LM) can describe this change in time. In Ref. [2], we made it clear that the communication between individuals is imitation; people who do not possess (non-adopters) purchase the product quickly after they met people who already possess it (adopters). However, the communication is unnatural, because there are not only trend-conscious people, but also cautious people in our society. We considered that a memory of how many adopters a non-adopter met is essential for him/her to decide to purchase the product. Accordingly, we proposed a model including this effect, which is the *hierarchical logistic model* (HLM) [2].

Another essential element for fashion diffusion is an advertisement. The Bass model (BM) [3] includes the element, which is represented by

$$\frac{\mathrm{d}P(t)}{\mathrm{d}t} = \left\{b + \frac{a}{N}P(t)\right\}\{N - P(t)\} \ , \tag{1}$$

where $P(t)$ is the number of those possessing the product at $t$ and $N$ is the total population. This differential equation coincides with LM by setting $b = 0$. Indeed, this model does not take account of the memory we mentioned above. So, in this study, we shall propose a model including the memory, which is the *hierarchical Bass model* (HBM).

## 2. Bass model

In this section, we derive BM in the same way as Ref. [2].

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Let us put $N$ random walkers on $n \times n$ $(> N)$ lattices. They move to one of the nearest lattices with the same provability at each discretized time step. Furthermore, we shall apply the following rules for the random walkers: i) If non-adopters meet adopters, they start to adopt a product quickly; ii) There are no interactions among (non-)adopters; iii) Adopters do not part with it. Here, we define sharing a lattice with someone as meeting the person. By these rules, we can derive LM [2]. In order to derive BM, we need to consider another influence on non-adopters from other than adopters, that is an advertisement like a billboard. We assume that the number of advertisements does not change in time. In addition, we apply the forth rule: iv) Non-adopters who met advertisements start to adopt the product quickly. As a final assumption, we set the density of people to be so low that three or more people cannot share a lattice at the same step, as if three or more body collisions are ignored in a rarefied gas. Therefore, it is natural to consider that a probability of meeting adopters, non-adopters, or advertisements is proportional to each number of them.

Let us set the number of adopters and non-adopters at the $i$th step as $P_i$ and $Q_i$, respectively. Indeed, $P_i + Q_i = N$. Moreover, we set the number of advertisements $B$ $(< n^2)$. Then, the probability of meeting adopters, non-adopters and advertisements can be represented by $P_i/n^2$, $Q_i/n^2$ and $B/n^2$, respectively. According to the first and the forth rule, $(B/n^2)\times Q_i + (P_i/n^2)\times Q_i$ people of non-adopters becomes adopters at the next step, and then, the following recursion formulae can be obtained.

$$P_{i+1} = P_i + \frac{B+P_i}{n^2}Q_i \ , \ Q_{i+1} = Q_i - \frac{B+P_i}{n^2}Q_i \ . \tag{2}$$

We shall define the number of them at $t$ as $P(t) = P(i\cdot\Delta t) \equiv P_i$ and $Q(t) = Q(i\cdot\Delta t) \equiv Q_i$. Here, we take the limits as $\Delta t \to 0$ and $n \to \infty$ with $n^2\Delta t$ fixed. By setting the fixed value as $\frac{a}{N} = \lim_{\substack{\Delta t\to 0\\ n\to\infty}} \frac{1}{n^2\Delta t}$, defining $Ba/N$ as $\lim_{\substack{\Delta t\to 0\\ n\to\infty}} \frac{B}{n^2\Delta t} = \frac{Ba}{N} \equiv b$, and using $P(t) + Q(t) = N$, the following differential equation is derived:

$$\frac{\mathrm{d}P(t)}{\mathrm{d}t} = \left\{b + \frac{a}{N}P(t)\right\}\{N - P(t)\} , \tag{3}$$

which describes BM. If we use the penetration rate $p(t) \equiv P(t)/N$, the above differential equation becomes

$$\frac{\mathrm{d}p(t)}{\mathrm{d}t} = \{b + ap(t)\}\{1 - p(t)\} \ . \tag{4}$$

## 3. Hierarchical Bass model

In the previous section, it is unveiled that BM reflects quick influences on non-adopters from adopters and advertisements. However, it is unnatural because all the people in our society are not easily influenced by others and advertisements. We consider that a memory of how many adopters or advertisements a non-adopter met is important for his/her decision to purchase the product. Therefore, as in Ref. [2], we shall include the memory in BM by the following way: We set the number of non-adopters starting to adopt the product after they meet $\mu$ adopters and advertisements at $i$th step as $Q_i^\mu$, in which we call $\mu$ as *remaining adopters and advertisements number* (RAAN). Indeed, if a non-adopter, whose RAAN is $\mu$, meet one of adopters or advertisements, his/her RAAN becomes $\mu - 1$ at the next step. We do not alter other rules.

If the maximum of RAAN is $m$, the recursion formulae change into

$$P_{i+1} = P_i + \frac{B+P_i}{n^2}Q_i^1 \ , \ Q_{i+1}^1 = Q_i^1 - \frac{B+P_i}{n^2}Q_i^1 + \frac{B+P_i}{n^2}Q_i^2 \ , \ \dots \ , \ Q_{i+1}^m = Q_i^m - \frac{B+P_i}{n^2}Q_i^m \ . \tag{5}$$

These become the following differential equations by the previous continuation of space and time.

$$\frac{\mathrm{d}P(t)}{\mathrm{d}t} = \left\{b + \frac{a}{N}P(t)\right\} Q^1(t)\ ,\ \frac{\mathrm{d}Q^1(t)}{\mathrm{d}t} = \left\{b + \frac{a}{N}P(t)\right\} \left\{Q^2(t) - Q^1(t)\right\}\ ,$$
$$\dots\ ,\ \frac{\mathrm{d}Q^m(t)}{\mathrm{d}t} = -\left\{b + \frac{a}{N}P(t)\right\} Q^m(t) \tag{6}$$

where $Q^\mu(t) = Q^\mu(i\cdot\Delta t) \equiv Q_i^\mu$. We shall name this the *hierarchical Bass model* (HBM), because this model has the hierarchical structure in non-adopters.

If we use ratios of adopters and non-adopters to the total number $N$, HBM becomes

$$\frac{\mathrm{d}p(t)}{\mathrm{d}t} = \{b + ap(t)\}\, q^1(t)\ ,\ \frac{\mathrm{d}q^1(t)}{\mathrm{d}t} = \{b + ap(t)\}\left\{q^2(t) - q^1(t)\right\}\ ,$$
$$\dots\ ,\ \frac{\mathrm{d}q^m(t)}{\mathrm{d}t} = -\{b + ap(t)\}\, q^m(t) \tag{7}$$

where $q^\mu(t) \equiv Q^\mu(t)/N$.

## 4. Fitting iPod sales data

Let us apply HBM to a sales data of a product in this section. For it, we shall employ the iPod sales as in Ref. [2]. We can obtain the data from the official homepage of Apple Inc. which has been creating and marketing the iPod: http://www.apple.com/. We shall utilize the data from November 2001 to May 2006, for the interval includes only one peak of sales.

Setting November 2001 as the origin of the time axis, we construct the cumulative sales, and then, we fit the sales data by HBM, BM and LM. In doing so, if we minimize the residual sum of squares or the sum of the absolute value of error (SAE), the fit between the model and the data is good for high values of sales but not for low ones. On the other hand, if we minimize the sum of the absolute value of relative error (SARE), the fit for low values becomes better. We consider that this is because the maximum of iPod sales data is 3 digits different than the minimum. Thus, in order to resolve the problem, we shall minimize a product of SAE and SARE. The parameters minimizing this product are show in Tab. 1, together with ones of LM.

From this table, we can see that SARE diminish with increase of $m$: SARE with $m = 4$ reduces to nearly half that with $m = 1$. In other words, the average relative error for HBM with $m = 4$ is about as half as that for BM. Results of fitting the data by HLM are also shown in Tab. 1 by numbers in parentheses. From these, we can say that HBM considering the effect of advertisements can approximate the sales data better than HLM. However, it is not a reason that there is an extra fit parameter $b$ in HBM, because the degrees of agreement of LM and BM are comparable.

From the fit parameters of HBM with $m = 4$, we can guess the following facts: The market size producing the first peak of iPod sales is about 66 million people, the ratio of the trend-conscious people, $q^1(0)$, is about 29%, the ratio of the cautious people, $q^2(0)$, is about 64% and the ratio of more cautious people, $q^3(0) + q^4(0)$, is about 7%.

In order to compare the degrees of agreement of BM and HBM with $m = 4$, we plot the sales data with them in Fig. 1. The circles are the cumulative iPod sales and the (purple) dashed and the (light blue) full curves represent BM and HBM with $m = 4$, respectively. HBM can approximate sales data which BM cannot do.

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**Table 1.** Parameters of HBM and LM minimizing the product of SARE and SAE, and SAREs by them. Numbers in parentheses mean results of fitting the data by HLM. All values are rounded to a three-digit number, and so the sum of $p(0)$ and $p^{\mu}(0)$ is not equal to one.

| | LM | $m = 1$ (BM) | $m = 2$ | $m = 3$ | $m = 4$ |
|---|---|---|---|---|---|
| $a$ | 1.42 | 1.42 | 4.22 | 4.25 | 4.50 |
| $b$ | – | $2.19 \times 10^{-10}$ | $7.16 \times 10^{-4}$ | $6.93 \times 10^{-4}$ | $7.10 \times 10^{-4}$ |
| $N$ [$\times 10^6$] | $1.28 \times 10^5$ | $1.07 \times 10^5$ | 65.7 | 65.6 | 66.4 |
| $p(0)$ | $9.26 \times 10^{-7}$ | $1.11 \times 10^{-6}$ | 0.00189 | 0.00193 | 0.00188 |
| $q^1(0)$ | 1.00 | 1.00 | 0.309 | 0.305 | 0.290 |
| $q^2(0)$ | – | – | 0.689 | 0.686 | 0.640 |
| $q^3(0)$ | – | – | – | 0.00739 | 0.00800 |
| $q^4(0)$ | – | – | – | – | 0.0600 |
| SARE | 1.84 | 1.84 | 0.999 (1.12) | 0.989 (1.02) | 0.984 (1.02) |

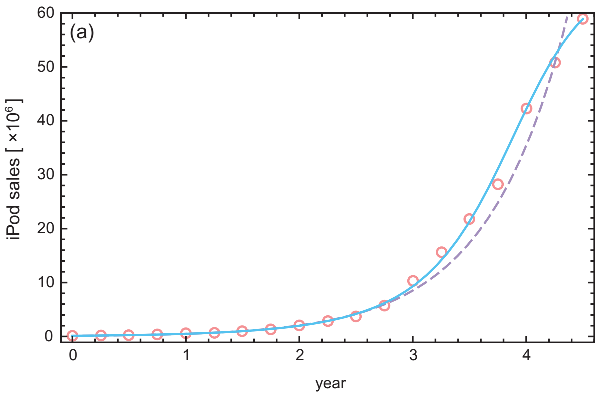


**Figure 1.** (color online) Cumulative iPod sales data represented by circles and fitting curves. The (purple) dashed and the (light blue) full curves are BM and HMB with $m = 4$, respectively.

## 5. Concluding remarks

In this paper, we made it clear that BM is based on imitation among people and influence from advertisements, and proposed the new model, HBM, including a memory of how many adopters or advertisements a non-adopter met which BM lacks. Additionally, we utilized the model to fit the iPod sales data, and so the better agreement was obtained than BM. The iPod sales increased gradually until they peaked for the first time in November 2005. HBM is far more suitable for describing such a slow pace than BM, because all the non-adopters in our society did not purchase the iPod quickly after they met adopters or advertisements.

### *5.1. Acknowledgments*

We would like to thank Suzuko Kawakami and members of astrophysics laboratory at Ochanomizu University for extensive discussions. This work was supported by JSPS Grant-in-Aid for Exploratory Research 12849926.

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