Journal of Physics: Conference Series **490** (2014) 012154 doi:10.1088/1742-6596/490/1/012154

Breakdown of the equivalence between gravitational mass and energy for a composite quantum body

Andrei G. Lebed

Department of Physics, University of Arizona, 1118 E. 4-th Street, Tucson, AZ 85721, USA and L.D. Landau Institute for Theoretical Physics, RAS, 2 Kosygina Street, Moscow, Russia

E-mail: `lebed@physics.arizona.edu`

Abstract. The simplest quantum composite body, a hydrogen atom, is considered in the presence of a weak external gravitational field. We define an operator for the passive gravitational mass of the atom in the post-Newtonian approximation of the general relativity and show that it does not commute with its energy operator. Nevertheless, the equivalence between the expectation values of the mass and energy is shown to survive at a macroscopic level for stationary quantum states. Breakdown of the equivalence between passive gravitational mass and energy at a microscopic level for stationary quantum states can be experimentally detected by studying unusual electromagnetic radiation, emitted by the atoms, supported by and moving in the Earth's gravitational field with constant velocity, using spacecraft or satellite

1. Introduction

The notion of gravitational mass for a composite body is known to be non-trivial in general relativity and related to several paradoxes. The role of the classical virial theorem in establishing of the equivalence between averaged over time gravitational mass and energy is discussed in detail in Refs.[1,2] for different types of classical composite bodies. In particular, for electrostatically bound two bodies, it is shown that the gravitational field is coupled to a combination $3K + 2U$, where K is the kinetic energy and U the potential energy. Since the classical virial theorem states that the following time average is equal to zero, $\langle 2K + U\rangle_t = 0$, we can conclude that averaged over time gravitational mass is proportional to the total amount of energy, E [1,2]:

$$\langle m^g\rangle_t = \langle 3K + 2U\rangle_t/c^2 = \langle K + U\rangle_t/c^2 = E/c^2. \tag{1}$$

2. Goal

The main goal of our paper is to study the quantum problem concerning the passive gravitational mass of a composite body. As the simplest example, we consider a hydrogen atom in the Earth's gravitational field, where we take into account only kinetic and Coulomb potential energies of an electron in a curved spacetime. We claim three main results in the paper (see also Refs. [3,4]). Our first result is that the equivalence between passive gravitational mass and energy in the absence of gravitational field survives, at the macroscopic level, in the quantum case. More strictly speaking, we show that the expectation value of the mass is equal to E/c^2 for stationary quantum states due to the quantum virial theorem [5]. Our second result is a breakdown of the equivalence between passive gravitational mass and energy, at the microscopic level, for stationary quantum states due to the fact that passive gravitational mass operator does not commute with energy operator. As a result, there exist a non-zero probability that a measurement of passive gravitational mass can yield a value, which is different from E/c^2, given by the Einstein's equation. Our third result is a suggestion of a realistic experiment to detect this inequivalence by measurement of electromagnetic radiation, emitted by a macroscopic ensemble of hydrogen atoms, supported by and moving in the Earth's gravitational field, using spacecraft or satellite.

3. Gravitational mass in classical physics

Here, we derive the Lagrangian and Hamiltonian of a hydrogen atom in the Earth's gravitational field, taking into account couplings of kinetic and potential Coulomb energies of an electron with a weak gravitational field. Note that we keep only terms of the order of $1/c^2$ and disregard all tidal effects. Therefore, we can write the interval, using a weak field approximation [6]:

$$ds^2 = -(1 + 2\phi/c^2)(cdt)^2 + (1 - 2\phi/c^2)(dx^2 + dy^2 + dz^2), \ \phi = -GM/R, \tag{2}$$

where G is the gravitational constant, c is the velocity of light, M is the Earth mass, R is the distance between center of the Earth and center of mass of a hydrogen atom (i.e., a proton). Then, in the local proper spacetime coordinates, which in the first approximation in ϕ/c^2 are

$$x' = (1 - \phi/c^2)x, \ y' = (1 - \phi/c^2)y, z' = (1 - \phi/c^2)z, \ t' = (1 + \phi/c^2)t, \tag{3}$$

the classical Lagrangian and action of an electron have the following standard forms:

$$L' = -m_e c^2 + \frac{1}{2} m_e (\mathbf{v}')^2 + \frac{e^2}{r'} \ , \quad S' = \int L' dt', \tag{4}$$

where m_e is the bare electron mass, e and $\mathbf{v}'$ are the electron charge and velocity, respectively; r' is a distance between electron and proton. It is possible to show that the Lagrangian (4) can be rewritten in coordinates (x, y, z, t) as

$$L = -m_e c^2 + \frac{1}{2} m_e \mathbf{v}^2 + \frac{e^2}{r} - m_e \phi - \left(3 m_e \frac{\mathbf{v}^2}{2} - 2 \frac{e^2}{r} \right) \frac{\phi}{c^2}, \tag{5}$$

which corresponds to the following Hamiltonian:

$$H = m_e c^2 + \frac{\mathbf{p}^2}{2m_e} - \frac{e^2}{r} + m_e \phi + \left(3 \frac{\mathbf{p}^2}{2m_e} - 2 \frac{e^2}{r} \right) \frac{\phi}{c^2}. \tag{6}$$

4. Gravitational mass in quantum physics

The Hamiltonian (6) can be quantized by substituting a momentum operator, $\hat{\mathbf{p}} = -i\hbar\partial/\partial\mathbf{r}$, instead of canonical momentum, $\mathbf{p}$:

$$\hat{H} = m_e c^2 + \frac{\hat{\mathbf{p}}^2}{2m_e} - \frac{e^2}{r} + \hat{m}_e^g \phi \ , \tag{7}$$

where we introduce passive gravitational mass operator of an electron to be proportional to its weight operator in a weak gravitational field (2),

$$\hat{m}_e^g = m_e + \left(\frac{\hat{\mathbf{p}}^2}{2m_e} - \frac{e^2}{r} \right) \frac{1}{c^2} + \left(2 \frac{\hat{\mathbf{p}}^2}{2m_e} - \frac{e^2}{r} \right) \frac{1}{c^2} \ . \tag{8}$$

Note that the Hamiltonian (7),(8) contains the virial contribution (i.e., the last term) to the mass operator and, therefore, does not commute with electron energy operator, taken in the absence of the field. Below, we discuss some consequences of Eqs.(7),(8). Suppose that we have a macroscopic ensemble of hydrogen atoms with each of them being in a ground state with energy E_1. Then the expectation value of the electron mass operator (8) per atom is

$$< \hat{m}^g_e >= m_e + \frac{E_1}{c^2} + \left\langle 2\frac{\hat{\mathbf{p}}^2}{2m_e} - \frac{e^2}{r} \right\rangle \frac{1}{c^2} = m_e + \frac{E_1}{c^2}, \tag{9}$$

where the third term in Eq.(9) is zero in accordance with the quantum virial theorem [5]. Therefore, we conclude that the equivalence between passive gravitational mass and energy survives at a macroscopic level for stationary quantum states. Let us discuss how Eqs.(7),(8) break the equivalence between passive gravitational mass and energy at a microscopic level. Here, we illustrate the above mentioned inequivalence, using the following thought experiment. Suppose that at $t = 0$ we create a ground state wave function of a hydrogen atom, corresponding to the absence of gravitational field,

$$\Psi_1(r,t) = \Psi_1(r)\exp(-iE_1 t/\hbar)\ . \tag{10}$$

In gravitational field (2), wave function (10) is not anymore a ground state. It is possible to show that a general solution of the Schrödinger equation, corresponding to the Hamiltonian (7),(8), can be written as

$$\Psi(r,t) = (1-\phi/c^2)^{3/2}\sum_{n=1}^{\infty} a_n \Psi_n[(1-\phi/c^2)r]\exp[-im_e c^2(1+\phi/c^2)t/\hbar]\exp[-iE_n(1+\phi/c^2)t/\hbar]. \tag{11}$$

We wish to draw attention to the fact that wave function (11) is a series of eigenfunctions of passive gravitational mass operator (8). $\Psi_n(r)$ is a normalized wave function of an electron in a hydrogen atom in the absence of gravitational field, corresponding to energy E_n.

In accordance with quantum mechanics, probability that, at $t > 0$, an electron occupies excited state with energy $m_e c^2(1+\phi/c^2) + E_n(1+\phi/c^2)$ is

$$\begin{aligned} P_n = |a_n|^2,\ a_n = \int \Psi_1^*(r)\Psi_n[(1-\phi/c^2)r]d^3\mathbf{r} = -(\phi/c^2)\int \Psi_1^*(r) r \Psi_n'(r) d^3\mathbf{r}, \\ \int \Psi_1^*(r) r \Psi_n'(r) d^3\mathbf{r} = V_{n,1}/(\hbar\omega_{n,1}),\ \hbar\omega_{n,1} = E_n - E_1,\ n \neq 1, \\ V_{n,1} = \int \Psi_1^*(r)\hat{V}(\mathbf{r})\Psi_n(r)d^3\mathbf{r},\ \ \hat{V}(\mathbf{r}) = 2\frac{\hat{\mathbf{p}}^2}{2m_e} - \frac{e^2}{r}. \end{aligned} \tag{12}$$

Let us discuss Eq.(12). We stress that it directly demonstrates that there is a finite probability,

$$P_n = |a_n|^2 = (\phi/c^2)^2\ [V_{n,1}/(E_n - E_1)]^2\ ,\ n \neq 1, \tag{13}$$

that, at $t > 0$, an electron occupies n-th ($n \neq 1$) energy level, which breaks the expected Einstein's equation, $m^g_e = m_e + E_1/c^2$. In fact, this means that quantum measurement of passive gravitational mass in a quantum state with a definite energy (10) gives the following quantized values:

$$m^g_e(n) = m_e + E_n/c^2\ . \tag{14}$$

5. Suggested experiment

Here, we describe a realistic experiment: We consider a hydrogen atom being in its ground state at $t = 0$, located at distance R' from the center of the Earth and having the wave function:

$$\tilde{\Psi}_1(r,t) = (1-\phi'/c^2)^{3/2}\Psi_1[(1-\phi'/c^2)r]\exp[-im_e c^2(1+\phi'/c^2)t/\hbar]\exp[-iE_1(1+\phi'/c^2)t/\hbar]. \quad (15)$$

The atom is supported by in the Earth's gravitational field and moving from the Earth at constant velocity, $u \ll \alpha c$ (where α is the fine structure constant), by spacecraft or satellite. It is possible to show that the electron wave function and the time dependent perturbation for the Hamiltonian (7)-(8), in the coordinate system related to center of mass of the atom, can be expressed as

$$\tilde{\Psi}(r,t) = (1-\phi'/c^2)^{3/2}\sum_{n=1}^{\infty}\tilde{a}_n(t)\Psi_n[(1-\phi'/c^2)r]\exp[-im_e c^2(1+\phi'/c^2)t/\hbar]$$
$$\exp[-iE_n(1+\phi'/c^2)t/\hbar], \quad (16)$$

$$\hat{U}(\mathbf{r},t) = \frac{\phi(R'+ut)-\phi(R')}{c^2}\left(3\frac{\hat{\mathbf{p}}^2}{2m_e} - 2\frac{e^2}{r}\right). \quad (17)$$

We wish to draw attention that in a spacecraft (satellite), which moves with constant velocity, gravitational force, which acts on each individual hydrogen atom, is compensated by some non-gravitational forces. This causes very small changes of a hydrogen atom energy levels and is not important for our calculations. Therefore, the atoms do not feel directly gravitational acceleration, $\mathbf{g}$, but feel, instead, gravitational potential, $\phi(R") = \phi(R'+ut)$.

It is important that, if excited levels of a hydrogen atom were strictly stationary, then probability to find the passive gravitational mass to be quantized with $n \neq 1$ (14) would be

$$\tilde{P}_n = (V_{n,1}/\hbar\omega_{n,1})^2[\phi(R')/c^2]^2\ , n \neq 1,\ |\phi(R'+ut)| \ll |\phi(R')|, u \ll \omega_{n,1}R, \quad (18)$$

which coincides with Eq. (13). In reality, the excited levels spontaneously decay with time and, therefore, it is possible to observe the quantization law (14) indirectly by measuring electromagnetic radiation from a macroscopic ensemble of the atoms. In this case, Eq.(18) gives a probability that a hydrogen atom emits a photon with frequency $\omega_{n,1} = (E_n - E_1)/\hbar$ during the time of the experiment. To estimate the probabilities (18) we use the following numerical values of the Earth's mass, $M \simeq 6 \times 10^{24} kg$, and its radius, $R_0 \simeq 6.36 \times 10^6 m$. It is important that, although the probabilities (18) are small, the number of photons, N, emitted by macroscopic ensemble of the atoms, can be large. For instance, for 1000 moles of hydrogen atoms, N is estimated as

$$N(n \to 1) = 2.95 \times 10^8\ [V_{n,1}/(E_n - E_1)]^2, \quad N(2 \to 1) = 0.9 \times 10^8, \quad (19)$$

which can be experimentally detected. [Here, $N(n \to 1)$ stands for a number of photons, emitted with frequency $\omega_{n,1} = (E_n - E_1)/\hbar$.]

Acknowledgments

We are thankful to N.N. Bagmet (Lebed), V.A. Belinski, Steven Carlip, Li-Zhi Fang, Douglas Singleton, and V.E. Zakharov for useful discussions. This work was supported by the NSF under Grant DMR-1104512.

References

[1] Nordtvedt K 1994 *Class. Quantum Grav.* **11**, A119
[2] Carlip S 1998 *Am. J. Phys.* **66**, 409
[3] Lebed A G 2011 *Preprint* arXiv:1111.5365v1
[4] Lebed A G 2012 *Preprint* arXiv:1205.3134v1
[5] Park D 2005 *Introduction to the Quantum Theory, 3-rd Edition* (Dover Publications, New York)
[6] Misner C W, Thorne K S and Wheeler J A 1973 *Gravitation* (W.H. Freeman and Co, New York)