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Towards the Dressing Phase in the AdS_3/CFT_2 Duality

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Abstract. In this paper we report some recent progresses and open problems in the determination of the first quantum correction at strong coupling of the dressing phases appearing in the Bethe Ansatz equation which have been conjectured to describe the spectrum of string theory on the $AdS_3 \times S^3 \times T^4$ background.

1. Introduction

Besides the two most studied examples of integrable theories in the AdS/CFT duality ($AdS_5 \times S^5$ and $AdS_4 \times CP^3$), other backgrounds allow for (classically) integrable string theories; recently the background $AdS_3 \times S^3 \times T^4$ got much attention. A first set of Bethe Ansatz (BA) equations describing the asymptotic spectrum of the theory has been conjectured in [1] followed by a second proposal in [2]. These equation, as in the $AdS_{5,4}$ cases, contain a "dressing factor" (two different phases appear in the BA for $AdS_3 \times S^3 \times T^4$), which is not completely fixed by the symmetry of the problem; at strong coupling, the leading term of the dressing phase is required to be the AFS phase [3], to match the classical finite gap limit of the model. The aim of this paper is to report some progresses [4, 5] in understanding the first quantum correction of the dressing phases at strong coupling, i.e. the determination of the $c^{(1)}_{r,s}$ coefficients appearing in the expansion of the scattering phase of elementary magnons with momenta p_j and p_k:

$$\vartheta(p_j, p_k) = 2\sum \; c_{r,s}(\lambda)\left(\frac{\lambda}{16\,\pi^2}\right)^{\frac{r+s-1}{2}}\Big[q_r(p_j)\,q_s(p_k) - q_s(p_j)\,q_r(p_k)\Big]$$

where $q_n(p)$ is the elementary magnon n-th charge,

$$c_{r,s}(\lambda) = c^{(0)}_{r,s} + \frac{1}{\sqrt{\lambda}}\,c^{(1)}_{r,s} + \dots$$

and the $c^{(0)}_{r,s}$ are the classical AFS phase.

2. Algebraic Curve determination of the $c^{(1)}_{r,s}$

The 1-loop correction to the dressing phase and the $c^{(1)}_{r,s}$ coefficients can be computed, in principle in a very general way, without refering to any particular solution, in the Algebraic Curve (AC) framework, following the approach of [6]. We will need anyway an independent calculation of the 1-loop energy, done for the particular solution of a rigid circular string using the World-Sheet (WS) method: we need this second result both to understand regularization issues and to check the independence of the result from the missing modes in the AC, realtive to the T^4 component of the background.

In the semiclassical quantisation of the AC, the first correction to the phase is encoded in a set of potentials V_I correcting the classical quasimomenta that characterize the curve: these potentials compute the effect of the quantum fluctuations around the classical solution; for each quasi-momentum p_I the correction V_I is obtained summing over all the fluctuations connecting the sheets of the AC. The total phase corrections to the Bethe equations are obtained by evaluating $\mathcal{V} = V_I - V_J$.For the middle node 2 equation (the other BA equations are not corrected) $\mathcal{V}(x) \equiv V_2(x) - V_3(x)$

$$\mathcal{V}(x) = \int_{-1}^{1} \frac{dy}{2\pi} \left[\left(G_2(y) + \overline{G}_{\overline{2}}(y) \right)' \frac{\widehat{\alpha}(x)}{x-y} + \left(\overline{G}_2(y) + G_{\overline{2}}(y) \right)' \frac{\widehat{\alpha}(1/x)}{1/x - y} \right]. \tag{1}$$

where x is the spectral parameter,

$$\widehat{\alpha}(x) = \frac{4\pi}{\sqrt{\lambda}} \frac{x^2}{x^2-1} , \tag{2}$$

and G the discrete resolvent defined in terms of the Bethe roots $x_{a,k}$ as:

$$G_a(x) = \sum_{k=1}^{K_a} \frac{\widehat{\alpha}(x_{a,k})}{x - x_{a,k}}, \qquad \overline{G}(x) = G(1/x) \tag{3}$$

Expanding the resolvent in terms of the conserved charges Q_r, we get:

$$\mathcal{V}_2(x) = \frac{\widehat{\alpha}(x)}{2\pi} \sum_{r=2}^{\infty} \sum_{s=1}^{\infty} \widehat{c}_{r,s} \frac{Q_r}{x^s} , \qquad \widehat{c}_{r,s} = -4 \left(1 - \frac{1}{2}\delta_{s,1} \right) \frac{1-(-1)^{r+s}}{2} \frac{r-1}{r+s-2} .$$

This result can't be interpreted as a phase, since the coefficient are not antisymmetric in the r, s indeces; moreover, if we apply the AC method to the case of the $SU(2)$ circular string, we get for the 1-loop contribution of the dressing phase to the energy ($\mathcal{J}$ is the angular momentum of the string, m the winding number):

$$\delta E_1^{AC} = \underbrace{\frac{1}{\sqrt{m^2+\mathcal{J}^2}} \left(m^2 + \mathcal{J}^2 \log \frac{\mathcal{J}^2}{m^2+\mathcal{J}^2} \right)}_{\delta E_1^{AdS_3}} - \underbrace{\frac{m^2 \left(2\mathcal{J} \left(\mathcal{J} - \sqrt{\mathcal{J}^2+m^2} \right) + m^2 \right)}{2\,(\mathcal{J}^2+m^2)^{3/2}}}_{\Delta E_1} ,$$

While, computing the 1-loop correction to the energy of the circular string in the WS framework, and isolating the dressing part as in [7], the result is

$$\delta E_1^{AdS_3} = \frac{1}{\sqrt{\mathcal{J}^2+m^2}} \left(m^2 + \mathcal{J}^2 \log \frac{\mathcal{J}^2}{m^2+\mathcal{J}^2} \right) \tag{4}$$

where ΔE_1 is a discrepancy between the two approaches.

Both problems, the non-antisymmetry of the coefficients and the mismatch ΔE_1, can be traced back to a regularization issue in the sum over the frequencies of the quantum fluctuations: in the WS approach the natural cut-off is a common mode number N, while in the AC approach the cut-off is a common radius for the contour integral in the spectral plane defining the potentials; this difference is translated in a reordering of the terms in the sum over the frequences to compute the 1-lopp energy $E_1 \sim \sum_{n\in Z}(\omega_n^B - \omega_n^F)$, and is the origin of the ΔE_1 term. We can repeat the calculation of $\mathcal{V}$ enforcing the regularization of the AC curve to reproduce the WS result, solving both problems at once: indeed, with the new prescription, we get rid of the ΔE_1 term obtaining full agreement with WS prediction and the the results for the new potential $\widehat{\mathcal{V}}$ and the coefficients change in:

$$\widehat{\mathcal{V}}(x) = \frac{\widehat{\alpha}(x)}{2\,\pi}\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{c^{(1)}_{r,s}\,Q_r - \overline{c}^{(1)}_{r,s}\,\overline{Q}_r}{x^s},$$
$$c^{(1)}_{r,s} = 2\,\frac{1-(-1)^{r+s}}{2}\,\frac{s-r}{r+s-2}\;,\quad \overline{c}^{(1)}_{r,s} = -2\,\frac{1-(-1)^{r+s}}{2}\,\frac{r+s-2}{s-r} \tag{5}$$

Using the new regularization, we can now compute the scattering phases between magnons, following [6]: We identify the dressing phase contribution in the BA as:

$$e^{i\,\widehat{\mathcal{V}}(x_{2,i})} = \prod_{k\neq j}^{K_2} e^{i\,\vartheta(x_{2,j},x_{2,k})}\prod_{k}^{K_{\overline{2}}} e^{-i\,\widetilde{\vartheta}(x_{2,j},x_{\overline{2},k})}. \tag{6}$$

and the result for the two phases is

$$\vartheta(x,y) \;=\; -\frac{\widehat{\alpha}(x)\,\widehat{\alpha}(y)}{2\,\pi\,(x-y)^2}\Big[2\,\log\Big(\frac{x+1}{x-1}\,\frac{y-1}{y+1}\Big) + 2\,\frac{(x-y)\,(x^2+y^2-2)}{(x^2-1)(y^2-1)}\Big], \tag{7}$$
$$\widetilde{\vartheta}(x,y) \;=\; -\frac{\widehat{\alpha}(x)\,\widehat{\alpha}(y)}{2\,\pi\,(1-x\,y)^2}\Big[2\,\log\Big(\frac{x+1}{x-1}\,\frac{y-1}{y+1}\Big) - 2\,\frac{(x-y)\,(x^2y^2-1)}{(x^2-1)\,(y^2-1)}\Big]\;. \tag{8}$$

3. Folded string and open problems

The choice of regularization used in the previous section allows to separate the original potential $\mathcal{V} = \mathcal{V}_{phase} + \delta\mathcal{V}$, where only the first part can be consistently interpreted as a phase, while the second is understood as a regularization effect. Nevertheless the prescription is based on the agreement between the AC and WS approaches in the particular case of the $SU(2)$ curcular string. If we consider as a second example the $SL(2)$ folded string solution, we can repeat the same steps: computing the 1-loop dressing contribution to the energy for the folded string, the AC and WS give different result, and the mismatch is again due to the different regularizations. While we can remove the discrepancy and get a consistent result on the string theory side, if we compare the string prediction with the BA, using the phase derived in the previous section we still have a discrepancy.

$$\text{WS} \equiv \text{AC-reg. mismatch}: \qquad E_1^{\text{dressing}} = \frac{\coth^{-1}\Big(\sqrt{\mathcal{J}^2+1}\Big)}{2\mathcal{J}^3\sqrt{\mathcal{J}^2+1}}\,\mathcal{S}^2 + \mathcal{O}(\mathcal{S}^3),$$
$$\text{BA with } c_{r,s} \text{ coeff.}: \qquad E_1^{\text{dressing}} = \Big[\frac{\coth^{-1}\Big(\sqrt{\mathcal{J}^2+1}\Big)}{2\mathcal{J}^3\sqrt{\mathcal{J}^2+1}} + \frac{1}{2\,\mathcal{J}^4\,\sqrt{\mathcal{J}^2+1}}\Big]\,\mathcal{S}^2 + \mathcal{O}(\mathcal{S}^3).$$

where the first line is the string theory result for a folded string with semiclassical spin $\mathcal{S}$ and angular momentum $\mathcal{J}$, the second line is the result obtained from the BA equations, assuming the coefficients $c_{r,s}$ for the dressing phase.

This disagreement calls for a deeper understanding of the role of $\delta\mathcal{V}$; while the regularization/antisymmetrization prescription works perfectly in the $SU(2)$ sector, the comparison between string theory and BA predictions for more general cases is still problematic.

Another open issue is related to the fact that the phases in eq.(7) don't satisfy the crossing relations found in [2]: to get crossing respecting phases the coefficients $c_{r,s}$ with $r = 1$ should be modified with an additional factor 1/2 [8, 9]. The relative phases for the magnon scattering can be found starting from the original potential in eq.(1), and making the resulting phase antisymmetric by hand $\theta^{AC,asym}(x,y) = 1/2[\theta^{AC}(x,y) - \theta^{AC}(y,x)]$. But this phase, while crossing symmetric, gives a contribution to the 1-loop energy not in agreement with the string theory WS prediction, even in the case of the circular string solution.

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