Journal of Physics: Conference Series **490** (2014) 012111 doi:10.1088/1742-6596/490/1/012111

Numerical solution of unsteady generalized Newtonian and Oldroyd-B fluids flow by dual time-stepping method

R Keslerová, K Kozel

Czech Technical University in Prague, Faculty of Mechanical Engineering, Dep. of Technical Mathematics, Czech Republic

E-mail: Radka.Keslerova@fs.cvut.cz, Karel.Kozel@fs.cvut.cz

Abstract. This work deals with the numerical solution of viscous and viscoelastic fluids flow. The governing system of equations is based on the system of balance laws for mass and momentum for incompressible laminar fluids. Different models for the stress tensor are considered. For viscous fluids flow Newtonian model is used. For the describing of the behaviour of the mixture of viscous and viscoelastic fluids Oldroyd-B model is used. Numerical solution of the described models is based on cell-centered finite volume method in conjunction with artificial compressibility method. For time integration an explicit multistage Runge–Kutta scheme is used. In the case of unsteady computation dual-time stepping method is considered. The principle of dual-time stepping method is following. The artificial time is introduced and the artificial compressibility method in the artificial time is applied.

1. Mathematical model

The governing system of equations is the system of balance laws of mass and momentum for incompressible fluids. This system is completed by the equation for a viscoelastic part of stress tensor, [1]:

$$\text{div } \boldsymbol{u} = 0 \tag{1}$$

$$\rho\frac{\partial \boldsymbol{u}}{\partial t} + \rho(\boldsymbol{u}.\nabla)\boldsymbol{u} = -\nabla P + \text{div } \mathbf{T_s} + \text{div } \mathbf{T_e} \tag{2}$$

$$\frac{\partial \mathbf{T}_e}{\partial t} + (\boldsymbol{u}.\nabla)\mathbf{T}_e = \frac{2\mu_e}{\lambda_1}\mathbf{D} - \frac{1}{\lambda_1}\mathbf{T}_e + (\mathbf{W}\mathbf{T}_e - \mathbf{T}_e\mathbf{W}) + (\mathbf{D}\mathbf{T}_e + \mathbf{T}_e\mathbf{D}) \tag{3}$$

where P is the pressure, ρ is the constant density, $\boldsymbol{u}$ is the velocity vector. The symbols $\mathbf{T_s}$ and $\mathbf{T_e}$ represent the Newtonian and viscoelastic parts of the stress tensor and

$$\mathbf{T}_s = 2\mu_s\mathbf{D}, \qquad \mathbf{T}_e + \lambda_1\frac{\delta \mathbf{T}_e}{\delta t} = 2\mu_e\mathbf{D} \tag{4}$$

where $\mathbf{D}$ is symmetric part of the velocity gradient $\mathbf{D} = \frac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T)$ and $\mathbf{W}$ is antisymmetric part of the velocity gradient $\mathbf{W} = \frac{1}{2}(\nabla\boldsymbol{u} - \nabla\boldsymbol{u}^T)$.

Content from this work may be used under the terms of the Creative Commons Attribution 3.0 licence. Any further distribution of this work must maintain attribution to the author(s) and the title of the work, journal citation and DOI.
Published under licence by IOP Publishing Ltd

Both models could be generalized. In this case the viscosity μ is defined by viscosity function according to the cross model (for more details see [2])

$$\mu(\dot{\gamma}) = \mu_\infty + \frac{\mu_0 - \mu_\infty}{(1 + (\lambda\dot{\gamma})^b)^a}, \quad \dot{\gamma} = 2\sqrt{\frac{1}{2}\text{tr } \mathbf{D}^2} \tag{5}$$

where $\mu_0 = 1.6 \cdot 10^{-1} Pa.s, \mu_\infty = 3.6 \cdot 10^{-3} Pa.s, a = 1.23, b = 0.64, \lambda = 8.2s$.

2. Numerical solution

2.1. Steady case

Numerical solution of the described models is based on cell-centered finite volume method using explicit Runge–Kutta time integration. Steady state solution is achieved for $t \to \infty$. In this case the artificial compressibility method can be applied. It means that the continuity equation is completed by the time derivative of the pressure (for more details see e.g. [3], [4], [5]). The system of equations (including the modified continuity equation) could be rewritten in the vector form.

$$\tilde{R}_\beta W_t + F^c_x + G^c_y = F^v_x + G^v_y, \quad \tilde{R}_\beta = \text{diag}(\frac{1}{\beta^2}, 1, 1), \quad \beta \in \mathbf{R}^+ \tag{6}$$

where W is vector of unknowns, F^c, G^c are inviscid fluxes and F^v, G^v are viscous fluxes.

Eq. 6 is discretized in space by the finite volume method and the arising system of ODEs is integrated in time by the explicit multistage Runge-Kutta scheme [6], [7].

Steady boundary conditions The flow is modelled in a bounded computational domain where a boundary is divided into three mutually disjoint parts: an inlet, an outlet and a solid wall. At the inlet Dirichlet boundary condition for velocity vector is used. At the outlet the pressure value is given. The homogeneous Dirichlet boundary condition for the velocity vector is used on the wall.

2.2. Unsteady case

For the unsteady computation the dual-time stepping method is used. The principle of dual-time stepping method is following. The artificial time τ is introduced and the artificial compressibility method in the artificial time is applied. The system of Navier-Stokes equations is extended to unsteady flows by adding artificial time derivatives $\partial W/\partial\tau$ to all equations, for more details see [8]

$$\tilde{R}_\beta W_\tau + \tilde{R} W_t + F^c_x + G^c_y = F^v_x + G^v_y, \quad \tilde{R} = \text{diag}(0, 1, 1), \quad \tilde{R}_\beta = \text{diag}(\frac{1}{\beta^2}, 1, 1). \tag{7}$$

The derivatives with respect to the real time t are discretized using a three-point backward formula, it defines the form of unsteady residual

$$\tilde{R}_\beta \frac{W^{l+1} - W^l}{\Delta\tau} = -\tilde{R}\frac{3W^{l+1} - 4W^n + W^{n-1}}{2\Delta t} - \text{Res}(W)^l = -\overline{\text{Res}}(W)^{l+1}, \tag{8}$$

where $\Delta t = t^{n+1} - t^n$ and $\text{Res}(W)$ is the steady residual. The symbol $\overline{\text{Res}}(W)$ denotes unsteady residual.

Unsteady boundary condition In the inlet, in the solid wall and in one of the outlet part the steady boundary conditions are prescribed. In the branch going up the pressure value is prescribed by the function

$$p_{out} = \frac{1}{4}\left(1 + \frac{1}{2}\sin(\omega t)\right), \tag{9}$$

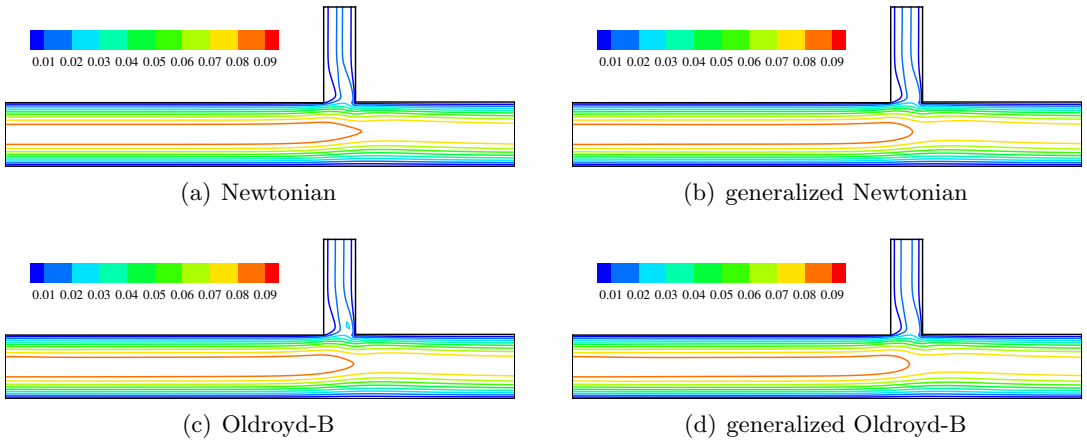
where ω is the angular velocity defined as $\omega = 2\pi f$, where f is the frequency.

3. Numerical results

3.1. Steady numerical results

In this section the steady numerical results of two dimensional incompressible laminar viscous and viscoelastic flows for generalized Newtonian and Oldroyd-B fluids flow are presented. The following model parameters are used: $\mu_e = 0.001Pa.s$, $\mu_s = 0.009Pa.s$, $\lambda_1 = 0.06s$, $U_0 = 0.1m.s^{-1}$, $L_0 = 0.01m$, $\rho = 1000kg.m^{-3}$.

In Fig. 1 velocity isolines for tested fluids cases are presented.



(a) Newtonian (b) generalized Newtonian

(c) Oldroyd-B (d) generalized Oldroyd-B

Figure 1. Velocity isolines of steady flows for generalized Newtonian and Oldroyd-B fluids.

3.2. Unsteady numerical results

The used method is the the dual-time stepping method with artificial compressibility coefficient $\beta = 1.0m.s^{-1}$. The frequency $f = 2H$ is used. In Figs. 2 and 3 graphs of velocity as the function of time and the velocity distribution are shown. First pictures show graphs of velocity as the function of the time. The square symbols mark positions in time of the snapshots shown in next three pictures during one period. As initial data the numerical solution of steady fully developed flow of generalized Newtonian fluid was used (see Figs. 1).

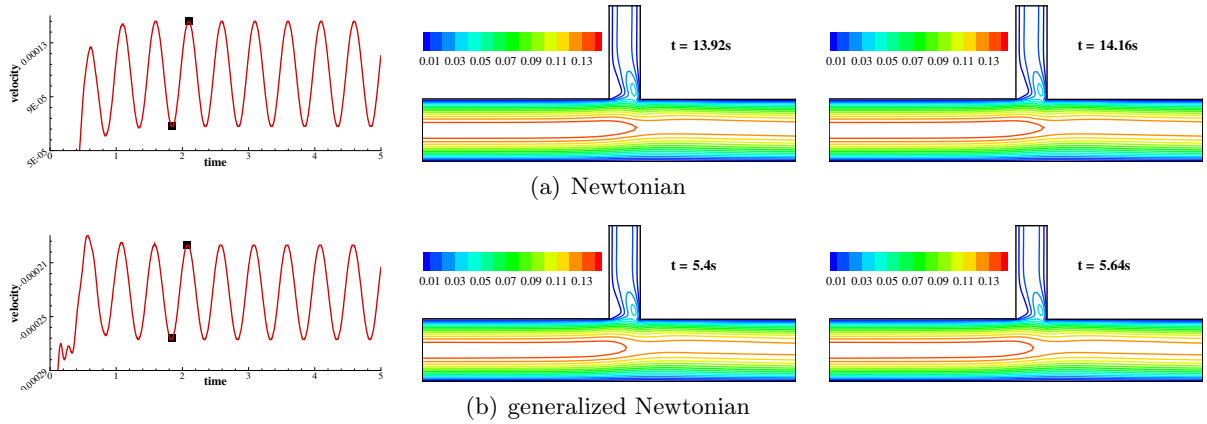
4. Conclusions

In this paper a finite volume solver for incompressible laminar viscous flows in the branching channel for two dimmensional case was described. Newtonian model was generalized for generalized Newtonian fluids flow. Power-law model with different values of power-law index were used. Three different values of this coefficient for corresponding Newtonian, shear thickening and shear thinning fluids flow were tested. The explicit Runge-Kutta method was considered for time integrating. The convergence history confirms robustness of the applied method. The numerical results obtained by this method were presented and compared.

Two unsteady approaches were considered, the artificial compressibility method and the dual-time stepping method. Both methods were applied for generalized Newtonian fluids with initial data obtained by steady numerical computation. In this computations two different values of frequency were used.

Acknowledgments

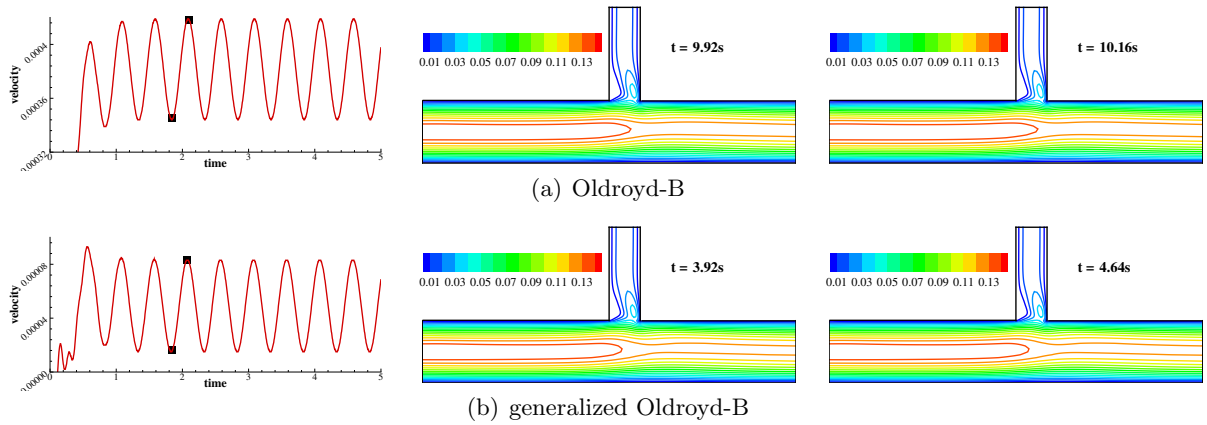
This work was partly supported by the grant GACR 201/09/0917 and SGS13/174/OHK2/3T/12.



(a) Newtonian

(b) generalized Newtonian

Figure 2. The graphs of the velocity as the function of time and velocity isolines of unsteady flow for Newtonian and generalized Newtonian fluids.



(a) Oldroyd-B

(b) generalized Oldroyd-B

Figure 3. The graphs of the velocity as the function of time and velocity isolines of unsteady flow for Oldroyd-B and generalized Oldroyd-B fluids.

References

[1] Bodnar T and Sequeira A 2010 *Advances in Mathematical Fluid Mechanics* ed R. Ranacher, A. Sequeira pp 83–104.
[2] Vimmr J and Jonášová A 2010 *Mathematics and Computers in Simulation* vol 80 pp 1324–1336
[3] Chorin AJ 1967 *Journal of Computational Physics* vol 135 pp 118–125
[4] Dvořák R and Kozel K 1996 *Mathematical Modelling in Aerodynamics* (CTU, Prague, Czech Republic)
[5] Louda P and Kozel K and Příhoda J and Beneš L and Kopáček T 2011 *Journal of Computers & Fluids* vol 46 pp 318–324
[6] Keslerová R and Kozel K 2010 *Journal of Mathematics and Computers in Simulation* vol 80 pp 1783–1794
[7] LeVeque R 2004 *Finite-Volume Methods for Hyperbolic Problems* (Cambridge University Press)
[8] Honzátko R 2007 *Numerical simulations of incompressible flows with dynamical and aeroelastic effects* (PhD thesis, CTU Prague, Czech Republic)