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Numerical estimates for the regularization of nonautonomous ill-posed problems

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Abstract. The regularization of ill-posed problems has become a useful tool in studying initial value problems that do not adhere to certain desired properties such as continuous dependence of solutions on initial data. Because direct computation of the solution becomes difficult in this situation, many authors have alternatively approximated the solution by the solution of a closely-defined well-posed problem. In this paper, we demonstrate this process of regularization for the backward heat equation with a time-dependent diffusion coefficient, among other nonautonomous ill-posed problems. In the process, we provide two different approximate well-posed models and numerically compare convergence rates of their solutions to a known solution of the original ill-posed problem.

1. Introduction

In this paper, motivated by the recent work of Trong and Tuan [9, 10], we illustrate regularization for certain nonautonomous ill-posed problems and demonstrate an array of numerical estimates for the regularization of the backward heat equation in $L^2[0,\pi]$ with a time-dependent diffusion coefficient, e.g.

$$\begin{aligned} \frac{\partial u}{\partial t} &= -e^t \frac{\partial^2 u}{\partial x^2}, \quad 0<x<\pi,\ 0<t<1 \\ u(0,t) &= u(\pi,t)=0, \quad 0<t<1 \\ u(x,0) &= \phi(x), \quad 0<x<\pi. \end{aligned} \tag{1}$$

Because solutions u may not exist or, if they do exist, generally do not depend continuously on initial data, numerical calculations are often difficult. The regularization of the problem offers an alternate method of obtaining information about such solutions u through comparison to the solution of an approximate well-posed problem. For example, for $\epsilon > 0$, one may consider an approximate solution v_ϵ where the model in (1) is replaced by Lattes and Lions's quasi-reversibility method $\frac{\partial v}{\partial t} = -e^t \frac{\partial^2 v}{\partial x^2} - \epsilon \frac{\partial^4 v}{\partial x^4}$ (cf. [5]).

Regularization for ill-posed problems, particularly the abstract Cauchy problem $\frac{du}{dt} = Au(t), \quad 0 \le t < T, \quad u(0) = \phi$, where A is an operator in a Banach space X, e.g. $A = -\Delta$ in $X = L^p(\mathbb{R})$, has been established in various settings by authors including Lattes and Lions [5], Miller [7], Showalter [8], and more recently Mel'nikova and Filinkov [6], Ames and Hughes [1], and Huang and Zheng [3, 4]. Very recently, error estimates for regularization of the backward heat equation have been scrutinized by Trong and Tuan (cf. [9, 10]). In [9], using the quasi-reversibility method, they determine a table of numerical estimates for $\|u(x,0.5)-v_\epsilon(x,0.5)\|_{L^2(0,\pi)}$ depending on different values of ϵ limiting to 0.

The tables that we provide in this paper extend the work of Trong and Tuan to the *nonautonomous* equation (1) and also encompass the difference $\|u(x,t)-v_\epsilon(x,t)\|_2$ for multiple values of t in $[0,1]$. Further, our paper illustrates two methods of regularization for other higher order nonautonomous ill-posed problems. The results of this paper find that the quasi-reversibility method yields a much faster convergence of the well-posed solution v_ϵ to the solution u of the ill-posed problem, as compared to a second method following Showalter [8].

2. Regularization

In this section, we provide background theory which explains the way in which problem (1), among other higher-order equations, may be regularized by the methods we propose. Consider the nonautonomous ill-posed problem

$$\frac{du}{dt} = a(t)A^k u(t), \quad 0 \le t < T, \quad u(0) = \phi \tag{2}$$

in a Hilbert space H where A is a positive, self-adjoint operator in H, k is a positive integer, and $a \in C([0,T] : \mathbb{R}^+)$. The regularization of this problem is defined as follows.

Definition 1. [4, Definition 3.1] A family $\{R_\epsilon(t) \mid \epsilon > 0,\ t \in [0,T]\}$ of bounded linear operators on X is called a *family of regularizing operators for the problem* (2) if for each solution $u(t)$ of (2) with initial data $\phi \in H$, and for any $\delta > 0$, there exists $\epsilon(\delta) > 0$ such that

(i) $\epsilon(\delta) \to 0$ as $\delta \to 0$,
(ii) $\|u(t) - R_{\epsilon(\delta)}(t)\phi_\delta\| \to 0$ as $\delta \to 0$ for $0 \le t \le T$ whenever $\|\phi - \phi_\delta\| \le \delta$.

In order to define a regularizing family, we consider an approximate well-posed problem

$$\frac{dv}{dt} = f_\epsilon(t,A)v(t), \quad 0 \le t < T, \quad v(0) = \phi \tag{3}$$

where $\epsilon > 0$ and $f_\epsilon(t,A)$ is defined by either of two examples $f_\epsilon(t,A) = a(t)A^k - \epsilon A^{k+1}$ or $f_\epsilon(t,A) = a(t)A^k(I + \epsilon A^k)^{-1}$, the first being a generalization of Lattes and Lions's quasi-reversibility method [5] and the second motivated by work of Showalter [8]. In [2], Fury and Hughes show under certain stabilizing conditions on a known solution $u(t)$ of (2), e.g. $\|u(T)\| \le M'$, that $\|u(t) - v_\epsilon(t)\| \le C\epsilon^{1-\frac{t}{T}}M^{\frac{t}{T}}, 0 \le t < T$ where $v_\epsilon(t) = e^{\int_0^t f_\epsilon(\tau,A)d\tau}\phi$ is the unique solution of (3) and C and M are constants independent of ϵ ([2, Theorem 3.9, Example 4.1, Example 4.2]).

Hence, with the aid of this inequality, we may define $R_\epsilon(t) = e^{\int_0^t f_\epsilon(\tau,A)d\tau}$ as a regularizing operator in which case $v_\epsilon(t) = R_\epsilon(t)\phi$. Regularization is then established by the calculation $\|u(t) - R_\epsilon(t)\phi_\delta\| \le \|u(t) - v_\epsilon(t)\| + \|R_\epsilon(t)\phi - R_\epsilon(t)\phi_\delta\| \to 0$ as $\delta \to 0$ where the first quantity tends to 0 if $\epsilon \to 0$ as $\delta \to 0$, while the second quantity on the right tends to 0 as $\delta \to 0$ because (3) is well-posed (and so satisfies continuous dependence on initial data). Choosing ϵ in terms of δ typically depends on the growth order of $\|R_\epsilon(t)\|$. Generally, this is not difficult and will be illustrated explicitly in the following section.

3. Numerical Estimates

Following recent work of Trong and Tuan [9, 10], we illustrate the process of regularization discussed in Section 2 for concrete partial differential equations represented by (2) and provide tables of numerical estimates for the regularization.

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Example 1. Let us consider, in $H = (L^2[0,\pi], \|\cdot\|_2)$ where $\|\phi\|_2 = (\int_0^\pi |\phi(x)|^2 dx)^{1/2}$ for $\phi \in L^2[0,\pi]$, the following nonautonomous partial differential equation by setting $A = -\Delta$ and $a(t) = e^t$ in (2):

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= (-1)^k e^t \frac{\partial^{2k} u}{\partial x^{2k}}, \quad 0 < x < \pi,\ 0 < t < 1 \\
u(0,t) &= u(\pi,t) = 0, \quad 0 < t < 1 \\
u(x,0) &= e \sin x, \quad 0 < x < \pi.
\end{aligned} \tag{4}
$$

The problem (4) is ill-posed with exact solution $u(x,t) = e^{e^t} \sin x$. In order to illustrate regularization for (4) we will consider a perturbed initial data $\phi_n(x) = e \sin x + \frac{1}{n}\sin(nx)$ which converges to $u(x,0) = e \sin x$ as $n \to \infty$. The approximate well-posed problem (3) according to the first approximation $f_\epsilon(t,A) = e^t A^k - \epsilon A^{k+1}$ becomes

$$
\begin{aligned}
\frac{\partial v}{\partial t} &= (-1)^k \left(e^t \frac{\partial^{2k} v}{\partial x^{2k}} + \epsilon \frac{\partial^{2k+2} v}{\partial x^{2k+2}} \right), \quad 0 < x < \pi,\ 0 < t < 1 \\
v(0,t) &= v(\pi,t) = 0, \quad 0 < t < 1 \\
v(x,0) &= e \sin x + \frac{1}{n}\sin(nx), \quad 0 < x < \pi
\end{aligned} \tag{5}
$$

with solution $v_\epsilon^n(x,t) = e^{e^t - \epsilon t} \sin x + \frac{1}{n} e^{(e^t-1)n^{2k} - \epsilon t n^{2k+2}} \sin(nx)$. For $0 < t \le 1$, assuming $n > 1$, it is easy to show that $\|u(x,t) - v_\epsilon^n(x,t)\|_2 = \sqrt{\frac{\pi}{2}} \sqrt{\left[e^{e^t}(1 - e^{-\epsilon t})\right]^2 + \left[\frac{1}{n} e^{(e^t-1)n^{2k} - \epsilon t n^{2k+2}}\right]^2}$. Because of the negative leading term $-\epsilon t n^{2k+2}$ in the exponent of the last exponential, ϵ may be chosen as $\epsilon = \frac{e}{n^2}$, in which case we have that $\epsilon \to 0$ as $n \to \infty$ and $\|u(x,t) - v_\epsilon^n(x,t)\|_2 = \sqrt{\frac{\pi}{2}} \sqrt{\left[e^{e^t}(1 - e^{-\frac{et}{n^2}})\right]^2 + \left[\frac{1}{n} e^{(e^t - 1 - et)n^{2k}}\right]^2} \to 0 \quad \text{as} \quad n \to \infty$ since $e^t - 1 - et < 0$ for $0 < t \le 1$. Note for the case that $t = 0$, we simply have $\|u(x,0) - v_\epsilon^n(x,0)\|_2 = \|e \sin x - \phi_n(x)\|_2 = \frac{1}{n}\sqrt{\frac{\pi}{2}} \to 0 \quad \text{as} \quad n \to \infty$. Hence, regularization is complete.

Note that when $k = 1$, (4) becomes the backward heat equation with time-dependent diffusion coefficient e^t. Table 1 shows numerical estimates for $\|u(x,t) - v_\epsilon^n(x,t)\|_2$ in this specific case ($k = 1$) based on different values of n and t and assuming the choice $\epsilon = \frac{e}{n^2}$.

Table 1.

$\|u(x,t) - v_\epsilon^n(x,t)\|_2$					
		n			
		1000	10000	100000	1000000
t	0	0.001253314	0.000125331	1.25331E-05	1.25331E-06
	0.1	1.02878E-06	1.02878E-08	1.02878E-10	1.02861E-12
	0.2	2.31117E-06	2.31118E-08	2.31117E-10	2.31126E-12
	0.3	3.94196E-06	3.94196E-08	3.94196E-10	3.94183E-12
	0.4	6.05767E-06	6.05767E-08	6.05767E-10	6.05789E-12
	0.5	8.85839E-06	8.85839E-08	8.85839E-10	8.85836E-12
	0.6	1.26427E-05	1.26428E-07	1.26428E-09	1.26424E-11
	0.7	1.78654E-05	1.78655E-07	1.78655E-09	1.78656E-11
	0.8	2.52339E-05	2.52339E-07	2.52339E-09	2.52335E-11
	0.9	3.58747E-05	3.58748E-07	3.58748E-09	3.58753E-11
	1	5.16284E-05	5.16285E-07	5.16285E-09	5.16282E-11

Table 2.

$\|u(x,t) - w_\epsilon^n(x,t)\|_2$					
		n			
		1000	10000	100000	1000000
t	0	0.001253314	0.000125331	1.25331E-05	1.25331E-06
	0.1	0.129960995	0.10662892	0.09040324	0.078463391
	0.2	0.301472976	0.248217982	0.210952415	0.183414183
	0.3	0.530421901	0.438378357	0.373542488	0.325406902
	0.4	0.839826665	0.696954498	0.59558013	0.519929335
	0.5	1.26361519	1.053350114	0.902955844	0.790084105
	0.6	1.852663844	1.551900529	1.334863161	1.170949288
	0.7	2.684706206	2.26073153	1.951776304	1.71681859
	0.8	3.880948905	3.286694025	2.848953561	2.513494718
	0.9	5.634602185	4.801148727	4.179848201	3.699671806
	1	8.261138225	7.085642684	6.197758763	5.505105575

Alternatively, we may consider the well-posed problem according to the second approximation

$f_\epsilon(t,A) = e^t A^k (I+\epsilon A^k)^{-1}$. Here, problem (3) becomes

$$\frac{\partial w}{\partial t} = (-1)^k \left(e^t \frac{\partial^{2k} w}{\partial x^{2k}} - \epsilon \frac{\partial^{2k+1} w}{\partial x^{2k} \partial t} \right), \quad 0<x<\pi,\ 0<t<1 \tag{6}$$
$$w(0,t) = w(\pi,t) = 0, \quad 0<t<1$$
$$w(x,0) = e \sin x + \frac{1}{n}\sin(nx), \quad 0<x<\pi$$

with solution $w_\epsilon^n(x,t) = e^{\frac{e^t+\epsilon}{1+\epsilon}} \sin x + \frac{1}{n} e^{\frac{(e^t-1)n^{2k}}{1+\epsilon n^{2k}}} \sin(nx)$. Note, for $0 < t \leq 1$, again assuming $n>1$, it may be shown that $\|u(x,t) - w_\epsilon^n(x,t)\|_2 = \sqrt{\frac{\pi}{2}} \sqrt{\left[e e^t - e^{\frac{e^t+\epsilon}{1+\epsilon}} \right]^2 + \left[\frac{1}{n} e^{\frac{(e^t-1)n^{2k}}{1+\epsilon n^{2k}}} \right]^2}$. This time, a reasonable choice is to choose ϵ so that $\frac{1}{n} e^{\frac{(e^t-1)n^{2k}}{1+\epsilon n^{2k}}}$ grows like $\frac{1}{\sqrt{n}}$. Equating these two quantities, and setting $t=1$, we arrive at $\epsilon = \frac{(e-1)n^{2k} - ln\sqrt{n}}{n^{2k}\ ln\sqrt{n}}$. Hence, $\epsilon \to 0$ as $n \to \infty$ and

$\|u(x,t) - w_\epsilon^n(x,t)\|_2 = \sqrt{\frac{\pi}{2}} \sqrt{\left[e e^t - e^{\frac{(e^t n^{2k}-1) ln\sqrt{n} + (e-1)n^{2k}}{(n^{2k}-1) ln\sqrt{n} + (e-1)n^{2k}}} \right]^2 + n^{\frac{e^t-1}{e-1}-2}} \to 0 \quad \text{as} \quad n \to \infty.$ Again for the case that $t=0$, we have $\|u(x,0) - w_\epsilon^n(x,0)\|_2 = \|e \sin x - \phi_n(x)\|_2 = \frac{1}{n}\sqrt{\frac{\pi}{2}} \to 0 \quad \text{as} \quad n \to \infty$. Regularization is thus accomplished in the case of the second approximation. Returning again to regularization of the backward heat equation, Table 2 shows estimates for the difference $\|u(x,t) - w_\epsilon^n(x,t)\|_2$ in the case that $k=1$, based on different values of n and t and assuming $\epsilon = \frac{(e-1)n^2 - ln\sqrt{n}}{n^2\ ln\sqrt{n}}$.

In comparing Table 1 and Table 2, we note that the convergence to 0 in the case of the second approximation is considerably slower than that in the first approximation.

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