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# The approximation of the normal distribution by means of chaotic expression

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**Abstract**. The approximation of the normal distribution by means of a chaotic expression is achieved by means of Weierstrass function, where, for a certain set of parameters, the density of the derived recurrence renders good approximation of the bell curve.

## 1. Introduction

Many biological and physical processes are described by sequences of numbers derived from the normal distribution. The numbers have various applications, for example, in simulations. Hence, many numerical generators (Gaussian Random Numbers Generators, GRNG) have been created, comprising the values of the numbers corresponding to the normal distribution.

The generators are based on: transformations [1-4], methods taking advantage of the cumulative density function inversion [5], rejection methods [6-8] and recursive methods [9].

The above mentioned transformations involve the conversion of the values of the numbers derived from the uniform distribution into the values corresponding to the normal distribution. In turn, methods based on inverse distributions match the values from the normal distribution by designating the values of the inverse cumulative distribution function with given argument in the form of a number corresponding to the uniform distribution. Rejection methods, similar to transformations, convert the numbers from the uniform distribution into the normal one, but, in addition, require the fulfillment of specific assumptions concerning the derived values of the numbers, while recurrences use the numbers obtained from the normal distribution and, in the next step, generate new values by means of linear expressions.

Alternative the above mentioned procedures can be divided into exact methods [1-2,6- 9] and approximation methods [3-5]. The exact methods, upon meeting certain criteria, enable the generation of numbers that ideally match the normal distribution. On the other hand, the approximation methods generate the values of numbers that only approximate the normal distribution.

The above mentioned GRNG methods require the numbers from the uniform distribution, with the exception of recursive method. In the case of the transformations based on the central limit theorem [4] at least several values from the uniform distribution are required to derive one numerical value from the normal distribution. On the other hand, the recursive methods use the values derived from the normal distribution to obtain new numbers.

The method discussed in the paper makes it possible to derive the values approximating the normal distribution without the need to take advantage of the uniform or normal distribution.

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## 2. Description of the method

The method utilizes Weierstrass function determined by the following equation[10-12]:

$$w(t)=\sum_{i=0}^{\infty}a^i\cos(b^i\pi t), \tag{1}$$

where $a$ and $b$ are the parameters fulfilling the conditions of $0<a<1$ and $ab>1+1.5\pi$. On the grounds of the recursive method:

$$t_{k+1}=w(t_k) \tag{2}$$

a sequence of numbers is derived, the density function of which, for parameter $a$ with the values close to 1, approximates the normal density distribution. The mentioned above recursion assumes the following form:

$$t_{k+1}=\sum_{i=1}^{n}a^i\cos\left(\pi b^i t_k\right) \quad . \tag{3}$$

## 3. Conducted analyses

*3.1 Analysis of the dynamic system*

Expression (3) is a chaotic representation. Due to the fact that function $w(t)$ is nowhere differential, Lyapunov's exponent – calculated in accordance with:

$$\lambda=\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\ln\left|\frac{dw(t)}{dt}\right|_{t=t_k} \tag{4}$$

- has an infinitely big positive value for $n\to\infty$. For example, the dependence of the exponent on parameter $a$ for $n=100$ terms of the sequence of Weierstrass function is shown in figure 1.

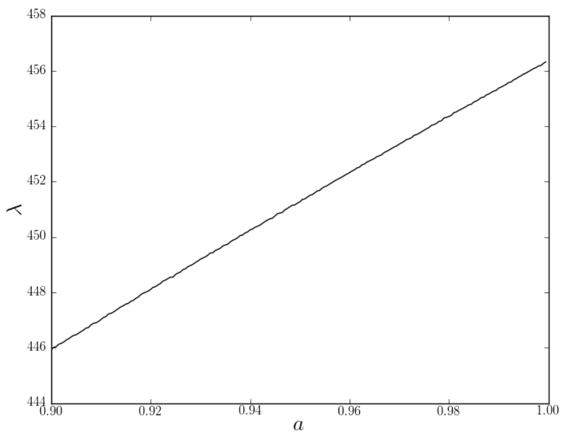


**Figure 1.** Lyapunov's exponent, $b=100, n=100$.

*3.2 Statistical analysis*

The density of the normal distribution is designated by the function:

$$\rho(x)=\frac{1}{\sigma\sqrt{2\pi}}\exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right) \tag{5}$$

where $\mu$ jest is the mean , whereas $\sigma$ is the standard deviation.

The above distribution is characterised, among other features, by the following measure dependencies:

- mean = median
- skewness coefficient = kurtosis = 0

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Assuming that parameter $a$ is close to number 1, the mean(figure 2), kurtosis(figure 3), median(figure 4) and skewness coefficient(figure 5) were derived from equation (3). The findings of the analysis indicated that the values of these parameters are convergent with those that correspond to the normal distribution. The standard deviation for the sequence of equation (3) is represented in figure 6. Furthermore, in figure 7 the numerical density function of equation (3) is illustrated, the shape of which approximates the bell curve with the square error equal to 4.223e-05. The influence of the number of the terms of the sequence on equation (3) as well as the influence of the values of parameter $b$ is shown in figures 8 and 9.

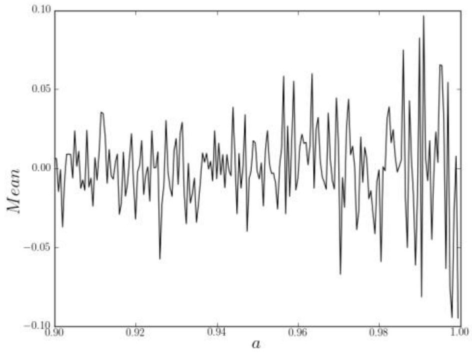


**Figure 2.** Mean, $b=100, n=100$.

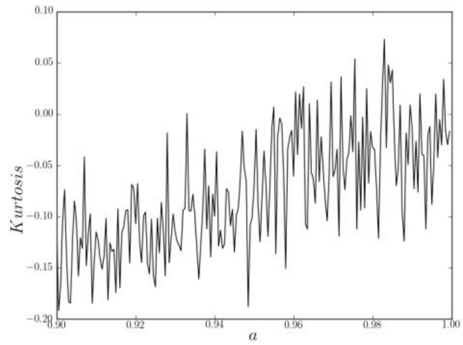


**Figure 3.** Kurtosis, $b=100, n=100$.

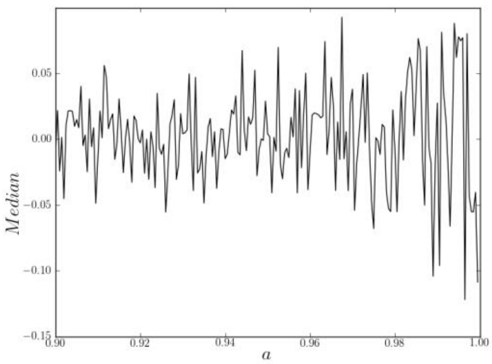


**Figure 4.** Median, $b=100, n=100$.

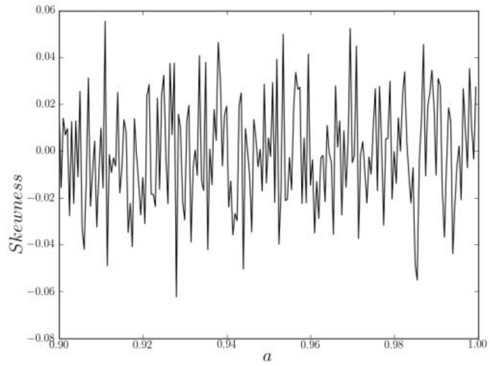


**Figure 5.** Skewness , $b=100, n=100$.

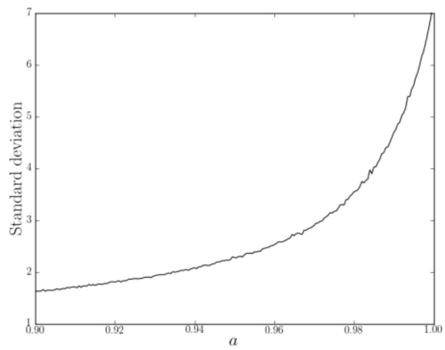


**Figure 6.** Standard deviation, $b=100,\ n=100$.

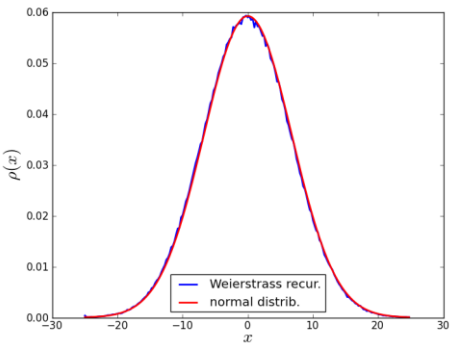


**Figure 7.** Exact and approximated density, $a=0.999, b=100, n=100$.

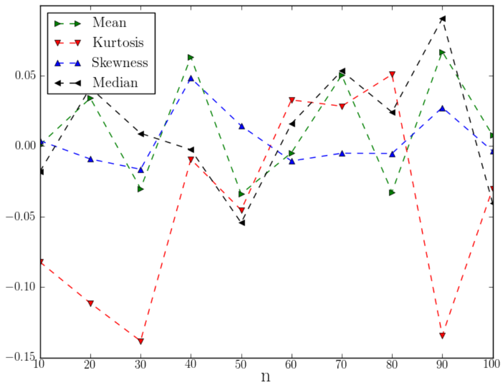


**Figure 8.** Different number of the terms of the sequence of (3), $a = 0.999, b = 100$.

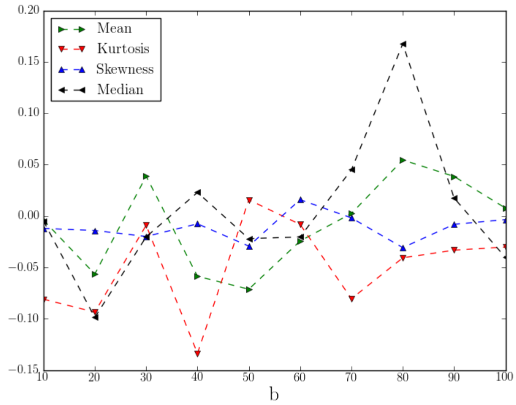


**Figure 9.** Different values of parameter $b, a = 0.999, n = 100$.

## 4. Conclusions

The method discussed in the paper enables the approximation of the normal distribution by means of a chaotic representation based on Weierstrass function. For the values of the parameter $a$ close to 1 in equation (3) the subsequent values of the dependence are derived from the normal distribution. Furthermore, the created recurrence does not require the numbers from the uniform distribution or normal distribution to generate new values.

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