Relativistic pseudospin and spin symmetries in physical systems - recent results

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Abstract. In this paper we revise the main features of pseudospin and spin symmetries of the Dirac equation with scalar and vector potentials and mention several of its applications to physical systems. These symmetries have been extensively researched in the last 15 years, especially pseudospin symmetry, mainly in its application in understanding certain nuclear structure features of heavy nuclei. The realization of both symmetries has also been studied using several mean-field scalar and vector potentials. For many classes of potentials, these symmetries allow to have analytical solutions of the Dirac equation which otherwise would not have been possible. We report here some recent results related to anti-fermions, Coulomb and confining potentials.

1. Introduction

Pseudospin symmetry has been a topic in nuclear physics since the late 60's, when it was introduced to explain the near degeneracy of some single-particle levels near the Fermi surface. The subject was revived in 1997 when Ginocchio was able to relate it with a symmetry of the Dirac equation with scalar S and vector V mean-field potentials such that $V = -S + C$ where C is a constant. However, this symmetry cannot be realized exactly in nuclei because the sum potential $V + S$ provides the binding of nucleons in nuclei. A related symmetry, the spin symmetry, was used to explain the suppression of spin-orbit splittings in states of mesons with a heavy and a light quark. In ref. [1] is reviewed the emergence of these symmetries as relativistic symmetries for the Dirac equation, as well as some of its applications. Those findings spurred much research about pseudospin symmetry in nuclei, including such topics as its very nature (how it is broken, whether is perturbative or not) [2, 3], the role of isospin [4, 5, 6] the effect of tensor potentials [7] and its realization under harmonic oscillator potentials [8]. Other theoretical studies showed the supersymmetric features of spin and pseudospin [9] and others were concerned with the realization of those symmetries for anti-fermions [10, 11, 12, 13]. More recently it was shown that pseudospin symmetry can be applied to ressonant states [14].

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In this paper we will review briefly the origin of spin and pseudospin symmetries in the Dirac equation, its generators, both for general potentials and radial potentials, and their quantum numbers. Finally we report about the main conclusions of recent works, namely about the perturbative nature of spin and pseudospin symmetries, how they are realized for fermion and anti-fermion systems with Coulomb potentials [15], and also how their realization depends on the asymptotic behaviour of the radial scalar and vector potentials in the Dirac equation [16].

2. Spin and pseudospin symmetries in the Dirac equation

The time–independent Dirac equation for a spin 1/2 particle with mass m and energy E, under the action of external scalar, S, and vector, V, potentials reads

$$H\psi = [\boldsymbol{\alpha}\cdot\hat{\boldsymbol{p}}\,c + \beta(mc^2+S)+V]\psi = E\psi\ , \quad \text{where} \quad \boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{1}$$

and $\boldsymbol{\sigma}$ are the Pauli matrices. Projecting the spinor ψ into its components $\psi_\pm = P_\pm\psi$, i.e.,

$$\psi_+ = \begin{pmatrix} \phi \\ 0 \end{pmatrix} \qquad \psi_- = \begin{pmatrix} 0 \\ \chi \end{pmatrix}\ , \tag{2}$$

where $P_\pm = [(I\pm\beta)/2]\,\psi$, and ϕ and χ are respectively the upper and lower two-component spinors, and applying them to the Dirac equation (1) we get two coupled equations for $\psi_\pm$:

$$c\,\boldsymbol{\alpha}\cdot\hat{\boldsymbol{p}}\,\psi_- + (\Sigma+mc^2)\,\psi_+ = E\psi_+ \qquad c\,\boldsymbol{\alpha}\cdot\hat{\boldsymbol{p}}\,\psi_+ + (\Delta-mc^2)\,\psi_- = E\psi_-\ , \tag{3}$$

where $\Sigma = V+S$ and $\Delta = V-S$.

2.1. *Spin symmetry*

If $\Delta = 0$ ($V = S$) and after multiplying it by $c\,\boldsymbol{\alpha}\cdot\hat{\boldsymbol{p}}$, the second equation in (3) becomes $\hat{\boldsymbol{p}}^2/(E/c^2+m)\,\psi_+ = (E-mc^2-\Sigma)\psi_+$, which is invariant under the transformation [17]

$$\delta\psi_+ = \frac{\boldsymbol{\epsilon}\cdot\tilde{\boldsymbol{\sigma}}}{2i}\psi_+\,, \qquad \tilde{\boldsymbol{\sigma}} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}\ . \tag{4}$$

Since $\psi_- = (c\,\boldsymbol{\alpha}\cdot\hat{\boldsymbol{p}})/(E+mc^2)\,\psi_+$, and defining $\delta\psi = \boldsymbol{\epsilon}\cdot\boldsymbol{\mathcal{S}}/(2i)\psi$, we can write the generators of this symmetry, called *spin symmetry*, as

$$\boldsymbol{\mathcal{S}} = \tilde{\boldsymbol{\sigma}}P_+ + \boldsymbol{\alpha}\cdot\hat{\boldsymbol{p}}\,\frac{\tilde{\boldsymbol{\sigma}}}{\hat{\boldsymbol{p}}^2}\,\boldsymbol{\alpha}\cdot\hat{\boldsymbol{p}}P_- = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}\,\frac{\boldsymbol{\sigma}}{\hat{\boldsymbol{p}}^2}\,\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}} \end{pmatrix}\ . \tag{5}$$

These generators commute with the Hamiltonian in (1) when $V = S$ and form an SU(2) algebra, i.e., $[\mathcal{S}_i,\mathcal{S}_j] = 2i\varepsilon_{ijk}\mathcal{S}_k$. The physical significance of this symmetry can be understood by looking at the second-order differential equation for ψ_+ when scalar and vector potentials are radial

$$\hat{\boldsymbol{p}}^2\,\psi_+ + \frac{\frac{2}{r}\,\Delta'\boldsymbol{L}\cdot\boldsymbol{S}\,\psi_+}{E-\Delta+mc^2} - \frac{\hbar^2\Delta'}{E-\Delta+mc^2}\,\frac{\partial\psi_+}{\partial r} = \frac{1}{c^2}(E-\Delta+mc^2)(E-\Sigma-mc^2)\psi_+\ , \tag{6}$$

where primes denote derivation with respect to r and $\boldsymbol{S} = (\hbar/2)\,\tilde{\boldsymbol{\sigma}}$, $\boldsymbol{L} = \boldsymbol{r}\times\hat{\boldsymbol{p}}$. From this last equation is clear that the physical significance of spin symmetry is *the disappearance of the spin-orbit coupling in a relativistic theory.* In this case there is another SU(2) symmetry connected to the orbital angular momentum, whose generators are $\boldsymbol{\mathcal{L}} = \boldsymbol{L}P_+ + \boldsymbol{\alpha}\cdot\hat{\boldsymbol{p}}\,\frac{\boldsymbol{L}}{\hat{\boldsymbol{p}}^2}\,\boldsymbol{\alpha}\cdot\hat{\boldsymbol{p}}\,P_-$. Indeed, one can check that one has $\boldsymbol{\mathcal{L}}^2\psi = \hbar^2\,\ell(\ell+1)\psi$, where ℓ is the orbital angular momentum quantum number of the upper component. This is true in spite of the fact that the upper and lower components of the Dirac spinor have different orbital angular momentum quantum numbers, since $\psi_+^\top = (i\,g_\kappa(r)/r\,\phi_{\kappa m_j}(\theta,\varphi)\quad 0\,)$ and $\psi_-^\top = (\,0 \quad -f_{\tilde\kappa}(r)/r\,\phi_{\tilde\kappa m_j}(\theta,\varphi)\,)$, where $\kappa = -(\ell+1)$ if $j = \ell + \frac{1}{2}$, $\kappa = \ell$ if $j = \ell - \frac{1}{2}$, and $\tilde\kappa = -\kappa$. The orbital angular momentum of the lower component, $\tilde\ell$, is given by $\tilde\ell = \ell - \kappa/|\kappa|$. This means that levels with the quantum numbers $(n, l, j = l - 1/2)$ and $(n, l, j = l + 1/2)$ are degenerate (relativistic fermion levels are classified according to the quantum numbers of the upper component of their Dirac spinor). Note that the results above would still be true if Δ were just a constant.

2.2. Pseudospin symmetry

If $\Sigma = 0$ ($V = S$) or a constant, one can repeat the arguments of the previous section for the spinor ψ_-, whose second-order equation would be a Schroedinger-like equation. The corresponding symmetry, the *pseudospin symmetry*, has the SU(2) generators $\tilde{\boldsymbol{\mathcal{S}}} = \tilde{\boldsymbol{\sigma}}P_- + \boldsymbol{\alpha}\cdot\hat{\boldsymbol{p}}\,\frac{\tilde{\boldsymbol{\sigma}}}{\hat{\boldsymbol{p}}^2}\,\boldsymbol{\alpha}\cdot\hat{\boldsymbol{p}}\,P_+ = \gamma^5\boldsymbol{\mathcal{S}}$. This symmetry implies that the spin-orbit coupling for the lower component of the Dirac spinor disappears, as can be seen from the second-order radial equation for ψ_-

$$\hat{\boldsymbol{p}}^2\,\psi_- + \frac{\frac{2}{r}\,\Sigma'\boldsymbol{L}\cdot\boldsymbol{S}\,\psi_-}{E-\Sigma-mc^2} - \frac{\hbar^2\Sigma'}{E-\Sigma-mc^2}\frac{\partial\psi_-}{\partial r} = \frac{1}{c^2}(E-\Delta+mc^2)(E-\Sigma-mc^2)\psi_- \;. \quad (7)$$

Again, in this case, there is another SU(2) symmetry, whose generators are $\tilde{\boldsymbol{\mathcal{L}}} = \boldsymbol{L}P_- + \boldsymbol{\alpha}\cdot\hat{\boldsymbol{p}}\,\frac{\boldsymbol{L}}{\hat{\boldsymbol{p}}^2}\,\boldsymbol{\alpha}\cdot\hat{\boldsymbol{p}}\,P_+ = \gamma^5\boldsymbol{\mathcal{L}}$. In this case the orbital angular momentum of the lower component is a good quantum number, i.e., $\tilde{\boldsymbol{\mathcal{L}}}^2\psi = \hbar^2\,\tilde\ell(\tilde\ell+1)\psi$, meaning that levels with the quantum numbers $(n', l+2, j = l-1/2)$ $(n, l, j = l+1/2)$ are degenerate. In nuclei, $n' = n-1$, and it was precisely the observation of the near-degeneracy of such levels in nuclei that led to the pseudospin concept.

3. Spin and pseudospin symmetries for Coulomb potentials

When the scalar and vector potentials are of Coulomb type, i.e., the sum and difference potentials are of the form $\Sigma = (\alpha_\Sigma/r)\,\hbar c$, $\Delta = (\alpha_\Delta)/r\,\hbar c$, there are analytical solutions to the respective Dirac equation. Details of the solutions can be found in refs. [9, 15]. There are two types of solutions, called $\pm$ solutions, corresponding to fermion and anti-fermion solutions in the weak coupling regime. If we expand those in terms of the α_Δ and α_Σ coefficients one gets, respectively,

$$\frac{E^+_{n,\kappa}}{mc^2} = 1 - \frac{2\alpha_\Sigma^2}{\alpha_\Sigma^2+4n^2} + \frac{4\alpha_\Sigma^3\,(\kappa^2-2n|\kappa|)}{\kappa^2\,(\alpha_\Sigma^2+4n^2)^2}\,\alpha_\Delta + \mathcal{O}\left(\alpha_\Delta^2\right) \quad , \quad \frac{E^-_{n,\kappa}}{mc^2} = -1 + \frac{\alpha_\Delta^2}{2n^2} + \mathcal{O}\left(\alpha_\Delta^3\right)\;, \quad (8)$$

$$\frac{E^+_{n,\kappa}}{mc^2} = 1 - \frac{\alpha_\Sigma^2}{2n^2} + \mathcal{O}\left(\alpha_\Sigma^3\right) \quad , \quad \frac{E^-_{n,\kappa}}{mc^2} = -1 + \frac{2\alpha_\Delta^2}{\alpha_\Delta^2+4n^2} - \frac{4\alpha_\Delta^3\,(\kappa^2-2n|\kappa|)}{\kappa^2\,(\alpha_\Delta^2+4n^2)^2}\,\alpha_\Sigma + \mathcal{O}\left(\alpha_\Sigma^2\right) \quad (9)$$

where n is the principal quantum number and κ is defined above. One can see immediately that for fermion (+) solutions one cannot have bound solutions in pseudospin symmetry conditions ($\alpha_\Sigma \to 0$), but for spin symmetry ($\alpha_\Delta \to 0$) there are bound solutions. The opposite is true for anti-fermion ($-$) solutions, that is, there are bound solutions in pseudospin symmetry conditions but not in spin symmetry conditions. These results agree with what has been found for nuclear mean-field Woods-Saxon-like potentials, both for nucleons [1] and for anti-nucleons [10, 11, 12, 13]. These facts have been related to perturbative or non-perturbative nature of spin and pseudospin symmetries [2, 3, 5]. One may notice also from (8) and (9) that the energy for the bound states in case of spin symmetry (fermion states) and pseudospin symmetry (anti-fermion states) depends only on the principal quantum number. It is interesting to relate those to the well-known non-relativistic solutions of hidrogenic atoms. For spin symmetry, one has

$$\frac{\boldsymbol{p}^2}{2m}\,\psi_+ = (\mathcal{E}' - \Sigma')\,\psi_+ \ , \tag{10}$$

where $\mathcal{E}' = [\mathcal{E}/(2mc^2)+1]\mathcal{E}$, $\Sigma' = [\mathcal{E}/(2mc^2)+1]\Sigma$ and $\mathcal{E} = E - mc^2$. Equation (10) is just the Schroedinger equation for a hydrogenic atom of "atomic number" $Z' = -(\mathcal{E}/(2mc^2)+1)\alpha_\Sigma/\alpha$, where α is the fine structure constant. The energy of this "atom" is just $\mathcal{E}' = -mc^2[\mathcal{E}/(2mc^2)+1]^2\,\alpha_\Sigma^2/n^2$, the same as $E^+_{n,\kappa}$ in (8) when $\alpha_\Delta = 0$. A similar reasoning can be made for the ψ_- equation for pseudospin symmetry ($\alpha_\Sigma = 0$). One will get then a Schroedinger equation for a hydrogenic anti-atom of "atomic number" $Z' = -[\mathcal{E}/(2mc^2)+1]\alpha_\Delta/\alpha$ where $\mathcal{E} = -E - mc^2$. Notice that for the positive energy solutions we have $\alpha_\Sigma < 0$ while for negative energy solutions one has $\alpha_\Delta > 0$. The fact that the energy only depends on n implies that there is an extra degeneracy: levels with the same n and $|\kappa| \le n$ are degenerate [15]. These features of Coulomb potentials for spin and pseudospin symmetries have to do with a particular symmetry of the Dirac equation with scalar and vector Coulomb potentials (see [9, 15]).

4. Realization of spin and pseudospin symmetries - final considerations

As referred above, for potentials like Woods-Saxon potentials or Coulomb potentials one cannot realize pseudospin symmetry for positive energy states and spin symmetry for negative energy states. This has been observed also by many authors for a great variety of scalar and vector potentials in the Dirac equation, which have the common feature of going to zero at large distances. However, for harmonic oscillator potentials one is able to find bound solutions for both symmetries for positive and negative energy states [8]. Recently, it was shown that this behaviour is shared with general radial potentials going to infinity at large distances [16], establishing that the asymptotic behaviour of scalar and vector potentials in the Dirac equation is what determines whether spin and pseudospin symmetry can be realized for bound relativistic fermion (or anti-fermion) systems. The study of these relativistic symmetries is still very much a active subject, and its application to physical systems other than nuclei seems promising.

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