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Numerical solution of the problems for plates on partial internal supports of complicated configurations

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Abstract. Very recently in the work "Simple Iterative Method for Solving Problems for Plates with Partial Internal Supports, *Journal of Engineering Mathematics* , DOI: 10.1007/s10665-013-9652-7 (in press)", we proposed a numerical method for solving some problems of plates on one and two line partial internal supports (LPIS). In the essence they are problems with strongly mixed boundary conditions for biharmonic equation. Using this method we reduced the problems to a sequence of boundary value problems for the Poisson equation with weakly mixed boundary conditions, which are easily solved numerically. The advantages of the method over other ones were shown.

In this paper we apply the method to plates on internal supports of more complicated configurations. Namely, we consider the case of three LPIS and the case of the cross support. The convergence of the method is established theoretically and its efficiency is confirmed on numerical experiments.

1. Introduction

In this paper we consider problems of rectangular plates with three line partial internal supports (LPIS) and a cross internal support. The geometry of the problems is depicted in Fig. 1 (case of three LPIS) and Fig. 2 (case of a cross support). Suppose that the plates are subjected to a uniformly distributed load (q), their bottom and top edges are clamped, while the left and right edges are simply supported. Then due to the two-fold symmetry the problems are reduced to the solution of the biharmonic equation $\Delta^2 u = f$ for the deflection $u(x, y)$ in a quadrant of the plates, where $f = q/D$, D is the flexural rigidity of the plates, under corresponding boundary conditions.

Essentially, they are strongly mixed boundary value problems for the biharmonic equation in the sense that there is one or more points of change of types of boundary conditions in edges of the rectangle. Therefore, they belong to the class of the problems with boundary singularities. A brief review of the methods concerning these problems is given in our recent work [1], where we proposed an iterative method, combining the reduction of the equation order [2], [3] and domain

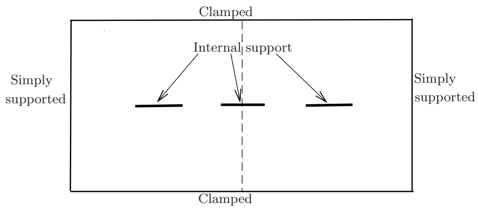

Figure 1. The plate with three internal line supports.

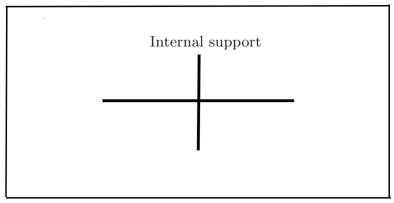

Figure 2. The plate with a cross internal support

decomposition [4] for the problems of plates on one and on two LPIS. It should be noticed that the motivation for our research was the paper of Sompornjaroensuk and Kiattikomol [6]. The present work is a further development of the method proposed in [1] to the problems of plates on three LPIS and on a cross support. The convergence of the method is established theoretically and verified on numerical examples.

2. Iterative method for solving the problem for plates with three LPIS

2.1. Description of the method

We consider the problem with boundary conditions in general form

$$\begin{aligned} \Delta^2 u &= f \quad \text{in } \Omega, \\ u &= g_0 \quad \text{on } S_B \cup S_D \cup S_F \cup S_G,\ \frac{\partial u}{\partial \nu} = g_1 \quad \text{on} \quad \Gamma \setminus S_G, \\ \Delta u &= g_2 \quad \text{on } S_G,\ \frac{\partial}{\partial \nu}\Delta u = g_3 \quad \text{on } S_A \cup S_C \cup S_E, \end{aligned} \tag{1}$$

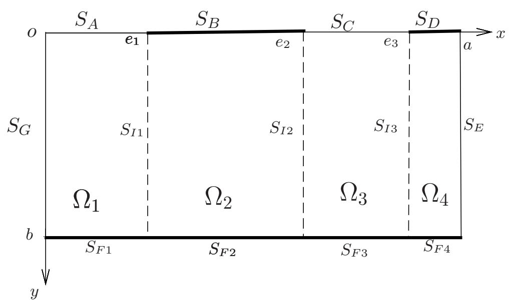

Figure 3. Domain decomposition for the problem with three LPIS considered in a quadrant of plate

where Ω is the rectangle $(0,a) \times (0,b)$, $S_A, S_B, S_C, S_D, S_E, S_F$ and S_G are parts of the boundary $\Gamma = \partial\Omega$ as shown in Fig. 3, Δ is the Laplace operator, f and g_i $(i = \overline{0,3})$ are functions given in Ω and on parts of the boundary Γ, respectively.

In the case if all boundary functions $g_i = 0\,(i = \overline{0,3})$ the problem models the bending of a quardrant of a rectangular plate on three LPIS.

For conciseness we put $\Gamma_1 = S_B \cup S_D \cup S_F$. Next we set $\Delta u = v$ in Ω, $v\,|_{\Gamma_1} = \varphi$.

For solving the problem we divide the domain Ω into three subdomains Ω_1, Ω_2, Ω_3 and Ω_4 by the lines $x = e_1$, $x = e_2$ and $x = e_3$, and denote the interfaces of these subdomains by S_{I_1}, S_{I_2} and S_{I_3} as depicted in Fig. 3.

Consider the following combined iterative method with the idea of simultaneous iterative update of φ on Γ_1 and $\xi_i = \partial v_2/\partial \nu_2$, $\eta_i = \partial u_2/\partial \nu_2$ on the interfaces S_{I_i} $(i = 1, 2)$, $\xi_3 = \partial v_4/\partial \nu_4$, $\eta_3 = \partial u_4/\partial \nu_4$ on the interface S_{I_3}. Here and afterwards we denote $u_i = u|_{\Omega_i}$, $v_i = v|_{\Omega_i}$ and ν_i denotes the outward normal to the boundary of Ω_i $(i = 1, ..., 4)$.

Step 1. Given $\varphi^{(0)} = 0$ on $S_B \cup S_D \cup S_F$; $\xi_i^{(0)} = 0$, $\eta_i^{(0)} = 0$ on S_{I_i}, $(i = 1, 2, 3)$.
Step 2. Knowing $\varphi^{(k)}$, $\xi_i^{(k)}$, $\eta_i^{(k)}$, $(k = 0, 1, ...)$, $(i = 1, 2, 3)$ solve consecutively problems for $v_2^{(k)}$ and $u_2^{(k)}$ in Ω_2, problems for $v_4^{(k)}$ and $u_4^{(k)}$ in Ω_4, problems for $v_1^{(k)}$ and $u_1^{(k)}$ in Ω_1 and problems for $v_3^{(k)}$ and $u_3^{(k)}$ in Ω_3:

$$\begin{cases} \Delta v_2^{(k)} = f & \text{in } \Omega_2, \\ v_2^{(k)} = \varphi^{(k)} & \text{on } S_B \cup S_{F2}, \\ \dfrac{\partial v_2^{(k)}}{\partial \nu_2} = \xi_i^{(k)} & \text{on } S_{I_i},\ (i = 1, 2), \end{cases} \qquad \begin{cases} \Delta u_2^{(k)} = v_2^{(k)} & \text{in } \Omega_2, \\ u_2^{(k)} = g_0 & \text{on } S_B \cup S_{F2} \\ \dfrac{\partial u_2^{(k)}}{\partial \nu_2} = \eta_i^{(k)} & \text{on } S_{I_i},\ (i = 1, 2), \end{cases} \tag{2}$$

$$\begin{cases} \Delta v_4^{(k)} = f & \text{in } \Omega_4, \\ v_4^{(k)} = \varphi^{(k)} & \text{on } S_D \cup S_{F_4}, \\ \dfrac{\partial v_4^{(k)}}{\partial \nu_4} = \xi_3^{(k)} & \text{on } S_{I_3}, \\ \dfrac{\partial v_4^{(k)}}{\partial \nu_4} = g_1 & \text{on } S_E, \end{cases} \qquad \begin{cases} \Delta u_4^{(k)} = v_4^{(k)} & \text{in } \Omega_4, \\ u_4^{(k)} = g_0 & \text{on } S_D \cup S_{F_4}, \\ \dfrac{\partial u_4^{(k)}}{\partial \nu_4} = \eta_3^{(k)} & \text{on } S_{I_3}, \\ \dfrac{\partial u_4^{(k)}}{\partial \nu_4} = g_1 & \text{on } S_E, \end{cases} \tag{3}$$

$$\begin{cases} \Delta v_1^{(k)} = f & \text{in } \Omega_1, \\ \dfrac{\partial v_1^{(k)}}{\partial \nu_1} = g_3 & \text{on } S_A, \\ v_1^{(k)} = \varphi^{(k)} & \text{on } S_{F1}, \\ v_1^{(k)} = g_2 & \text{on } S_G, \\ v_1^{(k)} = v_2^{(k)} & \text{on } S_{I_1}, \end{cases} \qquad \begin{cases} \Delta u_1^{(k)} = v_1^{(k)} & \text{in } \Omega_1, \\ \dfrac{\partial u_1^{(k)}}{\partial \nu_1} = g_1 & \text{on } S_A, \\ u_1^{(k)} = g_0 & \text{on } S_{F1} \cup S_G, \\ u_1^{(k)} = u_2^{(k)} & \text{on } S_{I_1}, \end{cases} \tag{4}$$

$$\begin{cases} \Delta v_3^{(k)} = f & \text{in } \Omega_3, \\ \dfrac{\partial v_3^{(k)}}{\partial \nu_3} = g_3 & \text{on } S_C, \\ v_3^{(k)} = \varphi^{(k)} & \text{on } S_{F3}, \\ v_3^{(k)} = v_2^{(k)} & \text{on } S_{I_2}, \\ v_3^{(k)} = v_4^{(k)} & \text{on } S_{I_3}, \end{cases} \qquad \begin{cases} \Delta u_3^{(k)} = v_3^{(k)} & \text{in } \Omega_3, \\ \dfrac{\partial u_3^{(k)}}{\partial \nu_3} = g_1 & \text{on } S_C, \\ u_3^{(k)} = g_0 & \text{on } S_D \cup S_{F3}, \\ u_3^{(k)} = u_2^{(k)} & \text{on } S_{I_2}, \\ u_3^{(k)} = u_4^{(k)} & \text{on } S_{I_3}, \end{cases} \tag{5}$$

Step 3. Compute the new approximation

$$\begin{aligned}
&\xi_1^{(k+1)} = (1-\theta)\xi_1^{(k)} - \theta\frac{\partial v_1^{(k)}}{\partial \nu_1}, \quad \eta_1^{(k+1)} = (1-\theta)\eta_1^{(k)} - \theta\frac{\partial u_1^{(k)}}{\partial \nu_1} \text{ on } \quad S_{I_1}\\
&\xi_2^{(k+1)} = (1-\theta)\xi_2^{(k)} - \theta\frac{\partial v_3^{(k)}}{\partial \nu_3}, \quad \eta_2^{(k+1)} = (1-\theta)\eta_2^{(k)} - \theta\frac{\partial u_3^{(k)}}{\partial \nu_3} \text{ on } \quad S_{I_2}\\
&\xi_3^{(k+1)} = (1-\theta)\xi_3^{(k)} - \theta\frac{\partial v_3^{(k)}}{\partial \nu_3}, \quad \eta_3^{(k+1)} = (1-\theta)\eta_3^{(k)} - \theta\frac{\partial u_3^{(k)}}{\partial \nu_3} \text{ on } \quad S_{I_3}\\
&\varphi^{(k+1)} = \varphi^{(k)} - \tau\Big(\frac{\partial u_i^{(k)}}{\partial \nu_i} - g_1\Big) \text{ on } \quad \Gamma_1,
\end{aligned} \tag{6}$$

where $\theta,\ \tau$ are iterative parameters to be chosen for the fast convergence of the iterative method.

The convergence of the above iterative method is proved in the same way as for the cases of one and of two LPIS in [1].

2.2. Numerical example

For numerical realization of the above iterative method each subdomain is covered by an uniform grid with the same number of nodes in the $y-$direction. On these grids the mixed BVPs for the Poisson equation (2)-(5) are discretized by difference schemes of second order approximation. After that the obtained systems of difference equations are solved by the method of complete reduction [5, Chapt. 3]. For computing the normal derivatives in (6) we also use formulas of second order error. We perform iterative process (2)-(6) until $\parallel u^{(k+1)} - u^{(k)} \parallel_\infty \leq \varepsilon$, where ε is a given accuracy taken of the same order as $O(\hat{h}^2)$, $\hat{h}$ being the step size of the grid.

The results of testing the convergence of the method on some exact solutions $u(x,y)$ with the corresponding boundary conditions and the right side function for the grids of 65×65 nodes on each subdomain show that for the values $\theta = 0.95,\ \tau = 0.55$ and some different combinations of e_1, e_2, e_3 after $40-60$ iterations the deviation of the approximate solution from the exact solution is less than 10^{-4}. Therefore, in sequel we use these values of θ and τ .

Now, we apply the proposed iterative method to the problem of the rectangular plates depicted in Fig. 3, where $u(x,y)$ is the deflection function, $f = q/D$, D being flexural rigidity of the plate which is expressed as $D = Eh^3/12(1-\nu^2)$, while h is the plate thickness, and ν and E are the Poisson's ratio and the Young's modulus of the plate, respectively.

The iterative process (2)-(6) for solving the problem for plates with three partial internal supports in domain $\Omega = [0,\pi/2]\times[0,\pi/2]$ on the grid 65×65 nodes for each subdomain with the given accuracy 10^{-4}, $h = 0.5$, $q = 0.3$, the iteration parameters $\theta = 0.95,\ \tau = 0.55$ converges after 48 iterations. Fig. 4 presents the normalized deflection surfaces of a quadrant and of the whole plate for $e_1/\pi = 0.15, e_2/\pi = 0.20, e_3/\pi = 0.45$.

3. Problem for a plate on a cross internal support

3.1. Description of the iterative method

Now we consider the following problem

$$\begin{aligned}
&\Delta^2 u = f \quad \text{in } \Omega,\\
&u = g_0 \quad \text{on } S_B\cup S_C\cup S_E,\ \frac{\partial u}{\partial \nu} = g_1 \quad \text{on} \quad S_A\cup S_B\cup S_C\cup S_D\cup S_E,\\
&\Delta u = g_2 \quad \text{on } S_F,\ \frac{\partial}{\partial \nu}\Delta u = g_3 \quad \text{on } S_A\cup S_D,
\end{aligned} \tag{7}$$

where Ω is the rectangle $(0,a)\times(0,b)$, S_A, S_B, S_C, S_D, S_E and S_F are parts of the boundary $\Gamma = \partial\Omega$ as shown in Fig. 5.

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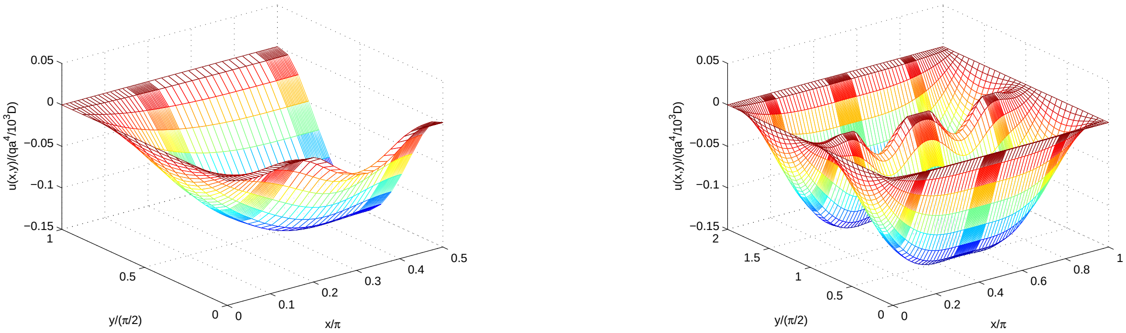

Figure 4. The surfaces of deflection of a quadrant (left) and of the whole plate (right)

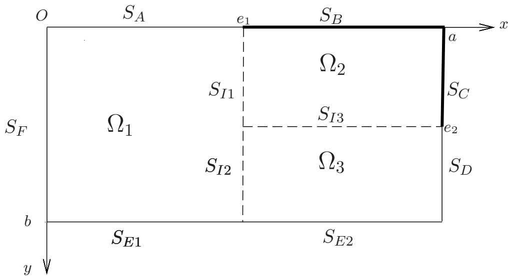

Figure 5. Domain decomposition for the problem with an internal cross support considered in a quadrant of plate

In the case if all boundary functions $g_i = 0\,(i = \overline{0,3})$ the problem models the bending of a quardrant of a rectangular plate.

For solving the problem we divide the domain Ω into three subdomains Ω_1, Ω_2 and Ω_3 by the lines $x = e_1$ and $y = e_2$, and denote the interfaces of these subdomains by S_{I_1}, S_{I_2} and S_{I_3} as depicted in Fig. 5. As usual, we denote $u_i = u|_{\Omega_i}$, $v_i = v|_{\Omega_i}$ and ν_i denotes the outward normal to the boundary of Ω_i $(i = 1, 2, 3)$

Consider the following *combined iterative method* for the problem (7):

Step 1. Given $\varphi^{(0)} = 0$ on $S_B \cup S_C \cup S_E$; $\xi_i^{(0)} = 0$, $\eta_i^{(0)} = 0$ on S_{I_i}, $(i = 1, 2, 3)$.

Step 2. Knowing $\varphi^{(k)}$, $\xi_i^{(k)}$, $\eta_i^{(k)}$, $(k = 0, 1, ...)$, $(i = 1, 2, 3)$ solve consecutively problems for $v_1^{(k)}$ and $u_1^{(k)}$ in Ω_1, problems for $v_2^{(k)}$ and $u_2^{(k)}$ in Ω_2, and problems for $v_3^{(k)}$ and $u_3^{(k)}$ in Ω_3:

$$\left\{\begin{array}{ll}
\Delta v_1^{(k)} & = f \quad \text{in } \Omega_1, \\
\dfrac{\partial v_1^{(k)}}{\partial \nu_1} & = g_3 \text{ on } S_A, \\
v_1^{(k)} & = \varphi^{(k)} \quad \text{on } S_{E1}, \\
v_1^{(k)} & = g_2 \quad \text{on } S_F, \\
\dfrac{\partial v_1^{(k)}}{\partial \nu_1} & = \xi_i^{(k)} \text{ on } S_{I_i},\ (i = 1, 2),
\end{array}\right.
\qquad
\left\{\begin{array}{ll}
\Delta u_1^{(k)} & = v_1^{(k)} \quad \text{in } \Omega_2, \\
\dfrac{\partial u_1^{(k)}}{\partial \nu_1} & = g_1 \text{ on } S_A, \\
u_1^{(k)} & = g_0 \quad \text{on } S_{E1} \cup S_F \\
\dfrac{\partial u_1^{(k)}}{\partial \nu_1} & = \eta_i^{(k)} \text{ on } S_{I_i},\ (i = 1, 2),
\end{array}\right. \tag{8}$$

$$\begin{cases} \Delta v_2^{(k)} &= f \quad \text{in } \Omega_2, \\ v_2^{(k)} &= \varphi^{(k)} \quad \text{on } S_B \cup S_C, \\ \dfrac{\partial v_2^{(k)}}{\partial \nu_2} &= \xi_3^{(k)} \quad \text{on } S_{I_3}, \\ v_2^{(k)} &= v_1^{(k)} \quad \text{on } S_{I_1}, \end{cases} \qquad \begin{cases} \Delta u_2^{(k)} &= v_2^{(k)} \quad \text{in } \Omega_2, \\ u_2^{(k)} &= g_0 \quad \text{on } S_B \cup S_C, \\ \dfrac{\partial u_2^{(k)}}{\partial \nu_2} &= \eta_3^{(k)} \quad \text{on } \; S_{I_3}, \\ u_2^{(k)} &= u_1^{(k)} \quad \text{on } S_{I_1}, \end{cases} \tag{9}$$

$$\begin{cases} \Delta v_3^{(k)} &= f \quad \text{in } \Omega_3, \\ v_3^{(k)} &= v_2^{(k)} \quad \text{on } \; S_{I_3}, \\ v_3^{(k)} &= \varphi^{(k)} \quad \text{on } S_{E2}, \\ v_3^{(k)} &= v_1^{(k)} \quad \text{on } \; S_{I_2}, \\ \dfrac{\partial v_3^{(k)}}{\partial \nu_3} &= g_3 \quad \text{on } S_D, \end{cases} \qquad \begin{cases} \Delta u_3^{(k)} &= v_3^{(k)} \quad \text{in } \Omega_3, \\ u_3^{(k)} &= u_2^{(k)} \quad \text{on } \; S_{I_3}, \\ u_3^{(k)} &= g_0 \quad \text{on } S_{E2}, \\ u_3^{(k)} &= u_1^{(k)} \quad \text{on } \; S_{I_2}, \\ \dfrac{\partial u_3^{(k)}}{\partial \nu_3} &= g_1 \quad \text{on } \; S_D, \end{cases} \tag{10}$$

Step 3. Compute the new approximation

$$\begin{aligned} \xi_1^{(k+1)} &= (1-\theta_1)\xi_1^{(k)} - \theta_1 \frac{\partial v_2^{(k)}}{\partial \nu_2}, \quad \eta_1^{(k+1)} = (1-\theta_2)\eta_1^{(k)} - \theta_2 \frac{\partial u_2^{(k)}}{\partial \nu_2} \text{ on } \quad S_{I_1} \\ \xi_2^{(k+1)} &= (1-\theta_1)\xi_2^{(k)} - \theta_1 \frac{\partial v_3^{(k)}}{\partial \nu_3}, \quad \eta_2^{(k+1)} = (1-\theta_2)\eta_2^{(k)} - \theta_2 \frac{\partial u_3^{(k)}}{\partial \nu_3} \text{ on } \quad S_{I_2} \\ \xi_3^{(k+1)} &= (1-\theta_1)\xi_3^{(k)} - \theta_1 \frac{\partial v_3^{(k)}}{\partial \nu_3}, \quad \eta_3^{(k+1)} = (1-\theta_2)\eta_3^{(k)} - \theta_2 \frac{\partial u_3^{(k)}}{\partial \nu_3} \text{ on } \quad S_{I_3} \\ \varphi^{(k+1)} &= \varphi^{(k)} - \tau \Big(\frac{\partial u_i^{(k)}}{\partial \nu_i} - g_1 \Big) \text{ on } \quad S_B \cup S_C \cup S_E, \end{aligned} \tag{11}$$

where θ_1, θ_2 and τ are iterative parameters to be chosen for guaranteeing the convergence of the iterative process.

3.2. Numerical example

As for the problem of plate on three LPIS we verify the convergence of the discrete analogy of the iterative process (8)-(11) on some exact solutions for some sizes of the cross support. Performed experments show that the convergence rate depends on the sizes (e_1, e_2) and the values of the iteration parameters θ_1, θ_2 and τ. In application to the problem when $e_1/\pi = 0.25, e_2/(\pi/2) = 0.4$ with the iterative parameters $\theta_1 = 0.75, \theta_2 = 0.9$ and $\tau = 0.55$ for reaching the accuracy 10^{-4} the number of iterations needed is 46. The surfaces of deflection of a quadrant and of the whole plate in this case are depicted in Fig. 6.

4. Concluding remarks

In this paper we developed an iterative method for solving the problems for a plate having three line partial internal supports and for a plate having a cross internal support. The idea of the method is to reduce the problems to a sequence of problems for the Poisson equation with strongly mixed boundary conditions, which can be efficiently solved by a domain decomposition method. The convergence of the method is proved and tested on examples. The computations by the method are simple due to the use of our software for solving the Poisson equation with different weakly mixed boundary conditions. The method can be applied to the problems of plates on internal supports of more complex configurations such as a system of more than three LPIS, a system of two symmetric cross supports, a system of a cross and line internal supports or a system of parallel internal line supports.

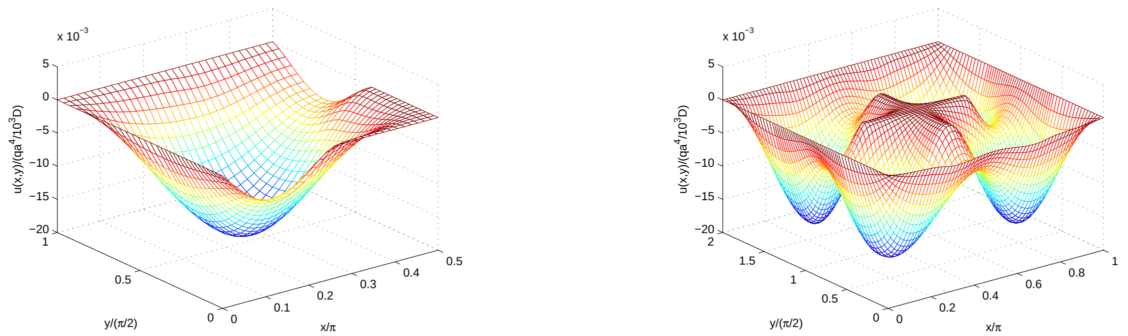

Figure 6. The surfaces of deflection of a quadrant (left) and of the whole plate (right) for $e_1/\pi = 0.25, e_2/\pi = 0.40$

Acknowledgments

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