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On equations with infinitely many derivatives: integral transforms and the Cauchy problem

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Abstract. We analyze initial value problems for ordinary differential equations with infinitely many derivatives such as (linearized versions of) nonlocal field equations of motion appearing in particle physics, nonlocal cosmology and string theory. We show that the corresponding initial value problem on a half line is well-posed and that it requires only a finite number of initial conditions. We also investigate nonlinear pseudo-equations defined by functions of the Laplace operator, that is, nonlinear partial differential equations with infinitely many derivatives.

1. Introduction

In this report we outline some recent developments in the analytic understanding of differential equations with an infinite number of derivatives, also called *nonlocal equations*. These equations have appeared recently as field equations of motion in particle physics, in string theory, and in gravity and cosmology (see for instance [13, 7, 16, 1, 3, 4, 12]). An important example is

$$p^{a\,\partial_t^2}\phi = \phi^p\ , \qquad a > 0\ . \tag{1}$$

Equation (1) describes the dynamics of the open p-adic string for the scalar tachyon field (see [1, 12, 14, 15] and references therein) and it can be understood, at least formally, as an equation in an infinite number of derivatives if we expand the left hand side as a power series in ∂_t^2.

Our first aim is to investigate analytic properties of nonlocal *linear* ordinary differential equations of the form

$$f(\partial_t)\phi = J(t)\ , \quad t > 0\ , \tag{2}$$

including the initial value problem. The importance of such a study has been recently stressed by Aref'eva and Volovich [1]: classical versions of Big Bang cosmological models contain singularities at the beginning of time, and therefore the time variable appearing in the field equations should vary only over a half line. Then, we wish to move on to *partial* differential equations in an infinite number of derivatives such as

$$\Delta e^{-c\Delta}\phi = U(x,\phi)\ , \quad c > 0\ , \tag{3}$$

on Euclidean space and on compact Riemannian manifolds (M, g). "Wick rotated" versions of these equations appear naturally in cosmology and string theory, see for instance [4, 5, 8, 16].

Our approach is, as explained in [8, 9, 10], to emphasize the role played by integral transforms in the analysis of Equations (2) and (3). This point of view allows us to prove that if some a priori data connected with our functional calculus is specified, the initial value problem for Equation (2) is well-posed and it requires only a *finite number* of initial conditions. It also allows us to move on to equations such as (3) posed on on Euclidean space or compact Riemannian manifolds, as we show in [8, 10].

2. Lorentzian functional calculus and linear equations

We fix a number $\omega \in \mathbb{R}$ and we consider the Banach space $L^\infty_\omega(\mathbb{R}_+)$ of all complex-valued and exponentially bounded functions on $\mathbb{R}_+$, and the *Widder space*, $C^\infty_W(\omega,\infty)$, defined as the image of the Laplace transform

$$\phi \in L^\infty_\omega(\mathbb{R}_+) \mapsto \mathcal{L}(\phi)(s) = \int_0^\infty e^{-st}\phi(t)dt \in C^\infty_W(\omega,\infty) \; .$$

As proven in [2], functions in $C^\infty_W(\omega,\infty)$ extend analytically to the region $Re(s) > \omega$ and the Laplace transform $\mathcal{L}(\phi)$ converges for $Re(s) > \omega$. Moreover, $\mathcal{L}$ is an isometric isomorphism from $L^\infty_\omega(\mathbb{R}_+)$ onto $C^\infty_W(\omega,\infty)$.

Definition 1. *Let f be an entire function and let $\mathcal{R}$ be the space of analytic functions on $\mathbb{C}$. We consider the subspace D_f of $L^\infty_\omega(\mathbb{R}_+)\times\mathcal{R}$ consisting of all the pairs (ϕ,r) such that $\widehat{(\phi,r)} = f\,\mathcal{L}(\phi) - r$ belongs to the Widder space $C^\infty_W(\omega,\infty)$. The domain of $f(\partial_t)$ as a linear operator on $L^\infty_\omega(\mathbb{R}_+)\times\mathcal{R}$ is D_f. If $(\phi,r)\in D_f$ then $f(\partial_t)\,(\phi,r) = \mathcal{L}^{-1}(\;\widehat{(\phi,r)}\;) = \mathcal{L}^{-1}(f\,\mathcal{L}(\phi) - r)$.*

Now, let us assume that a function $r \in \mathcal{R}$ has been fixed. We think of an equation such as $f(\partial_t)(\phi,r) = J$ as an equation for ϕ such that $(\phi,r)\in D_f$, and we simply write $f(\partial_t)\phi = J$.

Definition 2. *Let us fix a function $r\in\mathcal{R}$. We say that $\phi\in L^\infty_\omega(\mathbb{R}_+)$ is a solution to Equation $f(\partial_t)\phi = J$ if and only if*

(i) $\widehat{\phi} = f\,\mathcal{L}(\phi) - r \in C^\infty_W(\omega,\infty)$; (*i.e.*, $(\phi,r)\in D_f$);
(ii) $f(\partial_t)(\phi) = \mathcal{L}^{-1}(f\,\mathcal{L}(\phi) - r) = J$.

Theorem 1. *Let us fix an entire function f and a function $J\in L^\infty_\omega(\mathbb{R}_+)$. We assume that the function $(\mathcal{L}(J)+r)/f$ is in Widder space. Then, the linear equation $f(\partial_t)\phi = J$ can be uniquely solved for $\phi\in L^\infty_\omega(\mathbb{R}_+)$. The solution is given by the explicit formula*

$$\phi = \mathcal{L}^{-1}\left(\frac{\mathcal{L}(J)+r}{f}\right) \; . \tag{4}$$

3. The initial value problem

Formula (4) tells us that –expanding the analytic function r appearing in (4) as a power series– ϕ depends in principle on an infinite number of arbitrary constants. However, this fact *does not* mean that the equation itself is superfluous, as (4) depends essentially on f and J. We think of r as a "generalized initial condition".

Definition 3. *A generalized initial condition for the equation*

$$f(\partial_t)\phi = J \tag{5}$$

is an analytic function r_0 such that $(\phi,r_0)\in D_f$ for some $\phi\in L^\infty_\omega(\mathbb{R}_+)$. A generalized initial value problem is an equation such as (5) together with a generalized initial condition r_0. A

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solution to a given generalized initial value problem $\{(5), r_0\}$ *is a function* ϕ *satisfying the conditions of Definition* 2 *with* $r = r_0$.

Thus, given a generalized initial condition, we find a unique solution for (5) much in the same way as given *one* initial condition we find a unique solution to a first order linear ODE. *There is no reason to believe that for a given* r *the solution* (4) *will be analytic*: we can only conclude that the solution is an integrable exponentially bounded function, and it follows that classical initial value problems do not even exist in full generality!

Definition 4. A classical initial value problem is an equation $f(\partial_t)\phi = J$ *together with a finite set of conditions* $\phi(0) = \phi_0, \phi'(0) = \phi_1, \cdots, \phi^{(k)}(0) = \phi_k$. *A solution to a classical initial value problem is a pair* $(\phi, r_0) \in D_f$ *satisfying the conditions of Definition* 2 *with* $r = r_0$ *such that* ϕ *is differentiable at zero and the* $k+1$ *conditions above for* $\phi(0)$ *hold.*

Theorem 2. *Let* f *be entire and take* J *in* $L^\infty_\omega(\mathbb{R}_+)$ *such that* $\mathcal{L}(J)/f \in C^\infty_W(\omega, \infty)$. *Fix also a number* $N \geq 0$, *a finite number of points* ω_i *to the left of* $Re(s) = \omega$, *and (if* $N > 0$*) a finite number of positive integers* r_i, $i = 1, ..., N$. *Set* $K = \sum_{i=1}^N r_i$. *Then, generically, given* K *initial conditions,* $\phi_0, \ldots, \phi_{K-1}$, *there exists an analytic function* r_0 *such that*

(α) $\dfrac{r_0}{f}$ *has a finite number of poles* ω_i *of order* r_i, $i = 1, \ldots, N$ *to the left of* $Re(s) = \omega$ *;*

(β) $\dfrac{\mathcal{L}(J) + r_0}{f} \in C^\infty_W(\omega, \infty)$ *;*

(γ) $\left|\dfrac{r_0(s)}{f(s)}\right| \leq \dfrac{M}{|s|^p}$ *for some* $p \geq 1$ *and* $|s|$ *sufficiently large.*

Moreover, the unique solution ϕ *to* $f(\partial_t)\phi = J$ *with* $r = r_0$ *is of class* C^K *and it satisfies* $\phi(0) = \phi_0, \ldots, \phi^{(K-1)}(0) = \phi_{K-1}$.

The proof of Theorem 2 (see [9]) depends on the ability to differentiate certain function of J and f, which we present in [9], at least K times, and on the assumption that a system of K linear equations has a solution. In this sense we are solving initial value problems "generically". The proof also shows that it is essential to give *a priori* a finite number of points ω_i in order to have classical initial value problems. If no points ω_i are present, the solution to $f(\partial_t)\phi = J$ is simply $\phi = \mathcal{L}^{-1}(\mathcal{L}(J)/f)$, a formula which fixes completely the values of all the derivatives of ϕ at zero.

4. On nonlocal equations of elliptic type

Now we consider equations of the form (3). More generally, in [10] we investigate equations of the form

$$f(\Delta)u = U(x, u(x)) \ , \tag{6}$$

in which f is in principle an arbitrary *measurable* function defined on the whole Euclidean space $\mathbb{R}^n$ or on a compact Riemannian manifold (M, g). As examples, we note that Barnaby, [3], studied Equation (6) in a Lorentzian context using his "method of steps" and delay differential equations, and that in the paper [6] Dragovich considers (6) in which f is the Riemann ζ-function.

At this level of generality neither pseudo-differential operators, [11], nor generalized functions are used: we attach a meaning to the nonlocal operator $f(\Delta)$ directly by using Fourier transform, and we solve Equation (6) on a family of Hilbert spaces $\mathcal{H}^\beta(f)$, $\beta > 0$, of differentiable functions on which the operator $f(\Delta) + I$ acts naturally. For example, [8], we define the action of the operator $L_c = \Delta e^{-c\Delta} - Id$ which appears in the analysis of (3) as

$$L_c u = -\mathcal{F}^{-1}\left(\mathcal{F}(u) + |\,\xi|^2 e^{c|\,\xi|^2}\mathcal{F}(u)\right) \ .$$

These Fourier-based definitions allow us to show the existence of *regular* solutions to Equation (3) and more generally to equations of the type (6). We quote from [10]:

Theorem 3. *Let us assume that f is such that the function $s \mapsto f(-s^2)$ is non-negative, and that there exist real numbers $\beta, R, M > 0$ satisfying*

$$M(1+|\xi|^2)^{\frac{\beta}{2}} \leq f\left(-|\xi|^2\right) \text{ for all } \xi \text{ with } |\xi| > R$$

for $\beta > \frac{n}{2}\left(\frac{\alpha-1}{\alpha}\right)$. Suppose that U is a spherically symmetric function with respect to x, and assume also that there exist a constant $\alpha > 1$, and functions $h \in L^2(\mathbb{R}^n)$, $g \in L^{\frac{2\alpha}{\alpha-1}}(\mathbb{R}^n)$ such that the following two inequalities hold:

$$|U(x,y)+y| \leq C(|h(x)|+|y|^{\alpha}), \qquad \left|\frac{\partial}{\partial y}(U(x,y)+y)\right| \leq C\left(|g(x)|+|y|^{\alpha-1}\right)$$

for some constant $C > 0$. Then, there exist $0 < \epsilon < 1$ and $0 < \rho_\epsilon < 1$ such that whenever $\|h\|_{L^2(\mathbb{R}^n)} < \rho_\epsilon$, there is a smooth radial solution u to Equation (6) satisfying $\|u\|_{L^{2\alpha}(\mathbb{R}^n)} \leq \epsilon$.

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