Interaction between a drift and a fractional power of a Laplacian in semi-group theory

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Abstract. We give an interaction between a drift and a fractional power of a degenerated Laplacian such that the involved semi-group has a density by using the Malliavin Calculus for boundary processes translated by ourself in semi-group theory in [1].

1. Introduction and statement of the main theorem

We consider m vector fields on R^d with bounded derivatives at each order $X_1, .., X_m$ and the diffusion generator

$$L = \frac{\partial}{\partial s} + 1/2 \sum_{i>0} (X_i)^2 \tag{1}$$

on R^{d+1}. We could add a drift in (1), but it is done to simplify the proof. We consider a vector field on R^d D with bounded derivatives at each order. Bismut [2] considers the generator

$$A = D - 1/2\sqrt{-2L} \tag{2}$$

For the theory of fractional powers of Laplacian, we refer to the book of Yosida [3]. Let us recall quickly its definition. L generates a semi-group P_s acting on bounded continuous functions f on R^{d+1}:

$$\frac{\partial}{\partial t} P_t f(s, x) = LP_t(s, x) \tag{3}$$

Then

$$\sqrt{-2L} = C \int_0^\infty s^{-3/2}(P_s - I)ds \tag{4}$$

A generates a Markovian semi-group $\exp[tA]$ acting on continuous functions f on R^{1+d}:

$$\frac{\partial}{\partial t} \exp[tA]f(s, x) = A \exp[tA]f(s, x) \tag{5}$$

There is a stochastic representation of this semi-group (See [2]). Let $(B_1, .., B_m, z_t)$ be a Brownian motion on R^{m+1} starting from the origin. Let L_t be the local time associated to

z_t ([4]) and A_t its right inverse process. We introduce the stochastic differential equation in Stratonovitch sense issued from x:

$$dx_t = \sum_1^m X_i(x_t)dB_i + D(x_t)dL_t \tag{6}$$

We consider the subordinated process $(x_{A_t}, A_t + s)$. Unlike x_t, this process is not continuous but is still a Markov process. We have the main relation

$$\exp[tA]f(s,x) = E[f(A_t + s, x_{A_t})] \tag{7}$$

This paper follows the probabilistic intuition which comes from this stochastic representation of the semi-group. But in [2] and [5], only stochastic differential equations appear which explain that we can expulse the probabilistic language of [2] and [5].

The natural question is to know if the semi-group has an heat-kernel:

$$\exp[tA]f(s,x) = \int_{R^{1+d}} q_t(s,x,s',x')f(s',x')ds'dx' \tag{8}$$

This problem was solved by Bismut by using the Malliavin Calculus and a stochastic representation of it ([2]) in the elliptic case. We applied Bismut's technic to state an Hoermander theorem for fractional powers of Laplacians in [5] We have translated Malliavin Calculus of Bismut for Boundary processes in semi-group theory in [1] and state a regularity result for the semi-group associated to A in the elliptic case. We do now an Hoermander type hypothesis. We put

$$G_1 Y = Y \tag{9}$$

$$G_l Y = \cup_{i\geq 0} \cup_{Z\in G_{l-1}} ([Z, X_i]) \cup G_{l-1}Z \tag{10}$$

We put:

$$E_l = \cup_{j\leq l} \cup_{i>0} (G_j X_i) \tag{11}$$

The following theorem was proved in [5] by using the Calculus of Boundary Process of Bismut. We prove it again by using the Malliavin Calculus of Bismut type in semi-group theory of [1]:

Theorem 1 Let us suppose that the uniform Hoermander's hypothesis is checked:

$$\inf_{x\in R^d, \|f\|=1} \sum_{Y\in E_{l_0}} < Y, f >^2 + < [D, Y], f >^2 > C > 0 \tag{12}$$

Then the heat-kernel on R^{d+1} $q_t(0,x,s,y)$ exists.

Remark: It should be possible to show that $(s,y) \to q_t(0,x,s,y)$ is smooth.

Remark: It is possible to replace (12) by the general hypothesis (3.5) of [5].

2. The main ingredient of the proof

Let $E_d = R^{1+d} \times G_d \times M_d$ where G_d denotes the set of invertible matrices on R^d and M_d the set of symmetric matrices on R^d. (s,x,U,V) is the generic element of E_d. V is called the Malliavin matrix.

On E_d we consider the vector fields:

$$\hat{D} = (0, D, DD(x)U, 0) \tag{13}$$

$$\hat{X}_i = (0, X_i, DX_i(x)U, 0) \tag{14}$$

$$\hat{Y} = (0, 0, 0, \sum_i^m < U^{-1}X_i, . >^2) \tag{15}$$

We consider the Malliavin generator $\hat{L}$ on E_d:

$$\hat{L} = \frac{\partial}{\partial s} + 1/2 \sum_{i=1}^m (\hat{X}_i)^2 + \hat{Y} \tag{16}$$

and the square root associated $\sqrt{-\hat{L}}$. This semi-group on this bigger space is defined according the line of (5).

We consider

$$\hat{A} = \hat{D} - 1/2\sqrt{-2\hat{L}} \tag{17}$$

and the Malliavin semi-group $\exp[t\hat{A}]$.

An adaptation of one of main result of Léandre [1] is the following:

Theorem 2 Let us suppose that the Malliavin condition is checked: for all $p \in N$, all $s > 0$

$$\exp[t\hat{A}][V^{-p}1_{[0,s_0]}](0, x, I, 0) < \infty \tag{18}$$

then

$$\exp[tA]f(0, x) = \int_{R^{1+d}} f(s, y)q_t(s, y)dsdy \tag{19}$$

where $q_t(s, y) \geq 0$.

Remark: The proof follows the proof of Theorem 2.1 of [1]. Following the general strategy of the Malliavin Calculus, it is enough to show the theorem to get integration by parts formulas. If f is with compact support

$$|\exp[tA](df)(s, x)| \leq C\|f\|_\infty \tag{20}$$

where C depends only from the support of f and $\|f\|_\infty$ denotes the supremum norm of f. For that, we integrate by parts under the underlying diffusion P_s and the Brownian motions B_i as in part 3 of [1]. This allows to remove the space derivatives of f. In order to remove the time derivatives in $df(s, x)$, we integrate by parts on the subordinators A_t as it was done in part 4 of [1]. The main difference with [1] is that $\hat{D}$ appears when we take the variation of the subordinated semi-group. It is the only change in the abstract theorem of part 5 of [1]. When we get this abstract theorem, the drift D will appear another time in the inversion of the Malliavin matrix V.

3. Inversion of the Malliavin matrix in semi-group theory

Let be

$$F_l(x, U, \xi) = \sum_{Y \in G_l} < U^{-1}Y(x), \xi >^2 \tag{21}$$

where ξ is of modulus one. A simple adaptation of Lemma 3 of [6] shows:

Lemma 3 Let us suppose that

$$\exp[t_0\hat{A}][1_{[0,s_0]}; F_l(I, ., \xi) > Ct_0^\alpha](0, x_0, U_0, 0) > C > 0 \tag{22}$$

for all $x_0 \in R^d$, $\|U_0\| < t_0^{-\epsilon}$ for a small ϵ and some positive β. Then (22) remains true on an interval of length t_0^β for another β.

Since $\hat{A}$ is Markovian, $\exp[t\hat{A}]$ is represented by a stochastic process X_t following the same line of the representation of $\exp[tA]$ by a stochastic process. X_t is a Markov jump process. It has a Levy measure [7]. The main remark is the following: if the Levy measure of a jump process is enough concentrated in small jumps, there are a lot of small jumps.We get:

Definition 4 If f is a function from R^+ into R^+, we consider the Levy measure associated with the Malliavin matrix where ξ is of norm 1:

$$\mu_\xi(f) = C\int_0^{s_0} \frac{ds}{s^{3/2}}\hat{P}_s[f(V'(\xi) - V(\xi))](0, x, U, V) \tag{23}$$

We recall (See Lemma 3 of [6]):

Lemma 5 Let us suppose that $F_1(x, U, \xi) \geq \rho$ for $|U| + |U^{-1}| < \rho^{-\epsilon}$ for some small ϵ. Then $\mu_\xi[z > \rho^\alpha] \geq C\rho^\beta$ for some positive α and some negative β.

The theorem will follow as in the proof of Theorem 1 (38), (39), (40) in [6] if we show the next proposition:

Proposition 6 Let us suppose that $|U| + |U^{-1}| < \rho^{-\epsilon}$ for some small ϵ . There exists α such that

$$\exp[\rho^\alpha \hat{A}][F_1(x, U, \xi) \geq \rho; 1_{[0,s_0]}](0, x, U, 0) > C > 0 \tag{24}$$

We can state an analog of Lemma 2 of [6]:

Lemma 7 Let us suppose that

$$\exp[\rho^\alpha \hat{A}][F_l(x, U, \xi) \geq \rho^\beta; 1_{[0,s_0]}](0, x_0, U_0, 0) > C > 0 \tag{25}$$

where $|U_0| + |U_0^{-1}| < \rho^{-\epsilon}$ for some small ϵ.. Then (25) remains true for $l - 1$ for others α and β.

By using this lemma, it is enough to show the following propopsition in order to show Proposition 6:

Proposition 8 If we take $l_0 + 2$, (25) is checked if $|U| + |U^{-1}| < \rho^{-\epsilon}$ for some small ϵ.

Proof of proposition 8 Let us suppose that

$$F_{l_0}(x_0, U_0, \xi) \leq \rho^\beta \tag{26}$$

and

$$F_{l_0+2} \leq \rho^{\beta_1} \tag{27}$$

By Hypothesis (12), we can find a $Y \in E_{l_0}$ such that

$$< [D, Y](x_0), \xi >> C > 0 \tag{28}$$

We choose $C > 0$ to simplify the exposition and we choose $\epsilon = 0$ in order to simplify the expostion of the proof.

We remark that if $u < t$ and if $\gamma < 1/2$ that

$$\exp[u\hat{A}][1_{[t^\gamma,\infty[}](0, x_0, U_0, 0) \leq Ct^r \tag{29}$$

for some $r > 0$. So it is enough to estimate

$$\exp[u\hat{A}][F_{l_0}(., ., \xi) > \rho^\alpha; 1_{[0,t^\gamma]}](0, x_0, U_0, 0) \tag{30}$$

for some well choosed α We put

$$G(x, U, \xi) =< U^{-1}Y, \xi > g(\frac{F_{l_0+2}(x, U, \xi)}{\rho^{\beta_1}}) \tag{31}$$

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where g is a smooth function from R^+ into $[0,1]$ equals to 1 in a neighborhood of 0 and to 0 in a neighborhood of the infinity.

Let us suppose that

$$|x-x_0|+|U-U_0|<C\rho^{\beta/2} \tag{32}$$

Let us estimate for $s'\leq t^\gamma$

$$\sqrt{-\hat{L}}[G(.,.,\xi)1_{[0,t^\gamma]}](s',x,U,0) \tag{33}$$

For that we look at

$$f(s)=\hat{P}_s[G(.,.,\xi)1_{[0,t^\gamma]}](s',x,U,0) \tag{34}$$

where $\hat{P}_s$ is the semi-group generated by $\hat{L}$.

$$f'(s)=\hat{P}_s[\hat{L}[G(x',U',\xi)1_{[0,t^\gamma]}]](s',x,U,0) \tag{35}$$

We distinguish if

$$|x-x'|+|U-U'|<C.\rho^{\beta/2} \tag{36}$$

or not. If yes

$$|\hat{L}[G(x',U',\xi)1_{[0,t^\gamma]}]|\leq \rho^{\beta/2}1_{[0,t^\gamma]} \tag{37}$$

If not we remark that

$$\hat{P}_s[|x-x'|+|U-U'|\geq \rho^{\beta/2}](s',x,U,0)\leq Cs\rho^{-\beta} \tag{38}$$

In conclusion, we deduce that

$$|\sqrt{-\hat{L}}[G(.,.,\xi)1_{[0,t^\gamma]}](s',x,U,0)|\leq \rho^{\beta/2}t^{\gamma/2}+\rho^{-\beta}\rho^{-2\beta_1}t^{\frac{3\gamma}{2}} \tag{39}$$

Let us consider the case

$$|x-x_0|+|U-U_0|>C\rho^{\beta/2} \tag{40}$$

In such a case

$$|f'(s)|\leq C\rho^{-2\beta_1}1_{[0,t^\gamma]} \tag{41}$$

Therefore

$$|\sqrt{-\hat{L}}[G(.,.,\xi)1_{[0,t^\gamma]}](s',x,U,0)|\leq C\rho^{-\beta_1}t^{\gamma/2}1_{[0,t^\gamma]} \tag{42}$$

On the other hand

$$|\sqrt{-\hat{L}}[|U'-U_0|^2+|x'-x_0|^2;1_{[0,t^\gamma]}](s',x,U,0)| \\ \leq A1_{[0,t^\gamma]}\int_0^{t^\gamma}ds\frac{s}{s^{3/2}}\leq Ct^{\gamma/2}1_{[0,t^\gamma]} \tag{43}$$

This shows that

$$exp[u\hat{A}][|U-U_0|^2+|x-x_0|^2;1_{[0,t^\gamma]}|](0,x_0,U_0,0)\leq Cut^{\gamma/2} \tag{44}$$

Therefore

$$exp[u\hat{A}][|U-U_0|+|x-x_0|>\rho^{\beta/2};1_{[0,t^\gamma]}|](0,x_0,U_0,0)\leq ut^{\gamma/2}\rho^{-\beta} \tag{45}$$

By putting all together, we deduce that if $\gamma<1/2$

$$|\exp[u\hat{A}][\sqrt{-\hat{L}}[G(.,.,\xi)1_{[0,t^\gamma]}]](0,x_0,U_0,0)| \\ C(\rho^{\beta/2}t^{\gamma/2}+\rho^{-\beta}\rho^{-2\beta_1}t^{\frac{3\gamma}{2}}+ut^{\gamma/2}\rho^{-2\beta_1}\rho^{-\beta}) \tag{46}$$

Let us now estimate

$$\exp[u\hat{A}][\hat{D}G(x,U,\xi)1_{[0,t^\gamma]}](0,x_0,U_0,0) \tag{47}$$

We suppose first of all that

$$|x-x_0|+|U-U_0|<C\rho^{\beta/2} \tag{48}$$

In such a case we have a lower bound in $C>0$ of the expression. If the previous inequality is not checked we have an estimate by using the previous considerations in $Cut^{\gamma/2}\rho^{-\beta_1}\rho^{-\beta}$ By using the semi-group property for $\exp[u\hat{A}]$ we deduce that if $u<t^\gamma$

$$g'(u)=\frac{\partial}{\partial u}\exp[u\hat{A}][G(.,.,\xi)1_{[0,t^\gamma]}](0,x_0,U_0,0)\geq$$
$$C-Ct^{3\gamma/2}\rho^{-\beta_1}\rho^{-\beta}-C(\rho^{\beta/2}t^{\gamma/2}+\rho^{-\beta}\rho^{-2\beta_1}t^{\frac{3\gamma}{2}} \tag{49}$$

We distinguish if

$$|g(0)|<C\rho^{3\beta} \tag{50}$$

or not. If (50) is not checked, we can apply Lemma (7). If not we have $\beta^1<\beta$. We deduce that

$$g(t)\geq Ct-C\rho^{3\beta}-C\rho^{-\beta}\rho^{-2\beta_1}t^{1+3\gamma/2}-C\rho^{\beta/2}t^{1+\gamma/2} \tag{51}$$

We choose $t=C_1\rho^\beta$ for a big C_1.From (51), we deduce that

$$g(C_1\rho^\beta)\geq C\rho^\beta-C\rho^{-\beta}\rho^{-2\beta_1}\rho^{(\beta)(1+3\gamma/2)} \tag{52}$$

to choose γ close from 1/2 and β_1 very small in order to deduce that

$$g(C_1\rho^\beta)\geq C\rho^\beta \tag{53}$$

We deduce that

$$\exp[C_1\rho^\beta\hat{A}][<U^{-1}Y(x),\xi>>C\rho^\beta](0,x_0,U_0)>C>0 \tag{54}$$

Therefore the result holds.

$\diamondsuit$.

Remark: The principle of the proof is very simple: we establish a criterium in order to show that the Levy measure associated to $V(\xi)$ is very concentrated in small jumps. If the Levy measure is very concentrated in small jumps, there are a lot of small jumps which obliges that $V(\xi)$ to be not very small. If this criterium is not satisfied, another criterium will obliged it to be satisfied. In [6], this criterium comes from the interaction between two Levy measures. Here it come from the interaction between a Levy measure and the drift D.

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