Journal of Physics: Conference Series **490** (2014) 012038 doi:10.1088/1742-6596/490/1/012038

The Complex Angle in Normed Spaces

VOLKER THÜREY
Rheinstr. 91
28199 Bremen, Germany [1]

Abstract. We consider a generalized angle in complex normed vector spaces. Its definition corresponds to the definition of the well known Euclidean angle in real inner product spaces. Not surprisingly it yields complex values as 'angles'. This 'angle' has some simple properties, which are known from the usual angle in real inner product spaces. But to do ordinary Euclidean geometry real angles are necessary. We show that even in a complex normed space there are many pure real valued 'angles'. The situation improves yet in inner product spaces. There we can use the known theory of orthogonal systems to find many pairs of vectors with real angles, and to do geometry which is based on the Greeks 2000 years ago.

Keywords and phrases: complex normed space, generalized angle, complex Hilbert space.

There are some attempts to generalize the well known Euclidean angle of real inner product spaces to real normed spaces, see, for instance, the references [2], [6] [7], [9], [10] [11], [12]. Also angles in complex inner product spaces were considered, see [3], [4], [8]. In this paper for the first time a complex-valued 'angle' is defined for all complex normed spaces, and we investigate its properties. We believe that this concept of a complex-valued angle is superior to the enforcement of pure real angles, since real angles may suppress some true properties of a complex normed vector space.

It is well known that the inner product $< . \mid . >$ in an inner product space $(X, < . \mid . >)$ over the complex field $\mathbb{C}$ can be expressed by its norm, i.e. for arbitrary $\vec{x}, \vec{y} \in X$ we can write $< \vec{x} \mid \vec{y} > = (1/4) \cdot \left[\|\vec{x}+\vec{y}\|^2 - \|\vec{x}-\vec{y}\|^2 + \mathbf{i} \cdot \left(\|\vec{x}+\mathbf{i}\cdot\vec{y}\|^2 - \|\vec{x}-\mathbf{i}\cdot\vec{y}\|^2 \right) \right]$, where the symbol '$\mathbf{i}$' means the imaginary unit. We use this fact, and we normalize all vectors $\vec{x}, \vec{y} \neq \vec{0}$ to generate a continuous product in all complex normed vector spaces $(X, \|\cdot\|)$, which is just the inner product in the special case of a complex inner product space.

Definition 1. *Let $\vec{x}, \vec{y}$ be two arbitrary elements of X. In the case of $\vec{x} = \vec{0}$ or $\vec{y} = \vec{0}$ we set $< \vec{x} \mid \vec{y} > := 0$, and if $\vec{x}, \vec{y} \neq \vec{0}$ (i.e. $\|\vec{x}\| \cdot \|\vec{y}\| > 0$) we define the complex number*

$$< \vec{x} \mid \vec{y} > :=$$

$$\|\vec{x}\| \cdot \|\vec{y}\| \cdot \frac{1}{4} \cdot \left[\left\| \frac{\vec{x}}{\|\vec{x}\|} + \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 - \left\| \frac{\vec{x}}{\|\vec{x}\|} - \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 + \mathbf{i} \cdot \left(\left\| \frac{\vec{x}}{\|\vec{x}\|} + \mathbf{i} \cdot \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 - \left\| \frac{\vec{x}}{\|\vec{x}\|} - \mathbf{i} \cdot \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 \right) \right] .$$

□

[1] T: 49 (0)421/591777, E-mail: volker@thuerey.de

Content from this work may be used under the terms of the Creative Commons Attribution 3.0 licence. Any further distribution of this work must maintain attribution to the author(s) and the title of the work, journal citation and DOI.
Published under licence by IOP Publishing Ltd

It is easy to show that the product fulfils the conjugate symmetry ($< \vec{x}|\vec{y} > \; = \; \overline{< \vec{y}|\vec{x} >}$), where $\overline{< \vec{y}|\vec{x} >}$ means the complex conjugate, the positive definiteness ($< \vec{x}|\vec{x} > \geq \; 0$ and $< \vec{x}|\vec{x} >= 0$ only for $\vec{x} = \vec{0}$), and the homogeneity for real numbers, ($< r \cdot \vec{x} \,|\, \vec{y} > = \; r \cdot < \vec{x} \,|\, \vec{y} >$), and the homogeneity for pure imaginary numbers, $(< r \cdot \mathbf{i} \cdot \vec{x} \,|\, \vec{y} > = \; r \cdot \mathbf{i} \cdot < \vec{x} \,|\, \vec{y} >)$, for $\vec{x}, \vec{y} \in X, \; r \in \mathbb{R}$. We express the norm of $\vec{x}$ by $\|\vec{x}\| = \sqrt{< \vec{x} \,|\, \vec{x} >}$. Further we have the identity $< e^{\mathbf{i} \cdot \varphi} \cdot \vec{x} \,|\, \vec{x} > = \; e^{\mathbf{i} \cdot \varphi} \cdot < \vec{x} \,|\, \vec{x} >$, but for distinct $\vec{x}, \vec{y} \in X$ generally it holds the inequality $< e^{\mathbf{i} \cdot \varphi} \cdot \vec{x} \,|\, \vec{y} > \neq \; e^{\mathbf{i} \cdot \varphi} \cdot < \vec{x} \,|\, \vec{y} >$.

If we switch for a moment to real inner product spaces $(X, < . \,|\, . >_{real})$ we recall that we have for all $\vec{x}, \vec{y} \neq \vec{0}$ the usual Euclidean angle $\angle_{Euclid}(\vec{x}, \vec{y}) \; = \; \arccos\left(\frac{<\vec{x} \,|\, \vec{y}>_{real}}{\|\vec{x}\| \cdot \|\vec{y}\|}\right)$.

For all pairs $\vec{x}, \vec{y} \neq \vec{0}$ from a complex normed vector space $(X, \|\cdot\|)$ we define an 'angle'. It coincides with the definition of the Euclidean angle in real inner product spaces.

Definition 2. *Let $\vec{x}, \vec{y} \in X \backslash \{\vec{0}\}$, where $(X, \|\cdot\|)$ is a complex normed space. We define*

$$\angle(\vec{x}, \vec{y}) \; := \; \arccos\left(\frac{< \vec{x} \,|\, \vec{y} >}{\|\vec{x}\| \cdot \|\vec{y}\|}\right) .$$

This number $\angle(\vec{x}, \vec{y}) \in \mathbb{C}$ is called the angle of the pair $(\vec{x}, \vec{y})$. □

We state that the angle $\angle(\vec{x}, \vec{y})$ is defined for all $\vec{x}, \vec{y} \neq \vec{0}$. Since we deal with complex vector spaces it is not surprising that we get complex numbers as 'angles'. For the definition we need the extension of the cosine and arccosine functions on complex numbers. See as a reference, e.g., the books [1] or [5]. We use two subsets $\mathcal{A}$ and $\mathcal{B}$ of the complex plane $\mathbb{C}$, where
$\mathcal{A} \; := \; \{a + \mathbf{i} \cdot b \; \in \mathbb{C} \mid 0 < a < \pi, \; b \in \mathbb{R}\} \; \cup \; \{0, \pi\}, \quad \mathcal{B} \; := \; \mathbb{C} \backslash \{r \in \mathbb{R} \,|\, r < -1 \;\; \text{or} \;\; r > +1\}$.
We have two known homeomorphisms $\cos : \mathcal{A} \xrightarrow{\cong} \mathcal{B}$ and $\arccos : \mathcal{B} \xrightarrow{\cong} \mathcal{A}$.

The cosines of the 'angles' are out of the 'complex square' $\mathcal{SQ} := \{r + \mathbf{i} \cdot s \in \mathbb{C} \mid \; -1 \leq r, s \leq +1\} \subset \mathcal{B}$. The values of the 'angles' are from its image $\arccos(\mathcal{SQ})$, which forms a symmetric hexagon (with concave sides) in $\mathcal{A}$. Its center is $\pi/2$, two corners are 0 and π. We get a third if we use one of four possibilities of $\pm 1 \pm \mathbf{i}$, for instance

$$\arccos(1 + \mathbf{i}) \; = \; (\pi/2) - (1/2) \cdot \left[\arccos\left(\sqrt{5} - 2\right) + \mathbf{i} \cdot \log\left(\sqrt{5} + 2 + 2 \cdot \sqrt{\sqrt{5} + 2}\right)\right] \; \approx \; 0.90 - \mathbf{i} \cdot 1.06 \; .$$

This 'angle' has eight comfortable properties (`An` 1) - (`An` 8) which are known from the Euclidean angle $\angle_{Euclid}$ in real inner product spaces.

Theorem 1. *In a complex normed space $(X, \|\cdot\|)$ the angle $\angle$ has the following properties:*

- *(`An` 1)* $\angle$ *is a continuous map from* $\left(X \backslash \{\vec{0}\}\right)^2$ *onto a subset of* $\arccos(\mathcal{SQ}) \subset \mathcal{A}$.
  For elements $\vec{x}, \vec{y} \neq \vec{0}$ it holds that
- *(`An` 2)* $\angle(\vec{x}, \vec{x}) = 0$,
- *(`An` 3)* $\angle(-\vec{x}, \vec{x}) = \pi$,
- *(`An` 4)* $\angle(\vec{x}, \vec{y}) = \overline{\angle(\vec{y}, \vec{x})}$,
- *(`An` 5) for real numbers $r, s > 0$ we have* $\angle(r \cdot \vec{x}, s \cdot \vec{y}) = \angle(\vec{x}, \vec{y})$,
- *(`An` 6)* $\angle(-\vec{x}, -\vec{y}) = \angle(\vec{x}, \vec{y})$,
- *(`An` 7)* $\angle(\vec{x}, \vec{y}) + \angle(-\vec{x}, \vec{y}) = \pi$.
- *(`An` 8) For any two linear independent vectors $\vec{x}, \vec{y}$ of $(X, \|\cdot\|)$ there is a continuous injective map* $\Theta : \; \mathbb{R} \longrightarrow \mathcal{A}, \quad t \mapsto \angle(\vec{x}, \vec{y} + t \cdot \vec{x})$. *The limits are*
  $\lim_{t \to -\infty} \; \Theta(t) = \pi$ *and* $\lim_{t \to \infty} \; \Theta(t) = 0$.

In the following table we list some angles and their cosines. We choose two elements $\vec{x},\vec{y}\neq\vec{0}$ of a complex normed space $(X,\|\cdot\|)$, and six suitable real numbers a,b,r,s,v,w, with $-\frac{\pi}{2}\le a,v\le\frac{\pi}{2}$ and $-1\le r,s\le 1$, such that $\angle(\vec{x},\vec{y})=\frac{\pi}{2}+a+\mathbf{i}\cdot b\in\mathcal{A}$, $\cos(\angle(\vec{x},\vec{y}))=r+\mathbf{i}\cdot s\in\mathcal{B}$, and $\angle(\mathbf{i}\cdot\vec{x},\vec{y})=\frac{\pi}{2}+v+\mathbf{i}\cdot w\in\mathcal{A}$.
Note that the cosines of all angles in the table (third column) have the same modulus $\sqrt{r^2+s^2}$.

pair of vectors	their angle $\angle$	the cosine of $\angle$	the angle for $\vec{x}=\vec{y}$	its cosine for $\vec{x}=\vec{y}$
$(\vec{x},\vec{y})$	$\frac{\pi}{2}+a+\mathbf{i}\cdot b$	$r+\mathbf{i}\cdot s$	0	1
$(-\vec{x},\vec{y})$	$\frac{\pi}{2}-a-\mathbf{i}\cdot b$	$-r-\mathbf{i}\cdot s$	π	-1
$(\vec{y},\vec{x})$	$\frac{\pi}{2}+a-\mathbf{i}\cdot b$	$r-\mathbf{i}\cdot s$	0	1
$(-\vec{y},\vec{x})$	$\frac{\pi}{2}-a+\mathbf{i}\cdot b$	$-r+\mathbf{i}\cdot s$	π	-1
$(\mathbf{i}\cdot\vec{x},\vec{y})$	$\frac{\pi}{2}+v+\mathbf{i}\cdot w$	$-s+\mathbf{i}\cdot r$	$\frac{\pi}{2}-\mathbf{i}\cdot\log\left[\sqrt{2}+1\right]$	$\mathbf{i}$
$(\vec{y},\mathbf{i}\cdot\vec{x})$	$\frac{\pi}{2}+v-\mathbf{i}\cdot w$	$-s-\mathbf{i}\cdot r$	$\frac{\pi}{2}+\mathbf{i}\cdot\log\left[\sqrt{2}+1\right]$	$-\mathbf{i}$
$(\vec{x},\mathbf{i}\cdot\vec{y})$	$\frac{\pi}{2}-v-\mathbf{i}\cdot w$	$s-\mathbf{i}\cdot r$	$\frac{\pi}{2}+\mathbf{i}\cdot\log\left[\sqrt{2}+1\right]$	$-\mathbf{i}$
$(\mathbf{i}\cdot\vec{y},\vec{x})$	$\frac{\pi}{2}-v+\mathbf{i}\cdot w$	$s+\mathbf{i}\cdot r$	$\frac{\pi}{2}-\mathbf{i}\cdot\log\left[\sqrt{2}+1\right]$	$\mathbf{i}$

With given two vectors $\vec{x},\vec{y}\in X$ we consider as before the angle $\angle(\vec{x},\vec{y})=\frac{\pi}{2}+a+\mathbf{i}\cdot b$ with suitable real numbers a,b. Now we express the complex number $\angle(\mathbf{i}\cdot\vec{x},\vec{y})$ in dependence of a and b. For the presentation the real valued cosine and hyperbolic cosine are used, and their inverses arccosine and area hyperbolic cosine. The result is as follows:

Theorem 2. *In a complex normed space $(X,\|\cdot\|)$ we take two elements $\vec{x},\vec{y}\neq\vec{0}$. We assume the angle $\angle(\vec{x},\vec{y})=\frac{\pi}{2}+a+\mathbf{i}\cdot b\in\mathcal{A}$, i.e. $-\frac{\pi}{2}\le a\le\frac{\pi}{2}$. (If $a=-\frac{\pi}{2}$ or $a=\frac{\pi}{2}$ since $\angle(\vec{x},\vec{y})\in\mathcal{A}$ it follows $b=0$.) We use abbreviations H_- and H_+. We get*

$$\angle(\mathbf{i}\cdot\vec{x},\vec{y}) \;=\; \frac{\pi}{2}+\frac{1}{2}\cdot\left[-\mathrm{sgn}(b)\cdot\arccos(\mathsf{H}_-)+\mathbf{i}\cdot\mathrm{sgn}(a)\cdot\mathrm{arcosh}(\mathsf{H}_+)\right],\quad \textit{where}$$

$$\mathsf{H}_\pm := \sqrt{\left[\cos^2\left(\frac{\pi}{2}+a\right)+\cosh^2(b)-2\right]^2+4\cdot\cos^2\left(\frac{\pi}{2}+a\right)\cdot\cosh^2(b)} \pm\left[\cos^2\left(\frac{\pi}{2}+a\right)+\cosh^2(b)-1\right].$$

It is worthwhile to look at special cases. We consider a real angle $\angle(\vec{x},\vec{y})=\frac{\pi}{2}+a$ (i.e. $b=0$), and an angle on the vertical axis $x=\frac{\pi}{2}$, this means $\angle(\vec{x},\vec{y})=\frac{\pi}{2}+\mathbf{i}\cdot b$ (i.e. $a=0$).

Corollary 1. *For a pure real angle $\angle(\vec{x},\vec{y})=\frac{\pi}{2}+a$ with $-\frac{\pi}{2}\le a\le\frac{\pi}{2}$, i.e. $b=0$, we get a complex angle $\angle(\mathbf{i}\cdot\vec{x},\vec{y})$ with a real part $\pi/2$,*

$$\begin{aligned}\angle(\mathbf{i}\cdot\vec{x},\vec{y}) &= \frac{\pi}{2}+\mathbf{i}\cdot\frac{1}{2}\cdot\mathrm{sgn}(a)\cdot\mathrm{arcosh}\left[2\cdot\cos^2\left(\frac{\pi}{2}+a\right)+1\right]\\ &= \frac{\pi}{2}+\mathbf{i}\cdot\mathrm{sgn}(a)\cdot\log\left[\sqrt{\cos^2\left(\frac{\pi}{2}+a\right)+1}+\left|\cos\left(\frac{\pi}{2}+a\right)\right|\right].\end{aligned}$$

Corollary 2. *For an angle $\angle(\vec{x},\vec{y})=\frac{\pi}{2}+\mathbf{i}\cdot b$ (i.e. $a=0$) we get a pure real angle $\angle(\mathbf{i}\cdot\vec{x},\vec{y})$,*

$$\angle(\mathbf{i}\cdot\vec{x},\vec{y}) \;=\; \frac{\pi}{2}-\frac{1}{2}\cdot\mathrm{sgn}(b)\cdot\arccos[3-2\cdot\cosh^2(b)] \;=\; \frac{\pi}{2}-\arcsin[\sinh(b)] \;=\; \arccos[\sinh(b)]\,.$$

Now we look for real valued angles in complex vector spaces. We can prove that in a complex normed vector space $(X, \|\cdot\|)$ for each pair $\vec{x}, \vec{y} \neq \vec{0}$ of non-zero vectors there is a real number $\varphi \in [0, 2\pi]$ such that the pair $(e^{\mathbf{i}\cdot\varphi} \cdot \vec{x}\,,\, \vec{y})$ has a pure real angle, i.e. $\angle(e^{\mathbf{i}\cdot\varphi} \cdot \vec{x}\,,\, \vec{y}) \in \mathbb{R}$.

This fact ensures the existance of many real valued angles even in complex normed spaces. The situation improves further in the special case of complex vector spaces provided with an inner product $< . \mid . >$. We discuss the case of complex inner product spaces $(X, < . \mid . >)$ now.

To do ordinary Euclidean geometry we need real valued angles. The idea is simple. We take an orthogonal basis T of $(X, < . \mid . >)$, and we generate $\mathcal{L}(\mathbb{R})(\mathsf{T})$, the set of all finite real linear combinations of elements of T. Let

$$\mathcal{L}(\mathbb{R})(\mathsf{T}) \;:=\; \left\{ \sum_{i=1}^{n} r_i \cdot \vec{x}_i \mid n \in \mathbb{N},\; r_1, r_2, \ldots, r_n \in \mathbb{R},\; \vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n \in \mathsf{T} \right\} .$$

If X is a Hilbert space, i.e. it is complete, we even can use Cauchy sequences from elements of $\mathcal{L}(\mathbb{R})(\mathsf{T})$. It means that we take the closure $\overline{\mathcal{L}(\mathbb{R})(\mathsf{T})}$ of $\mathcal{L}(\mathbb{R})(\mathsf{T})$ in X. This creates a real linear subspace $\overline{\mathcal{L}(\mathbb{R})(\mathsf{T})}$ of X, where all angles are real.

After that we consider complex inner product spaces X of finite complex dimension $n \in \mathbb{N}$. Their real dimension is $2 \cdot n$, and we state that the maximal dimension of a real subspace of X with all-real angles is n. The real span $\mathcal{L}(\mathbb{R})(\mathsf{T})$ of an orthogonal basis $\mathsf{T} \subset X$ yields an example.

Finally we remark that real angles are useful to do classical Euclidean geometry. We are still considering complex inner product spaces $(X, < \cdot \mid \cdot >)$, and we mention three well-known theorems of usual geometry. Let $\vec{x}, \vec{y} \neq \vec{0}$ be elements of X. We obtain the three identities $\angle(\vec{x}, \vec{y}) = \angle(\vec{x}, \vec{x}+\vec{y}) + \angle(\vec{x}+\vec{y}, \vec{y})$, (Addition of angles), $\angle(\vec{x}, \vec{y}) + \angle(-\vec{x}, \vec{y}-\vec{x}) + \angle(-\vec{y}, \vec{x}-\vec{y}) = \pi$, (The sum of inner angles in a triangle is π), $\|\vec{x}-\vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2 - 2 \cdot \|\vec{x}\| \cdot \|\vec{y}\| \cdot \cos(\angle(\vec{x}, \vec{y}))$, (The 'Law of Cosines'), if and only if the angle $\angle(\vec{x}, \vec{y})$, and therefore all angles in the above terms are pure real.

An extended paper with all proofs and additional informations is in preparation.

Acknowledgements: We like to thank Prof. Dr. Eberhard Oeljeklaus and Jeremy Goodsell for careful reading of the paper and suggesting some improvements, as well Verena Thürey and Wilfried Henning for technical aid.

References

[1] I.N. Bronstein, K.A. Semendjajew, *Taschenbuch der Mathemetik*, 19. Auflage, Harri Deutsch (1980)
[2] C.R. Diminnie, E.Z. Andalafte and R.W. Freese, *Angles in Normed Linear Spaces and a Characterization of Real Inner Product Spaces*, Math. Nachr. **129** (1986), 197-204
[3] A. Froda, *Sur l'angle complexe, orienté, de deux vecteurs d'un espace unitaire*, Atti Accad. Naz. Lincei, Cl. Sci. Fis. Mat. Natur. Rend. Lincei(8) Mat. Appl. **30** (1961), 845-853
[4] A Galántai, C.J. Hegedüs, *Jordan's Principal Angles in Complex Vector Spaces*, Numer. Linear Algebra Appl. **13** (2006), 589-598
[5] Alan Jeffrey, *Complex Analysis and Applications*, 2. Edition, Chapman and Hall/CRC(2006)
[6] P. M. Miličić, *Sur le* g*-angle dans un espace norme*, Matematik Vesnik **45** (1993), 43-48
[7] P. M. Miličić, *On the B-Angle and* g*-Angle in Normed Spaces*, J. Inequal. Pure Appl. Math. **8**, Iss. 4 (2007), 1-18
[8] Klaus Scharnhorst, *Angles in Complex Vector Spaces*, Acta Appl. Math. **69** (2001), 95-103
[9] Volker W. Thürey, *A Generalization of the Euclidean Angle*, J. Convex Anal. **20**, No 4 (2013), 1025-1042
[10] Volker W. Thürey, *Angles and a Classification of Normed Spaces*, Ann. Funct. Anal. **4**, Iss. 1 (2013), 114-137
[11] J.E. Valentine, C. Martin, *Angles in Metric and Normed Linear Spaces*, Colloq. Math. **34** (1976), 209-217
[12] J.E. Valentine, S.G. Wayment, *Wilson Angles in Linear Normed Spaces*, Pacific J. Math. **36**, No 1 (1971), 239-243