

# Quantum gravitational collapse

**C Vaz**

Department of Physics, University of Cincinnati, Cincinnati, OH 45221-0011, USA

E-mail: [Cenalo.Vaz@uc.edu](mailto:Cenalo.Vaz@uc.edu)

**Abstract.** We have outlined a canonical quantization of the LeMaître-Tolman-Bondi models, which describe the collapse of inhomogeneous, non-rotating dust. Although, there are many models of gravitational collapse, this particular class of models stands out for its simplicity and the fact that both black holes and naked singularity end states may be realized on the classical level, depending on the initial conditions. We have solved the Wheeler-DeWitt equation exactly after regularization on a spatial lattice, and discussed some consequences of the quantization. The solutions describe Hawking radiation and provide an elegant microcanonical description of black hole entropy, but they also raise some very interesting questions concerning the nature of gravity's fundamental degrees of freedom.

## 1. Introduction

So long as no generally agreed upon theory of quantum gravity exists, it is important to examine the quantization of particular models. This contribution addresses the quantization of the LeMaître-Tolman-Bondi (LTB) solutions, which describe the classical collapse of spherically symmetric, inhomogeneous, self-gravitating dust. The classical solutions were originally introduced by G. LeMaître [1] to study cosmology, where it has found interesting applications [2]. Our principal interest here will be to develop the Hamiltonian formalism for both the classical and quantum LTB models, as such we will describe a generalization of work by Kuchař [3], who has developed a midi-superspace quantization of the Schwarzschild black hole.

In any study of quantum collapse, one would like to be able to: (a) describe Hawking radiation during collapse for black holes and naked singularities, (b) recover a microscopic model of the eternal black hole that leads naturally to the Bekenstein-Hawking entropy, and (c) understand the role played by the cosmological constant (e.g., what is the origin of the Hawking-Page transition [4] for a negative cosmological constant and how do we effectively deal with two horizons when we have a positive cosmological constant.) In this paper, we have briefly addressed the first two of these, and argued that the presence of a (negative) cosmological constant requires a rethinking of the statistics obeyed by the fundamental degrees of freedom describing the black hole.

## 2. Classical models

The interior of a spherically symmetric, non-rotating and self-gravitating dust ball is described by the stress energy tensor  $T_{\mu\nu} = \varepsilon u_\mu u_\nu$  and the line element:

$$ds^2 = d\tau^2 - \frac{R'^2}{1 + 2E(\rho)} d\rho^2 - R^2 d\Omega^2, \quad (1)$$



where  $R = R(\tau, \rho)$  is the area radius and  $\varepsilon = \varepsilon(\tau, \rho)$  is the energy density, satisfying:

$$\dot{R}(\tau, \rho) = -\sqrt{2E(\rho) + \frac{F(\rho)}{R(\tau, \rho)}}, \quad \text{and} \quad \varepsilon(\tau, \rho) = \frac{F'(\rho)}{R(\tau, \rho)^2 R'(\tau, \rho)}. \quad (2)$$

The initial data are completely specified by the two functions  $F(\rho)$  and  $E(\rho)$  representing respectively, the weighted mass and the total energy contained within a shell of radius  $\rho$ , and generally referred to as the mass and energy functions. An analysis of the classical solutions for these models can be found in [5]. Here, we have discussed the canonical formalism for the so called “marginally bound” models, defined by  $E(\rho) = 0$  [6]. A generalization to non-marginally bound models is found in [7].

The canonical dynamics of the collapsing cloud is described by embedding the spherically symmetric ADM metric (with radial function  $L$ ) in the LTB spacetime in Eq. (1), and casting the action for the Einstein-dust system in canonical form. The phase-space of non-rotating dust is described by the dust proper time  $\tau$ , and its conjugate momentum, whereas the gravitational phase-space consists of the configuration space variables  $(R, L)$  and their conjugate momenta. Using a version of the canonical transformations developed by Kuchař [3], the configuration variable  $L$  is replaced by the mass density  $\Gamma = F'$ . In terms of the new chart  $(\tau, R, \Gamma, P_\tau, P_R, P_\Gamma)$ , the momentum and the Hamiltonian constraints read:

$$H_\tau = \tau' P_\tau + R' \bar{P}_R - \Gamma P_\Gamma' \approx 0, \quad \text{and} \quad H = \left( P_\tau^2 + \mathcal{F} \bar{P}_R^2 \right) - \frac{\Gamma^2}{4\mathcal{F}} \approx 0, \quad (3)$$

where the primes now refer to derivatives with respect to the ADM label  $r$ . The algebra of the constraints is not, however, of the standard form (given, for example, in [8]). A short calculation shows that the Poisson bracket of the Hamiltonian with itself vanishes, in contrast with the general case, in which it closes on the momentum constraint. The other brackets coincide with the general case. The transformations generated by the Hamiltonian constraint can, thus, no longer be interpreted as hypersurface deformations. They are in general not orthogonal to the hypersurfaces, but act along the flow lines of dust.

### 3. Quantization

Applying Dirac’s quantization procedure, the Hamiltonian constraint turns into the Wheeler-DeWitt equation and the momentum constraint ensures spatial diffeomorphism invariance. One can show, without much difficulty, that the momentum constraint is satisfied by any wave functional of the general form:

$$\Psi[\tau, R, \Gamma] = U \left( -\frac{i}{2} \int dr \Gamma(r) \mathcal{W}(\tau, R, F) \right), \quad (4)$$

where  $U$  is an arbitrary, complex valued function of its argument, provided that  $\mathcal{W}$  has no explicit dependence on  $r$ . The Hamiltonian constraint requires more work: As usual, it must be regularized and there are factor ordering ambiguities.

Regularization may be performed on a spatial lattice so long as  $\Gamma \neq 0$ . If, moreover, one demands that the wave-functional is factorizable on the lattice, then it is possible to obtain exact and unique solutions of the Wheeler-DeWitt equation together with a natural Hilbert space measure. One finds in the exterior, the positive energy solutions given by:

$$\Psi[\tau, R, \Gamma] = \prod_j \psi_j(\tau_j, R_j, \Gamma_j) = \prod_j e^{\omega_j b_j} \times \exp \left\{ -\frac{i\omega_j}{\hbar} \left[ \tau_j \pm 2F_j \left[ z_j - \tanh^{-1} \frac{1}{z_j} \right] \right] \right\}, \quad z > 1, \quad (5)$$

where  $j$  refers to the lattice site  $\omega_j = \sigma \Gamma_j$ ,  $\sigma$  is the lattice spacing  $z_j = \sqrt{R_j/F_j}$  and

$$\Psi[\tau, R, \Gamma] = \prod_j \psi_j(\tau_j, R_j, \Gamma_j) = \prod_j e^{\omega_j b_j} \times \exp \left\{ -\frac{i\omega_j}{\hbar} \left[ \tau_j \pm 2F_j \left[ z_j - \tanh^{-1} z_j \right] \right] \right\}, \quad z < 1, \quad (6)$$

*i.e.*, in the interior. The  $\psi_j$  represent shell wave functions, *i.e.*, wave functions describing shells of collapsing dust. In the exterior, the positive sign represents ingoing waves and the negative sign represents outgoing waves. The reverse holds in the interior. For these marginal models, the measure is uniquely given by  $\mu_j = z_j$  up to a constant scaling.

#### 4. Hawking radiation

This can be demonstrated by taking an approach that closely parallel to Hawking's original work by imagining a pre-existing, eternal black hole surrounded by tenuous dust, so that the mass function has the form  $F(r) = 2M\Theta(r) + f(r)$ , where  $\Theta(r)$  is the Heaviside function and  $f(r)$  represents the dust perturbation. Then the wave functional factorizes into a piece that represents the black hole at the centre and another that represents the dust propagating in the black hole geometry. A simple calculation of the Bogoliubov coefficient then shows that [9, 10]:

$$\langle \text{in} | \hat{N}_{\text{out}} | \text{in} \rangle = |\beta(f, \omega)|^2 \approx \prod_j \frac{2\pi M}{\Delta f_j} \left[ \frac{1}{e^{8\pi M \Delta f_j} - 1} \right], \quad (7)$$

where  $\Delta f_j = f_j - f_{j-1}$  is the mass energy associated with the lattice site  $j$ . This is interpreted as the eternal black hole being in equilibrium with a thermal bath at the Hawking temperature  $(8\pi M)^{-1}$ .

An alternative approach, better adapted to *collapse*, was considered in [11]. By matching the shell wave functions describing gravitational collapse across the apparent horizon, it was shown that an ingoing wave on one side of the apparent horizon is necessarily accompanied by an outgoing wave on the other side. Furthermore, the relative amplitude of the outgoing wave is suppressed by the square root of the Boltzmann factor at the ‘‘Hawking’’ temperature of the shell. Strictly speaking, the Hawking temperature  $T_H = (8\pi M)^{-1}$ , refers to an eternal black hole of mass  $M$ . The temperature appearing in the Boltzmann factor from matching shell wave functions across the horizon is  $T_H = (4\pi F)^{-1}$ , where  $F$  is the mass function and represents twice the mass contained within the shell. Diffeomorphism invariant wave functionals describing the collapse can also be matched and yields the same picture, but now the relative amplitude of the outgoing wave functional to the ingoing one is given by  $e^{-S/2}$ , where  $S$  is the entropy of the final state black hole. These results can be extended to non-marginal models with or without a (negative) cosmological constant and in any dimension.

#### 5. Entropy

One can think of an eternal black hole as a single shell represented by a mass function  $F(r) = 2M\Theta(r)$ . Then the black hole is, in fact, described by a free Klein-Gordon like equation, which is hyperbolic in the interior and elliptic in the exterior. The stationary states (in the interior) describe a spectrum of the form [12, 13]:

$$4GM_n L_{h,n} = A_{\text{Pl}} \left( n + \frac{1}{2} \right), \quad (8)$$

where  $A_{\text{Pl}}$  is the Planck area,  $n$  is a whole number and  $L_h$  is the proper radius of the black hole horizon. This expression holds true even in the presence of a (negative) cosmological constant and in any dimension. In 3+1 dimensions, and in the absence of a cosmological constant,

$L_h = \pi M$  and hence, one finds that the *area* of the horizon is quantized in half integer units. This is no longer true in the presence of a (negative) cosmological constant.

Now, we have treated the black hole as a single shell formed as the end state of many shells that have collapsed. Regardless of their history, we have assumed that each of the shells then occupies only the levels given above, contributing some multiple of the Planck area to the total horizon area  $A$ , of the final state. Thus, if the distribution is such that  $N_n$  shells occupy level  $n$ , then the horizon area becomes:

$$\frac{A}{4} = A_{\text{Pl}} \sum_n \left( n + \frac{1}{2} \right) N_n, \quad (9)$$

and the single shell solution of Eq. (8) is to be interpreted as an excitation by  $N = \sum_n N_n$  shells. It is now a simple matter to count the number of states  $\Omega$ , leading to a black hole of a given horizon area  $A$ , assuming that they are distinguishable. Holding  $A$  fixed and extremizing the entropy  $S = k_B \ln \Omega$  with respect to the number of shells, one easily recovers the Bekenstein-Hawking area law. In addition, the area quantization in Eq. (8) ensures that the entropy is effectively quantized in units of the Planck area.

The formalism can be used to describe AdS black holes as well. In 2+1 dimensions, one obtains  $L_h = \pi l/2$ , where  $l$  is the AdS length, and therefore, Eq. (8) leads to mass quantization [14]. Likewise, in higher dimensions, for large black holes, with a horizon radius that is much larger than the AdS length scale, it is not the area but the mass of the black hole that is quantized, whereas area quantization holds in the opposite limit [13]. The two limits are separated by the Hawking-Page phase transition [4].

## Comments

When mass and not area quantization holds, Bose and not Boltzmann statistics must be employed to recover the Bekenstein-Hawking entropy. Thus, the Hawking-Page transition describes a change in the nature of the fundamental degrees of freedom from geometric for small AdS black holes to field theoretic for large AdS black holes. This calls for a better understanding. It would be useful to examine the statistical mechanics of AdS black holes near the Hawking-Page transition point.

Hawking radiation from naked singularities, important for cosmic censorship, should be interesting to examine. Singularity avoidance could also be examined from this point of view via the construction of wave-packets, but this also remains to be done. Finally, we do not quite understand how to effectively deal with collapse in the presence of a positive cosmological constant.

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