Perturbations in loop quantum cosmology

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Abstract. The era of precision cosmology has allowed us to accurately determine many important cosmological parameters, in particular via the CMB. Confronting Loop Quantum Cosmology with these observations provides us with a powerful test of the theory. For this to be possible, we need a detailed understanding of the generation and evolution of inhomogeneous perturbations during the early, quantum gravity phase of the universe. Here, we have described how Loop Quantum Cosmology provides a completion of the inflationary paradigm, that is consistent with the observed power spectra of the CMB.

1. Open issues with inflation

The standard picture we have of early universe cosmology is that of inflation. This beautiful idea contains a very few basic assumptions, yet it produces, with spectacular agreement, the power spectra of density fluctuations observed in the Cosmic Microwave Background (CMB). Nevertheless, inflation remains a paradigm in search of a model, and there are several key questions that remain unanswered. Broadly speaking, these can be split into two categories: (i) Difficulties facing the particle physics interpretation of inflation, and (ii) those we expect to be related to quantum gravity. Examples of each type are given in Table 1.

Inflation is a period of quasi-de Sitter expansion, and typically the assumption is that slow-roll inflation began with the quantum state describing perturbations in its 'natural' vacuum state: the Bunch-Davies vacuum. A priori, this is an odd assumption, since it says that the quantum state is tuned to the subsequent (quasi-de Sitter) evolution of the geometry. Essentially, this implies that the pre-inflationary dynamics of the universe and the 'true' initial state conspired in such a way as to ensure that we arrived at the onset of slow-roll inflation with no particles present (relative to the Bunch-Davies vacuum). The intuition behind this assumption was that even if there were particles present at the onset of inflation, the exponential expansion would rapidly dilute them and hence, their consequences can safely be ignored. This intuition misses the important fact that quantum fields in a dynamical spacetime experience *both* spontaneous and simulated creation of particles [2]. The latter effect actually compensates for the exponential growth of the volume in such a way that the particle number density remains approximately constant.

In classical general relativity, inflation is inevitably preceded by the Big Bang and the only natural place to give initial conditions is at this singularity. An important open question then is: Can one find a quantum gravity completion to the inflationary paradigm? To be viable, such a completion should be non-singular and agree with current observations. It would also open up the exciting possibility that we can directly observe the pre-inflationary universe. If it were

Particle physics issues	Quantum gravity issues
What is the inflaton? Why is the potential flat? How does it couple to the standard model? What interactions are present?	What are the initial conditions? Is there a singularity? Why is the perturbation theory valid? What happens when the frequencies become trans-Planckian?

Table 1. Some examples of the unresolved issues facing inflation.

possible to see observational consequences of the quantum state at the onset of slow-roll, we would be able to probe the dynamics of the quantum gravity era of the universe.

2. Loop Quantum Cosmology

Loop Quantum Gravity is particularly a well developed approach to quantizing gravity [3]. It is a Hamiltonian quantization that maintains the fundamental relationship between geometry and gravity and is fully non-perturbative. However, a rigorous understanding of the dynamics of the theory are still lacking. One useful way to make progress is to consider simplified (truncated) systems of the full theory, which on the one hand, can be completely understood, and on the other are physically interesting. This approach has been applied with great success to the study of black holes (see Fernando Barbero's paper in this session), and graviton propagators within Loop Quantum Gravity. One can also consider the truncation of full general relativity to homogeneous systems, and then use Loop Quantum Gravity techniques to quantize these. This leads to Loop Quantum Cosmology (LQC) [4], which has turned out to be very successful at answering many of the difficulties facing classical cosmology. In particular, it has been shown in detail how the classical singularity is replaced by a 'Big Bounce' and how the late time, low energy, limit reproduces the standard expectations of general relativity. It has also been shown that the probability of having a sufficiently long phase of (single scalar field driven) slow-roll inflation, within LQC, is very close to one [5].

However, in order to make predictions about LQC effects on the CMB, first, one has to extend the underlying approach to include (perturbative) inhomogeneities. There have been several promising attempts to include such inhomogeneities based on consistent alterations of the homogeneous formulation of LQC. Here, we have described a systematic approach to extending the underlying formulation.

3. LQC and perturbations

The first step is to find a suitable truncation of the phase-space of classical general relativity that allows for cosmologies with perturbative inhomogeneities. Since we work in the Hamiltonian theory, we restrict our attention to cosmologies whose spatial slices are (flat) tori. One can then show that the full phase-space decomposes into homogeneous and *purely* inhomogeneous parts:

$$\Gamma_{\mathrm{Full}} = \Gamma_{\mathrm{H}} \times \Gamma_{\mathrm{IH}} \; . \tag{1}$$

The canonical variables of the homogeneous phase-space (Γ_{H}) are $(a, \pi_a, \phi, \pi_\phi)$, i.e., the scale factor a and the scalar field ϕ, and their conjugate momenta. The canonical variables of the inhomogeneous phase-space (Γ_{IH}) are the corresponding perturbations, $\left(h_{ab}\left(x\right), \pi^{ab}\left(x\right), \varphi\left(x\right), \pi_{\varphi}\left(x\right)\right)$, all of which are purely inhomogeneous.

All the canonical structures of the phase-space decompose in this way, in particular, the symplectic structure and the Possion brackets factor. One can now consider the inhomogeneous fields as perturbations, expand the constraints and find gauge invariant degrees of freedom [6]. In particular, the gauge invariant scalar modes of the perturbations satisfy the constraint:

$$\mathcal{C} = \frac{\pi_\phi^2}{2l^3} + \frac{m^2V^2}{2l^3} - \frac{3}{8\pi G l^3} b^2 V^2 + \int \mathrm{d}^3k \left(\frac{1}{2} P_k^2 + f\left(V, b, \phi, \pi_\phi; k\right) Q_k^2 \right) \approx 0, \tag{2}$$

where $V \sim a^3$ and $b \sim \pi_a/a^2$ are canonically conjugate background variables, l is the coordinate size of the 3-torus and (P_k, Q_k) are the gauge invariant, scalar degree of freedom (related to the Mukhanov variable). The important point is that gauge invariant scalar (and tensor) perturbations behave exactly as test scalar fields with a time dependent mass $f\left(V, b, \phi, \pi_\phi; k\right)$.

In [7], it was shown how to quantize such a system, and we schematically sketch this procedure here. Writing the background and perturbation parts of Eq. (2) as:

$$\mathcal{C} = \frac{\pi_\phi^2}{2l^3} + H_0^2 + \int \mathrm{d}^3k H_{\tau,k} \ , \tag{3}$$

where H_0 is the Hamiltonian for the background geometry (and the potential term) and $H_{\tau,k}$ is the (time dependent) Hamiltonian for the perturbations. These are then promoted to operators and we have de-parameterized with respect to the scalar field ϕ, to find,

$$-i\hbar\partial_\phi \Psi = \left(\widehat{H_0^2} + \int \mathrm{d}^3k \widehat{H}_{\tau,k} \right)^{1/2} \Psi \approx \left[\widehat{H}_0 + \widehat{H}_0^{-1/2} \left(\int \mathrm{d}^3k \widehat{H}_{\tau,k} \right) \widehat{H}_0^{-1/2} \right] \Psi, \tag{4}$$

where $\widehat{H_0^2} = \hbar^2 \widehat{\Theta}_0$ is the operator that governs the LQC evolution of the background [4], and the right hand side has been approximated via a series expansion.

This is the only approximation that is made in the procedure, and it can be viewed as a test-field approximation. Essentially, ones attention is to those states in which the $\widehat{H}_0$ is dominant, and $\widehat{H}_{\tau,k}$ does not alter the background dynamics[1]. The second term on the right hand side of Eq. (4) is precisely the Hamiltonian for the perturbations associated to the relational time defined by ϕ.

Finally, one decomposes the wave-function as a tensor product of background and perturbation pieces as:

$$\Psi\left(V, Q_k, \phi\right) = \Psi_0\left(V, \phi\right) \otimes \psi\left(Q_k, \phi\right), \tag{5}$$

and considers those states for which Ψ_0 is sharply peaked with respect to some particular classical background geometry at late times, i.e., a semi-classical background geometry. Taking expectation values of Eq. (4) with respect to Ψ_0, one arrives at the standard Hamiltonian for test quantum fields in a curved spacetime, with the background scale factor $a\left(\phi\right)$ replaced by $\left\langle \Psi_0 \middle| \hat{a} \middle| \Psi_0 \right\rangle\big|_\phi$.

4. Results and conclusions

In the previous Section, we have briefly sketched how one can extend the formulation of LQC to include (perturbative) inhomogeneities. With this in hand, one can now look for consequences of the pre-inflationary era and in particular, we can look for deviations from the standard initial conditions for inflation. The standard approach in inflation is to assume the existence of a scalar field with a suitable potential, which gives oneself some initial conditions for the quantum state of the perturbations and hence, calculate the late time power spectra. Here, we have taken exactly the same approach: We have specified initial conditions at the Big Bounce, and calculated the

[1] There are many important issues to do with ensuring that the integral over k results in a well defined operator that are not being described here (See [1] for details.

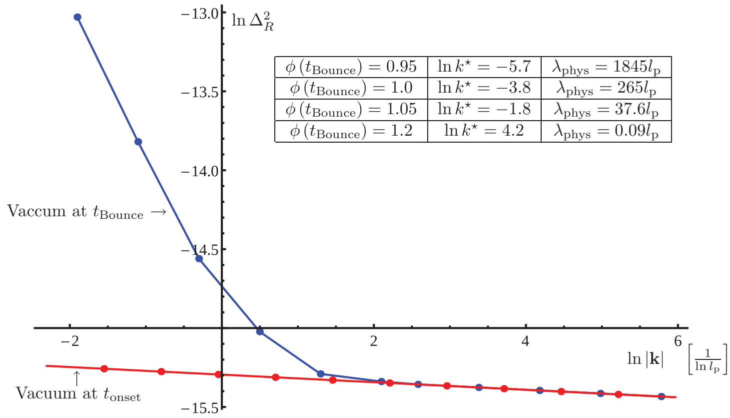

Figure 1. The power spectra for tensor perturbations at the end of slow-roll inflation, given an initial vacuum at the bounce. We have set $a(t_{\rm bounce}) = 1$ and hence, the physical scale at (for example) decoupling depends on the amount of inflation that has occurred. Note, in particular, the agreement with the predictions of standard inflation provided $\phi(t_{\rm bounce}) > 1.2$.

resulting power spectra (here, we have concentrated only on tensor perturbations). Since, at the bounce, the wavelength of all the observable modes is much smaller than the curvature scale, we have taken the initial state (for these modes) to be the Minkowski vacuum (see [1] for a discussion on this point).

Figure 1 shows the resulting late time (dimensionless) power spectra Δ^2_R, defined as:

$$\delta\left(\mathbf{k}+\mathbf{k}'\right)\frac{\Delta^2_R(k)}{4\pi|\mathbf{k}|^3} = \left\langle 0|\hat{R}_{\mathbf{k}}\hat{R}_{\mathbf{k}'}|0\right\rangle, \tag{6}$$

where the variable $R_{\mathbf{k}}$ is related to the Mukhanov variable $Q_{\mathbf{k}}$ by $R_{\mathbf{k}} = \frac{\dot{\phi}}{H}Q_{\mathbf{k}}$, and represents the gravitational potential on constant ϕ hyper-surfaces.

The important result, here, is that provided the background scalar field ϕ, at the bounce is large enough, i.e., $\phi(t_{\rm bounce}) \geq 1.2$, the power spectra is entirely consistent with that of standard inflation. Hence, we have a completion of the inflationary paradigm, including the quantum gravity era, which agrees with the observations. Note that there remains a (small but important) window ($0.95 < \phi(t_{\rm bounce}) < 1.2$) in which, we may hope to see corrections to the largest scales (smallest k) of the observed power spectra, and hence, directly observe pre-inflationary (i.e., quantum gravity) physics.

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References
[1] Agullo I, Ashtekar A and Nelson W *In preparation*
[2] Agullo I and Parker L 2011 *Phys. Rev. D* **83** 063526
[3] Ashtekar A, Bojowald M and Lewandowski J 2003 *Adv. Theor. Math. Phys.* **7** 233, Ashtekar A and Lewandowski J 2004 *Class. Quant. Grav.* **21** R53
[4] Ashtekar A and Singh P 2011 *Class. Quant. Grav.* **28** 213001, Bojowald M 2008 *Living Rev. Rel.* **11**
[5] Germani C, Nelson W and Sakellariadou M 2007 *Phys. Rev. D* **76** 043529, Ashtekar A and Sloan D 2011 *Gen. Rel. Grav.* **43** 3619, Corichi A and Karami A 2011 *Phys. Rev. D* **83** 104006
[6] David Langlois 1994 *Class. Quant. Grav.* **11** 387
[7] Ashtekar A, Kaminski W and Lewandowski J 2009 *Phys. Rev. D* **79** 064030