

Cosmological singularities, AdS/CFT and de Sitter deformations

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Abstract. We have reviewed aspects of certain time-dependent deformations of *AdS/CFT*, containing cosmological singularities and their gauge theory duals. Towards understanding these solutions better, we have explored similar singular deformations of de Sitter space and argued that these solutions are constrained, possibly corresponding to specific initial conditions.

1. Cosmological singularities and AdS/CFT

General relativity breaks down at cosmological singularities, with curvatures and tidal forces typically diverging: notions of spacetime, thus, break down. There is a rich history of string theory explorations of such singularities [1, 2]. We have focused on describing certain time-dependent deformations of AdS/CFT [3, 4, 5, 6], where the bulk gravity theory develops a cosmological singularity, and breaks down while the holographic dual field theory, a sensible Hamiltonian quantum system typically subjected to a time-dependent gauge coupling, can potentially be addressed in the vicinity of the singularity. The bulk string theory on $AdS_5 \times S^5$ (in Poincare slicing) with constant dilaton (scalar) is deformed to:

$$ds^2 = \frac{1}{z^2}(\tilde{g}_{\mu\nu}dx^\mu dx^\nu + dz^2) + d\Omega_5^2,$$

with $\tilde{g}_{\mu\nu}, \Phi$ functions of x^μ alone ($\Phi = \Phi(t)$ or $\Phi(x^+)$ then gives time-dependence). This is a solution if

$$\tilde{R}_{\mu\nu} = \frac{1}{2}\partial_\mu\Phi\partial_\nu\Phi, \quad \text{and} \quad \tilde{\square}\Phi = 0,$$

where $\tilde{\square} \equiv \frac{1}{\sqrt{-\tilde{g}}}\partial_\mu(\sqrt{-\tilde{g}}\tilde{g}^{\mu\nu}\partial_\nu)$: these include, *e.g.*, AdS-Kasner, FRW, BKL (based on the Bianchi classification), etc. In many cases, it is possible to find new coordinates such that the boundary metric $ds_4^2 = \lim_{z \rightarrow 0} z^2 ds_5^2$ is flat at least as an expansion about $z = 0$. This suggests that the dual is the $\mathcal{N}=4$ super Yang-Mills theory with the gauge coupling $g_{YM}^2 = e^\Phi$, deformed to have external time-dependence. It is useful to focus on sources approaching $e^\Phi \rightarrow 0$ at some finite point in time: For instance, a coupling of the form $g_{YM}^2 \rightarrow t^p$, $p > 0$, gives rise to $R_{tt} \sim \frac{1}{2}\dot{\Phi}^2 \sim \frac{1}{t^2}$, *i.e.*, a bulk singularity with curvatures and tidal forces diverging near $t = 0$.

Analyzing the gauge theory is possible in some cases. While at first sight, one might imagine that the dual in such cases to be weakly coupled, this is not the case and interactions are important in general [6]. For instance, the gauge kinetic terms $\frac{1}{g_{YM}^2(t)}Tr F^2$ can be transformed



to canonical ones for redefined gauge fields (as in standard perturbation theory), but this gives rise to new (tachyonic, divergent) mass-terms stemming from the time derivatives of the coupling, which ensure that the field variables get driven to large values as $t \rightarrow 0$. With gauge kinetic terms $\frac{1}{g_{YM}^2(t)} \text{Tr} F^2$, it turns out that the gauge theory Schrödinger wavefunctional near singularity ($t \rightarrow 0$) has a “wildly oscillating” phase for $p > 1$. Furthermore, the energy expectation value generically diverges as $\langle H \rangle \sim \frac{1}{g_{YM}^2(t)} \langle V \rangle$. This suggests that if the coupling vanishes strictly, $g_{YM}^2 = e^\Phi \rightarrow 0$, the gauge theory response is singular. Deforming the gauge coupling so that $g_{YM}^2 = e^\Phi$ is small but non-zero near $t = 0$ leads to finite, but large phase oscillation and energy production: $\dot{\Phi} \sim \frac{\dot{g}_{YM}}{g_{YM}}$ is now finite, so the bulk is also non-singular (but stringy). The eventual gauge theory endpoint depends on the details of energy production, but one might expect thermalization on long time scales if the sources turn off.

With the gauge coupling $g_{YM}^2 = e^{\Phi(x^+)}$ being a function of lightcone time x^+ , the physics is quite different [3, 4]: In this case, $\tilde{g}_{\mu\nu} = e^{f(x^+)} \eta_{\mu\nu}$ in the bulk, which is engineered to acquire a null singularity at, say, $x^+ = 0$ (with $e^\Phi \sim (x^+)^p$). Redefining A_μ to absorb the coupling gives canonical kinetic terms, and here, the potentially problematic mass-terms, in fact, vanish due to the lightlike coupling. With these new gauge theory variables, the interaction terms become unimportant as $e^\Phi \rightarrow 0$. The near singularity lightcone Schrödinger wavefunctional then appears regular, suggesting weakly coupled Yang-Mills theory. These variables appear to be dual to stringy objects in the bulk (see also related work on worldsheet string descriptions of certain null Kasner-like singularities [7]). We have noted the potential questions about renormalization effects, however, *e.g.*, introducing a “short-time” cutoff near singularity, and studying contributions to the gauge theory effective action from sufficiently high frequency modes (relative to $\dot{\Phi}$).

An interesting point in this discussion has to do with the initial conditions for the time evolution and eventual cosmological singularity. While in many cases, the sources can be seen to turn off in the far past, thus, suggesting the initial state is the vacuum (for the $\mathcal{N}=4$ SYM theory), this is subtle. The fact that there is a bulk curvature singularity in the deep interior ($z \rightarrow \infty$) [5] and the observation that the deformations are constrained (*e.g.*, $\tilde{R}_{\mu\nu} = \frac{1}{2} \partial_\mu \Phi \partial_\nu \Phi$, $\tilde{\square} \Phi = 0$), suggest that, in fact, the initial conditions are constrained or fine-tuned. Exploring this further turns out to be more fruitful in a related context, that of similar deformations in de Sitter space.

2. de Sitter boundary deformations and dS/CFT

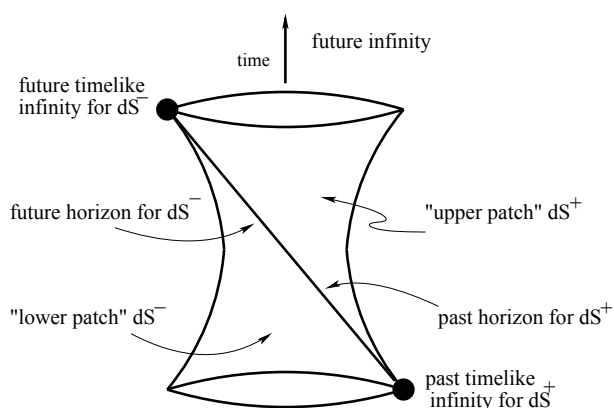


Figure 1. de Sitter space dS_{d+1} in the planar coordinates foliation has the metric $ds^2 = \frac{1}{\tau^2} [-d\tau^2 + \delta_{ij} dx^i dx^j]$: This covers half the space (dS^\pm), and $\tau \rightarrow \pm\infty$ with x^i fixed corresponds to past/future timelike infinity, while light rays $x^i \sim \tau$ give the horizon.

In the following, we have studied certain deformations of de Sitter space, containing cosmological (Big-Bang or Crunch) singularities. Consider de Sitter space dS_{d+1} in the planar

coordinates foliation (Figure 1), which is a solution to $R_{MN} = dg_{MN}$ (with positive cosmological constant), τ being conformal time. We have introduced possible deformations of the d -dim Euclidean boundary:

$$ds^2 = \frac{1}{\tau^2} [-d\tau^2 + \tilde{g}_{ij} dx^i dx^j], \quad (1)$$

where the spatial metric \tilde{g}_{ij} is a function solely of the spatial coordinates x^i (*i.e.*, not involving time τ). We have seen that these are in a sense analytic continuations of the AdS-cosmologies described above, although various physical features are quite different qualitatively. These metrics have $R_{\tau\tau} = -\frac{d}{\tau^2}$, $R_{ij} = \tilde{R}_{ij} + \frac{d}{\tau^2} \tilde{g}_{ij}$, and are, thus, solutions to Einstein's equations $R_{MN} = dg_{MN}$ (with no additional matter) if $\tilde{R}_{ij} = 0$, *i.e.*, the d -dim Euclidean metric is Ricci-flat. In general, we have considered regular \tilde{g}_{ij} . We have then seen that the generic spatial metric \tilde{g}_{ij} (even if regular) gives rise to singularities at $|\tau| \rightarrow \infty$ due to diverging $R^{ABCD} R_{ABCD} \sim \tau^4 \tilde{R}^{ijab} \tilde{R}_{ijab} + \dots$ ($|\tau| \rightarrow \infty$ is the analogue of $z \rightarrow \infty$ in the *AdS* interior).

Now, consider dS_{d+1} -deformations sourced by a background scalar field ϕ . If the scalar has purely spatial dependence $\phi(x^i)$, this is a solution if the d -dim part is an Einstein scalar system:

$$\tilde{R}_{ij} = \frac{1}{2} \partial_i \phi \partial_j \phi, \quad \text{and} \quad \tilde{\square} \phi = 0, \quad (2)$$

with $\tilde{\square} = \frac{1}{\sqrt{\tilde{g}}} \partial_i (\tilde{g}^{ij} \sqrt{\tilde{g}} \partial_j)$. This scalar is non-dynamical and could represent non-trivial initial or final conditions sourcing the deformation of the spacetime. Using Eq. (2), these solutions have the invariants:

$$R = d(d+1) + \frac{\tau^2}{2} (\tilde{\partial} \phi)^2, \quad \text{and} \quad R_{AB} R^{AB} = d^2(d+1) + \frac{3\tau^2}{2} (\tilde{\partial} \phi)^2 + \frac{\tau^4}{4} ((\tilde{\partial} \phi)^2)^2,$$

where $(\tilde{\partial} \phi)^2 = \tilde{g}^{ij} \partial_i \phi \partial_j \phi$. Thus, these invariants diverge at early/late times $|\tau| \rightarrow \infty$, sourced by the scalar (even if the scalar energy density itself is finite).

We have noticed now that purely gravitational dS_4 -deformations do not exist: $\tilde{R}_{ij} = 0$ has only trivial solutions, pure 3-dim gravity being trivial. Non-trivial dS_4 -deformations require a non-trivial source, *e.g.*, the scalar field $\phi(x^i)$ above¹: *e.g.*, consider the dS_4 -deformation:

$$ds^2 = \frac{1}{\tau^2} [-d\tau^2 + dr^2 + (r+C)d\varphi^2 + 2(r+C)dz^2], \quad \text{and} \quad e^\phi = \frac{1}{r+C}, \quad (3)$$

with $R = 12 + \frac{\tau^2}{2(r+C)^2}$, $R_{\mu\nu} R^{\mu\nu} = 36 + \frac{3\tau^2}{(r+C)^2} + \frac{\tau^4}{(r+C)^4}$, and $R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$ has similar structure. Then for finite r , these invariants diverge as $|\tau| \rightarrow \infty$, signalling a singularity at past/future timelike infinity. Light ray trajectories include geodesics of the form $\tau = r$, reaching the past horizon as $|\tau| \rightarrow \infty$, and here, these invariants are finite. If higher order invariants are also finite, this would give a spacelike singularity at $|\tau| \rightarrow \infty$.

It is interesting to compare these solutions with conventional investigations of cosmological perturbations about de Sitter space, *e.g.*, [8]. In an initial value formulation, the metric family $ds^2 = -N^2 dt^2 + h_{ij} dx^i dx^j$, with $N(t)$ the lapse function and h_{ij} , the spatial metric, has the action:

$$S = \int d^4x \sqrt{h} N \left[K_{ij} K^{ij} - K^2 + \tilde{R}^{(3)} - 2\Lambda + \frac{1}{2} N^{-2} \dot{\phi}^2 - \frac{1}{2} h^{ij} \partial_i \phi \partial_j \phi \right], \quad (4)$$

¹ For dS_5 -deformations (and higher dimensions), there are non-trivial solutions to $\tilde{R}_{ij} = 0$, *e.g.*, 4-dim Ricci-flat spaces, which include ALE spaces (non-compact $\mathbb{C}^2/\mathbb{Z}_N$ singularities and their complete resolutions which are smooth) and (compact) K3-surfaces. The dS_4 -deformation above has been found starting with the spatial metric $d\sigma^2 = dr^2 + (f(r))^2 d\varphi^2 + h(r) dz^2$, $\phi = \phi(r)$. $C \neq 0$ ensures regularity at $r = 0$; also ϕ is regular for large r .

with $R^{(3)}$ is the 3-curvature, the extrinsic curvature being $K_{ij} = -\frac{1}{2}(\nabla_i n_j + \nabla_j n_i) = -\frac{1}{2N}\partial_\tau h_{ij}$ ($K = h^{ij}K_{ij}$, and n is a unit normal to $t = \text{constant}$ surfaces). The action for small gravitational fluctuations $h_{ij} = a^2(t)(\delta_{ij} + \gamma_{ij})$ becomes:

$$S \sim \int d^4x \left(N a^d \frac{1}{4N^2} (\dot{\gamma}_{ij})^2 - N a^d \frac{1}{a^2} (\partial_k \gamma_{ij})^2 \right),$$

for dS_4 , using $N = a = \frac{1}{\tau}$ leads to the familiar Bunch-Davies vacua (after imposing appropriate regularity conditions). On the other hand, the solutions in Eq. (1) arise from the above action in the limit, where the γ_{ij} become essentially time-independent. The Hamiltonian constraint:

$$K_{ij}K^{ij} - K^2 = \tilde{R}^{(3)} - 2\Lambda - \frac{1}{2}N^{-2}\dot{\phi}^2 - \frac{1}{2}h^{ij}\partial_i\phi\partial_j\phi, \quad (5)$$

can be seen to be satisfied for Eq. (1) with $\tilde{R}_{ij} = \frac{1}{2}\partial_i\phi\partial_j\phi$, $\Lambda = \frac{d(d-1)}{2}$. The on-shell action for these solutions contains no divergence arising from the near singularity region $|\tau| \rightarrow \infty$. Similarly, the boundary term $S_B \sim \int d^3x \sqrt{g}K$ is also regular. This apparent non-singularity of the wavefunction of the universe $\Psi = e^{iS}$ could, of course, easily be invalidated by possibly singular higher derivative terms: one might expect stringy effects are important.

From the point of view of dS/CFT [9], using the discussion in [8], we expect that the wavefunction of the universe (related to the partition function for the dual field theory) is effectively an analytic continuation of that for similar solutions in Euclidean AdS , as they are correlation functions (while observables such as n-point functions of bulk fluctuations are not). The precise analytic continuation from Euclidean AdS_{d+1} to dS_{d+1} (in planar coordinates) is obtained by $z \rightarrow -i\tau$, $R_{AdS} \rightarrow -iR_{dS}$. For the deformations we are considering, in the limit of small deformations, we have seen that sources are turned on for the operators dual to bulk graviton and scalar ϕ modes. This leads us to guess that the dual CFT lives on the space \tilde{g}_{ij} and has been deformed by a source for the operator \mathcal{O}^ϕ (although for our solutions, the deformations are not small). In this perspective, these solutions are dual to corresponding deformations of the Euclidean CFT dual to de Sitter space.

Correlation functions for operators dual to bulk scalar modes can be calculated by differentiating with respect to the boundary sources. For instance, by using a Fourier decomposition in terms of eigen-mode functions on \tilde{g}_{ij} and evaluating the action, it can be seen that the momentum space 2-point function in these dS_4 -deformed theories is just as in dS_4 . The holographic stress tensor can be calculated using the usual counter term prescriptions for these theories using [10, 5], giving (on one of the dS -patches, Figure 1):

$$T^{ij} = \frac{1}{8\pi G_{d+1}} \left[K^{ij} - K h^{ij} + (d-1)h^{ij} + \frac{1}{2}G^{ij} - \frac{1}{4}\partial^i\phi\partial^j\phi + \frac{1}{8}h^{ij}(\partial\phi)^2 \right]. \quad (6)$$

For the present dS_{d+1} -deformations, with boundary metric $h_{ij} = \frac{1}{\tau^2}\tilde{g}_{ij}$, the extrinsic curvature is $K_{ij} = h_{ij}$, so that the stress tensor vanishes identically, using $\tilde{R}^{ij} = \frac{1}{2}\partial^i\phi\partial^j\phi$.

Recall that if a source for a bulk field is turned on, we generically expect a non-zero 1-point function for the dual operator as a response to the source. Here, with the metric source \tilde{g}_{ij} , we expect $\langle T_{ij} \rangle \neq 0$: This is at variance with the vanishing stress tensor above. In fact, requiring that the holographic stress tensor vanishes (relatedly, trace anomaly encoding conformality of the Euclidean CFT on the curved space) gives the conditions $\tilde{R}^{ij} = \frac{1}{2}\partial^i\phi\partial^j\phi$.

Now consider the ($\tau \sim 0$) Fefferman-Graham expansion for an asymptotically locally de Sitter spacetime:

$$ds^2 = -\frac{d\tau^2}{\tau^2} + \frac{1}{\tau^2} \left[g_{ij}^0(x^i) + \tau^2 g_{ij}^2(x^i) + \dots \right] dx^i dx^j.$$

With the leading source g_{ij}^0 , the sub-leading coefficients $g_{ij}^2, g_{ij}^4, \dots$ are generically non-zero [11], and encode information about the state of the dual CFT. Consider then the deformations above as applied to the lower patch dS^- : We have specified initial conditions for the bulk metric and equivalently, the initial conditions/state for the dual Euclidean CFT. In the present case, we have seen that, in fact, only $g_{ij}^0 \equiv \tilde{g}_{ij} \neq 0$ with $g_{ij}^n = 0$, $n > 0$, requiring that the higher order coefficients vanish, which leads to our solutions above. Using the small- τ Fefferman-Graham expansion for both metric and scalar $\phi = \tau^{(d-\Delta)/2}(\phi^0 + \tau^2\phi^2 + \dots)$, we need to solve $R_{MN} = dg_{MN} + \frac{1}{2}\partial_M\phi\partial_N\phi$ iteratively, and this gives:

$$g_{ij}^2 \sim R_{ij}^0 - \frac{1}{2}\partial_i\phi\partial_j\phi - \frac{1}{2(d-1)}\left(R - \frac{1}{2}(\partial\phi)^2\right)g_{ij}^0 \quad \text{and} \quad g_{ij}^2 = 0 \quad \Rightarrow \quad R_{ij}^0 = \frac{1}{2}\partial_i\phi\partial_j\phi, \quad (7)$$

(for a massless scalar $\Delta = d$) also implying the higher order coefficients vanish. Likewise, with \square^0 being the Laplacian with respect to g_{ij}^0 , we have also obtained $\phi^{(2)} \sim \square^0\phi^0$: thus $\phi^{(2)} = 0$ implies $\square^0\phi^0 = 0$. The vanishing of the stress tensor is also related to these sub-leading coefficients.

Our solutions have constrained these sub-leading pieces of the metric and scalar in the small τ expansion to vanish. These conditions on the g_{ij}^n , ϕ^n , $n > 0$, are highly non-generic and appear to be non-trivial constraints on the CFT state.

3. Discussion

We have argued that these deformations of AdS and dS are constrained from a Fefferman-Graham perspective, leading to certain singular structures. This suggests to turning on the sub-leading coefficients in appropriate fashion towards de-singularizing them, and we hope to explore this further. We believe these arguments also apply to the solutions in [12], which although, quite different in interpretation, are related to the AdS null solutions (section 1 above) by coordinate transformations [5]. Here, diverging tidal forces (with finite curvature invariants) lead to the large- z singularity [13].

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