

Detailed black hole state counting in loop quantum gravity

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Abstract. The combinatorial problems associated with the counting of black hole states in loop quantum gravity can be analyzed by using suitable generating functions. These can be used to perform a statistical analysis of the black hole degeneracy spectrum, study the interesting sub-structure found in the entropy of microscopic black holes, and its asymptotic behaviour for large horizon areas. They are also relevant for the discussion of the thermodynamic limit for black holes, and the understanding of sub-leading corrections to the Bekenstein-Hawking law.

1. Introduction

The two main goals of this contribution are to briefly describe the mathematical methods employed in the computation of black hole entropy in loop quantum gravity (LQG), and to discuss some applications of the resulting formalism. In the LQG framework, black holes are modelled with the help of isolated horizons [1, 2]. The quantization is carried out by combining the mathematical framework of LQG with the treatment of Chern-Simons models. By doing this, the Hilbert space appropriate for the model can be written as $\mathcal{H} = \mathcal{H}_{\text{bulk}} \otimes \mathcal{H}_{\text{surface}}$ and the entropy obtained from a maximally mixed density matrix by tracing over bulk states. In all the different proposals that have appeared in the literature [3, 4, 5] to obtain the black hole entropy (see [6, 7] for discussions on their relative merits), the actual computations can be phrased as well defined combinatorial problems involving the spin labels associated with the edges of spin networks piercing the BH horizon. These problems can be exactly solved by using number theoretic ideas. These methods are necessary for several reasons:

- (i) Interesting (quasi) periodic behaviour of the entropy, that should be explained, has been found by Corichi, Borja and Díaz Polo [8].
- (ii) It is important to have a detailed understanding of the entropy asymptotics.
- (iii) It is important to undertake precision studies of the entropy in the LQG framework. This type of study is, in fact, the object of a lot of activity in string inspired models at the present moment [9].
- (iv) They are useful to understand the thermodynamic limit, because they can be efficiently used to obtain partition functions in the area of canonical ensemble.

2. An assortment of combinatorial problems

In the following, we have enunciated the combinatorial problems that tell us how to obtain the black hole entropy for the three main proposals that have appeared in the literature:



- The Domagała-Lewandowski (DL) prescription: *The entropy $S_{\text{micro}}^{\text{DL}}(a)$ of a quantum horizon of the classical area a , according to quantum geometry and the Ashtekar-Baez-Corichi-Krasnov framework is:*

$$S_{\text{micro}}^{\text{DL}}(a) = \log \Omega^{\text{DL}}(a),$$

where $\Omega^{\text{DL}}(a)$ is one plus the number of all the finite sequences (m_1, \dots, m_N) of non-zero elements of $\frac{1}{2}\mathbb{Z}$, such that the following inequality and equality are satisfied:

$$\sum_{I=1}^N \sqrt{|m_I|(|m_I| + 1)} \leq \frac{a}{8\pi\gamma\ell_P^2}, \quad \text{and} \quad \sum_{I=1}^N m_I = 0,$$

where γ is the Immirzi parameter of quantum geometry.

- The Ghosh-Mitra (GM) prescription: *The entropy $S_{\leq}^{\text{GM}}(a)$ of a quantum horizon of the classical area a , according to the GM prescription, is defined by computing:*

$$S_{\leq}^{\text{GM}}(a) = \log \Omega^{\text{GM}}(a),$$

where $\Omega^{\text{GM}}(a)$ is one plus the number of all the finite, arbitrarily long, sequences $((j_1, m_1), \dots, (j_N, m_N))$ of ordered pairs of non-zero half integers j_I and spin components $m_I \in \{-j_I, -j_I + 1, \dots, j_I\}$, satisfying:

$$\sum_{I=1}^N \sqrt{j_I(j_I + 1)} \leq \frac{a}{8\pi\gamma\ell_P^2} \quad \text{and} \quad \sum_I m_I = 0.$$

- The Engle-Noui-Perez (ENP) prescription: *The entropy $S_{\text{micro}}^{\text{ENP}}(a)$ of a quantum horizon of the classical area a , is given by:*

$$S_{\text{micro}}^{\text{ENP}}(a) = \log \Omega^{\text{ENP}}(a),$$

where $\Omega^{\text{ENP}}(a)$ is one plus the number of all the finite, arbitrarily long, sequences (j_1, \dots, j_N) of non-zero positive half integers j_I satisfying:

$$\sum_{I=1}^N \sqrt{j_I(j_I + 1)} \leq \frac{a}{8\pi\gamma\ell_P^2},$$

and counted with a multiplicity given by the dimension of the invariant sub-space $\text{Inv}(\otimes_I [j_I])$.

The combinatorial problems associated with these three prescriptions can be solved by very similar methods that can be summarized in four steps [6]:

- Find the possible values of $|m_I|$ (or j_I) and their multiplicity for a given value of the area a (the so called configurations). Here, the inequalities that appear in the combinatorial problems are substituted by equalities.
- Compute, for each configuration, the number of possible re-orderings of the labels.
- Compute, for each re-ordering, the number of configurations compatible with the additional conditions specific of each proposal: Signs for DL, spin components for GM, or the ENP condition involving the dimension of the invariant sub-space.
- Add up the contributions to the black hole degeneracy for all the areas up to a , to take into account the inequalities that appear in the definitions of the entropy.

The solution for each of these problems can be conveniently encoded by the following generating functions (especially, useful to study the asymptotic limit of the entropy for large areas):

$$\begin{aligned} G^{\text{DL}}(z; x_1, x_2, \dots) &= \left(1 - \sum_{i=1}^{\infty} \sum_{\alpha=1}^{\infty} \left(z^{k_{\alpha}^i} + z^{-k_{\alpha}^i} \right) x_i^{y_{\alpha}^i} \right)^{-1}, \\ G^{\text{GM}}(z; x_1, x_2, \dots) &= \left(1 - \sum_{i=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{z^{k_m^i+1} - z^{-k_m^i-1}}{z - z^{-1}} \right) x_i^{y_m^i} \right)^{-1}, \text{ and} \\ G^{\text{ENP}}(z; x_1, x_2, \dots) &= -\frac{(z - z^{-1})^2}{2} \left(1 - \sum_{i=1}^{\infty} \sum_{\alpha=1}^{\infty} \left(\frac{z^{k_{\alpha}^i+1} - z^{-k_{\alpha}^i-1}}{z - z^{-1}} \right) x_i^{y_{\alpha}^i} \right)^{-1}. \end{aligned}$$

Here,

- The variables x_i are associated to square free integers p_i .
- The numbers $(k_{\alpha}^i, y_{\alpha}^i)$, $\alpha \in \mathbb{N}$ are solutions to the Pell equation associated to the square free p_i : $(k+1)^2 - p_i y^2 = 1$.
- The coefficient of the term $z^0 x_1^{q_1} \dots x_i^{q_i} \dots$ gives the number of sequences of non-zero half-integers, such that $a = \sum_i q_i \sqrt{p_i}$, and satisfying the additional constraints defining the DL, GM or ENP countings.
- For a given value of a , only finite number of variables and terms are needed.

The inequality involving the area a that appears in the definition of the entropy can be conveniently taken care of by using Laplace transforms given as:

$$\mathcal{L} \left[\sum_{n \in \mathbb{N}} \beta_n \theta(a - a_n); s \right] = \frac{1}{s} \sum_{n \in \mathbb{N}} \beta_n e^{-a_n s}.$$

This is so because, if the positions of the jumps (the area eigenvalues a_n) and their magnitudes (the black hole degeneracies β_n) can be encoded in a function that can be expanded as $\sum_{n \in \mathbb{N}} \beta_n e^{-a_n s}$, then we can get an integral representation for the BH entropy as an inverse Laplace transform. This can be done by using the generating functions given above, by substituting the x_i in $G(z; x_1, x_2, \dots)$ by $x_i = e^{-s \sqrt{p_i}/2}$ (and also $z = e^{i\omega}$), because $x_1^{q_1} \dots x_r^{q_r} \mapsto e^{-\frac{s}{2}(q_1 \sqrt{p_1} + \dots + q_r \sqrt{p_r})} = e^{-as}$, when $a = q_1 \sqrt{p_1} + \dots + q_r \sqrt{p_r}$ (an eigenvalue of the area operator). This way, we obtain:

$$\exp S^{\text{DL}}(a) = \frac{1}{(2\pi)^2 i} \int_0^{2\pi} \int_{x_0-i\infty}^{x_0+i\infty} \frac{e^{as}}{s} \left(1 - 2 \sum_{k=1}^{\infty} e^{-s \sqrt{k(k+2)}} \cos \omega k \right)^{-1} ds d\omega, \quad (1)$$

$$\exp S^{\text{GM}}(a) = \frac{1}{(2\pi)^2 i} \int_0^{2\pi} \int_{x_0-i\infty}^{x_0+i\infty} \frac{e^{as}}{s} \left(1 - \sum_{k=1}^{\infty} \frac{\sin((k+1)\omega)}{\sin \omega} e^{-s \sqrt{k(k+2)}} \right)^{-1} ds d\omega, \quad (2)$$

and

$$\exp S^{\text{ENP}}(a) = \frac{2}{(2\pi)^2 i} \int_0^{2\pi} \int_{x_0-i\infty}^{x_0+i\infty} (\sin^2 \omega) \frac{e^{as}}{s} \left(1 - \sum_{k=1}^{\infty} \frac{\sin((k+1)\omega)}{\sin \omega} e^{-s \sqrt{k(k+2)}} \right)^{-1} ds d\omega. \quad (3)$$

The asymptotic behaviour of the entropy as a function of the area can be extracted by looking at the pole structure of the integrands in the above formulas. By doing this, one easily finds the

Bekenstein-Hawking law $S(a) \propto a$. The $1/4$ pre-factor can be obtained by fixing the value of the Immirzi parameter γ . It is important to point out that the extra integration in ω gives rise to logarithmic corrections to the asymptotic behaviour for large areas. The results obtained from an asymptotic analysis along the lines just described are summarized in the following table:

Approach	γ	Logarithmic correction
DL	$\gamma_{\text{DL}} = 0.237 \dots$	$-1/2 \log(a/\ell_P^2)$
GM	$\gamma_{\text{GM}} = 0.274 \dots$	$-1/2 \log(a/\ell_P^2)$
ENP	$\gamma_{\text{ENP}} = \gamma_{\text{GM}}$	$-3/2 \log(a/\ell_P^2)$

3. Conclusion

The methods that we have described above can be used to obtain important results. It is shown in a plausible way that the staircase structure of the entropy that is found for microscopic black holes is not present in the large area limit [10]. This result can be derived by introducing new generating functions that isolate the configurations contributing to each step in the entropy. In any case, it is important to realize that the counting (or statistical) entropy considered up to this point is not strictly equivalent to the entropy of the system unless one goes to a suitable thermodynamic limit. In this limit, classic theorems [11] show that the entropy has some nice smoothness and concavity properties, necessary to ensure that the standard formalism of thermodynamics can be used, and to guarantee the stability of the system respectively.

The analysis of the thermodynamic limit in the LQG black holes description [12] can be carried out, because the partition function in the so called *canonical area ensemble* can be directly read off from Eqs. (1), (2) and (3). The main conclusions derived from this analysis are that the staircase behaviour of the entropy is absent in this limit (as is to be expected from general convexity results) and also that the sub-dominant corrections to the large area behaviour of the entropy differ from the ones obtained from the counting entropy. This means that one should exercise some care when deriving physical conclusions from this type of sub-dominant corrections, in particular, when several approaches to the computation of the black hole entropy are compared.

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