Spacetime inhomogeneity and gravitational collapse

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Abstract. A model for the non-adiabatic collapse of a spherically symmetric anisotropic distribution of matter accompanied with radial heat flux has been presented. This is an inhomogeneous generalization of the model, describing collapse of a homogeneous distribution of isotropic fluid and, therefore, provides a mechanism to investigate the impacts of inhomogeneity and anisotropic stresses on the collapse of a radiating star.

1. Introduction

In cosmology and astrophysics, there are many outstanding issues relating to the collapse of a self-gravitating system which continue to bother us even today [1, 2]. In the absence of any established theory governing collapse, it has been found to be a good idea to make separate investigations on the impact of various factors such as shear, inhomogeneity, anisotropy, electromagnetic field, etc. on collapse. Through such investigations, one hopes to get a proper understanding of the gross physical behaviour of a gravitationally collapsing body. In classical GR, solutions to Einstein's field equations for a self-gravitating system and their interpretation provide valuable insight into a collapsing system. The mechanism of solving Einstein's field equations describing a physically realistic dynamical system and predicting its subsequently evolutionary stages got a tremendous impetus when Vaidya [3] derived the metric corresponding to the exterior gravitational field of a radiating star, and Santos [4] presented the junction conditions joining the interior spacetime of the collapsing body to the Vaidya exterior metric.

The objective, here, is to examine the role of spacetime inhomogeneity and anisotropic stresses on the collapse of a radiating star. We have reported a model, which describes the evolution of a spherically symmetric inhomogeneous anisotropic fluid distribution radiating away its energy in the form of radial heat flux and shrinking in size as the collapse proceeds. The model is a generalization of an earlier work presented by Banerjee *et al* [5] describing collapse of a homogeneously distributed isotropic fluid. In the present work, inhomogeneity in the background spacetime has been introduced by considering the $t = constant$ hyperspace of the 4-D manifold as having a geometry of a 3-spheroid rather than a 3-sphere. Impacts of the inhomogeneous nature of the background spacetime and anisotropic stresses on the collapse have been examined by comparing the physical parameters in this set up to the behaviour of the corresponding parameters in Banerjee *et al* [5] model, admissible as a special class in this model.

2. An inhomogeneously collapsing model

The spacetime metric formulated by Maiti [6] represents the gravitational field of a spherically symmetric conformally flat shear-free and rotation-free fluid with heat flux as source. Making use of Maiti's prescription, Banerjee *et al* [5] have presented a simple model for the collapse of a homogeneously distributed isotropic fluid. Various aspects of the model have been extensively examined by Schäfer and Goenner [7], by giving constraints on the model parameters complying with various physical plausibility requirements. To generalize the model given in [5], let us assume that the collapsing configuration is an inhomogeneous distribution of anisotropic fluid with its background spacetime described by the metric:

$$ds_-^2 = -A_0^2(r)dt^2 + R^2(t)\left[\frac{1+\lambda kr^2}{1-kr^2}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]. \tag{1}$$

In Eq. (1), $k \neq 0$ is a constant and λ is a parameter measuring departure from homogeneous geometry. The $t = constant$ hypersurface of the spacetime metric (1) has a geometry of a 3-spheroid representing a perturbation from that of a 3-sphere [8, 9, 10, 11, 12]. The energy-momentum tensor of the fluid with anisotropic stresses filling the interior of the collapsing body is written explicitly in the form:

$$T_{\alpha\beta} = (\rho + p_t)u_\alpha u_\beta + p_t g_{\alpha\beta} + (p_r - p_t)\chi_\alpha\chi_\beta + q_\alpha u_\beta + q_\beta u_\alpha\ , \tag{2}$$

where ρ = energy density, p_r = radial pressure, p_t = tangential pressure, χ^i = unit space-like four vector along the radial direction, u^i = 4-velocity of the fluid, and $q^\alpha = q\delta_r^\alpha$ is the radially directed heat flux vector so that $q^\alpha u_\alpha = 0$. In view of Eqs.(1) and (2), Einstein's field equations lead to the following system of four independent equations (rendering $G = c = 1$):

$$8\pi\rho = \frac{1}{R^2}\left[\frac{1}{r^2} - \frac{1}{r^2B_0^2} + \frac{2B_0'}{rB_0^3}\right] + \frac{3\dot{R}^2}{A_0^2R^2}\ , \tag{3}$$

$$8\pi p_r = \frac{1}{R^2}\left[-\frac{1}{r^2} + \frac{1}{B_0^2r^2} + \frac{2A_0'}{rA_0B_0^2}\right] - \frac{1}{A_0^2}\left(\frac{\dot{R}^2}{R^2} + 2\frac{\ddot{R}}{R}\right)\ , \tag{4}$$

$$8\pi p_t = \frac{1}{R^2}\left[\frac{A_0''}{A_0B_0^2} + \frac{A_0'}{rA_0B_0^2} - \frac{B_0'}{rB_0^3} - \frac{A_0'B_0'}{A_0B_0^3}\right] - \frac{1}{A_0^2}\left[\frac{2\ddot{R}}{R} + \frac{\dot{R}^2}{R^2}\right]\ , \quad and \tag{5}$$

$$8\pi q = -\frac{2A_0'\dot{R}}{A_0^2B_0^2R^3}\ , \tag{6}$$

where, $B_0(r) = \sqrt{(1+\lambda kr^2)/(1-kr^2)}$. Combining Eqs. (4) and (5), a time-independent differential equation of the form:

$$\frac{A_0''}{A_0B_0^2} - \frac{A_0'}{rA_0B_0^2} - \frac{B_0'}{rB_0^3} - \frac{A_0'B_0'}{A_0B_0^3} - \frac{1}{B_0^2r^2} + \frac{1}{r^2} - \delta(r) = 0\ , \tag{7}$$

may be obtained, if it is assumed that anisotropy in this model evolves as:

$$8\pi(p_t - p_r) = \Delta(r,t) = \frac{\delta(r)}{R^2(t)}. \tag{8}$$

Eq. (7) may be rewritten as:

$$(1+\lambda k - \lambda kx^2)\frac{d^2A_0}{dx^2} + \lambda kx\frac{dA_0}{dx} + \left(\lambda k(\lambda k+1) - \frac{(1+\lambda k - \lambda kx^2)^2\delta(r)}{k(1-x^2)}\right)A_0 = 0\ , \tag{9}$$

where a transformation $x^2 = 1 - kr^2$ has been introduced. Eq. (9) admits a solution for $A_0(r)$, if the function $\delta(r)$ is specified as:

$$\delta(x) = \frac{k(1-x^2)[2\lambda k(\lambda k+1)(2\lambda k+1) - (4\lambda k+7)\lambda^2k^2x^2]}{4(1+\lambda k - \lambda kx^2)^3}\ , \tag{10}$$

which is regular at all interior points of the configuration. The solution for the metric function $A_0(r)$ is then obtained as:

$$A_0(r) = (1 + \lambda k^2 r^2)^{1/4}\Big(C + D\sqrt{1 - kr^2}\Big), \tag{11}$$

where C and D are arbitrary constants of integration. Consequently, the spacetime of the collapsing shear-free stellar body with anisotropic stresses is described by metric

$$ds_-^2 = -(1 + \lambda k^2 r^2)^{1/2}\Big(C + D\sqrt{1 - kr^2}\Big)^2 dt^2 + R^2(t)\left[\frac{1 + \lambda kr^2}{1 - kr^2}dr^2 + r^2 d\Omega^2\right]. \tag{12}$$

The model in [5] turns out to be a sub-class of a solution of the metric in Eq. (12), which follows on setting $\lambda = 0$ and $D = -1$ [7]. Thus, without any loss of generality we have set $D = -1$ and investigated the impact of inhomogeneity ($\lambda \neq 0$) on collapse.

The evolution of the collapse in the stellar body with interior spacetime metric in Eq. (12) is governed by the function $R(t)$, which can be determined from the boundary conditions across the boundary surface Σ of the collapsing configuration. The spacetime outside the collapsing body is appropriately described by the Vaidya [3] metric:

$$ds_+^2 = -\left(1 - 2m(v)/\bar{r}\right) dv^2 - 2dv d\bar{r} + \bar{r}^2 d\Omega^2, \tag{13}$$

where v denotes the retarded time and $m(v)$ represents the total mass of the collapsing star. The matching condition $(p_r)_\Sigma = (qR(t)B_0)_\Sigma$ [4], linking smoothly the interior and the exterior spacetimes across the boundary Σ, yields a differential equation of the form:

$$\ddot{R}R + \frac{1}{2}\dot{R}^2 - \alpha\dot{R} + \beta = 0, \tag{14}$$

where, α and β are constants evaluated at the surface. A simple solution of Eq. (14) has been obtained as [7]:

$$R(t) = nt, \quad n = \alpha - \sqrt{\alpha^2 - 2\beta}\ . \tag{15}$$

For a collapsing configuration, the model parameters should be so chosen that $\dot{R} < 0$. Similarly, the rate of expansion is given by:

$$\Theta = u^\alpha_{;\alpha} = \frac{3\dot{R}}{A_0 R} = \frac{3}{t(1 + \lambda k^2 r^2)^{1/4}(C - \sqrt{1 - kr^2})}\ , \tag{16}$$

should also be negative as it describes a contracting body. Therefore, during the collapse from time $t = -\infty$ to $t = 0$, we must have $(1 + \lambda k^2 r^2)^{1/4}(C - \sqrt{1 - kr^2}) > 0$, implying $C > 1$. The upper bound on C can be obtained by constraining $\alpha^2 > 2\beta$ so as to ensure that n remains real.

The dynamical variables of matter density, the radial and transverse pressures, and the heat flux parameter associated with the collapsing stellar structure are obtained as:

$$8\pi\rho = \frac{k(\lambda + 1)(3 + \lambda kr^2)}{n^2 t^2 (1 + \lambda kr^2)^2} + \frac{3}{t^2(1 + \lambda k^2 r^2)^{1/2}(C - \sqrt{1 - kr^2})^2}\ , \tag{17}$$

$$8\pi p_r = 8\pi\rho - \frac{4}{t^2(1 + \lambda k^2 r^2)^{1/2}(C - \sqrt{1 - kr^2})^2}\ , \tag{18}$$

$$8\pi p_t = 8\pi p_r + \frac{k^2 r^2\left[2\lambda k(1 + \lambda k)(1 + 2\lambda k) - \lambda^2 k^2(1 - kr^2)(4\lambda k + 7)\right]}{4n^2 t^2(1 + \lambda k^2 r^2)^3}\ , \quad and \tag{19}$$

$$8\pi q = -\frac{r\sqrt{1 - kr^2}\left[C\lambda k\sqrt{1 - kr^2} + (2 - \lambda k + 3\lambda k^2 r^2)\right]}{n^2 t^3 (1 + \lambda kr^2)(C - \sqrt{1 - kr^2})^2 (1 + \lambda k^2 r^2)^{5/4}}\ . \tag{20}$$

It can be shown that $\rho,\ p_r,\ p_t > 0$ and $\rho',\ p_r',\ p_t' < 0$, if the conditions $k > 0$ and $\lambda \geq 0$ are satisfied simultaneously. Moreover, the weak energy condition $\rho > p_r,\ pt$ is satisfied in this model.

The proper radius $[R(t)r]_\Sigma = ntr_\Sigma$ is infinite when collapse begins, positive at any later instant t and shrinks to zero at $t = 0$. The explicit expression for mass at any instant t within the boundary radius r_Σ is obtained as:

$$m(v) \stackrel{\Sigma}{=} m(r_\Sigma, t) = \frac{nr_\Sigma t}{2}\left[\frac{(1+\lambda)kr_\Sigma^2}{(1+\lambda kr_\Sigma^2)} + \frac{n^2r_\Sigma^2}{\left(C - \sqrt{1-kr_\Sigma^2}\right)^2\left(1+\lambda k^2 r_\Sigma^2\right)^{1/2}}\right], \tag{21}$$

which shows that the collapse begins with an infinite mass and size of the configuration at $t \to -\infty$ and it evaporates completely as the epoch $t = 0$ approaches. The collapse does not have any impact on the mass to size ratio $(2m(r,t)/rR)_\Sigma$ as it is independent of time, and therefore, remains constant throughout the evolution. Consequently, no event horizon is formed during the collapse, since $g_{tt} = 1 - [2m(r,t)/(rR)]_\Sigma = 1 - constant$, always remains positive. As the singularity is reached at $t = 0$, $m(r,t) = 0$, which implies that the collapsing body radiates all its mass energy in the process and the collapse terminates into a naked singularity. The Ricci curvature, in this model, diverges as $1/t^2$, which suggests that it is a weak curvature singularity.

3. Conclusion

In this paper, a general framework to investigate the impacts of inhomogeneous nature of background spacetime and anisotropic stresses on a gravitationally collapsing system has been formulated by generalizing the Banerjee *et al* [5] model. The set up provides a mechanism to investigate the impact of inhomogeneity on collapse by comparing the evolution of an inhomogeneous distribution ($\lambda \neq 0$) of anisotropic matter with the evolution of a homogeneous distribution ($\lambda = 0$) of isotropic matter [7]. The parameter λ is the measure of departure from homogeneity in this construction. Though, in general, the physical parameters evolve differently for two different background spacetimes, an in-depth analysis shows that the gross features of the evolving system remain unaffected by the inhomogeneous perturbation of the background spacetime.

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