

Variational principle for gravity in the extended phase-space

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Abstract. Variational formalism in the extended phase-space for fields is applied to gravity.

1. Introduction

The purpose of this paper is to show that torsion, even when zero, plays the crucial role of a mediating field between the stress-energy 3-form of matter fields and a 3-form related to the Einstein tensor, by coupling with both in the extended phase-space formalism recently given by the author. The extended phase-space formalism [1, 2] can be summarized as follows: When the evolution is along the one-dimensional time, the dynamics is determined by the stationary Poincare-Cartan or action 1-form $\Xi = p_i dq^i - H dt$ in the extended phase-space (EPS) with coordinates (t, q, p) . The EPS has a fiber bundle structure with time as the base manifold and the phase-space with coordinates (q, p) as the fiber. The evolution trajectories are *sections* from the one-dimensional base manifold into the EPS.

The Lie derivative of action is zero, i.e., $L_X \int \Xi = 0$, for a vector field X of variation of variables in the EPS. Using $L_X = i_X \circ d + d \circ i_X$, we see that we should first calculate $d\Xi$ and obtain equations of motion by contracting (i.e., applying i_X) with different independent variation fields X . The second term simply gives the Noether theorem $d(i_X \Xi) = 0$ for symmetry fields X , which satisfy $L_X \Xi = 0$.

2. Co-frames and frame-gauge invariance

We have described the gravitational field by giving a system of orthonormal co-frames. This involves four differential 1-forms $n^a = N_\mu^a dx^\mu$, $a = 0, 1, 2, 3$, such that $\langle n^a, n^b \rangle = \eta^{ab}$, where η is the matrix with $(-1, 1, 1, 1)$ on the diagonal and zero elsewhere. The co-frame n^a can be replaced by Lorentz rotated co-frame at each point without any change in the metric.

The essence of general theory of relativity is that physics is independent of the choice of local inertial frames represented by orthonormal co-frames. We have called this invariance, the “frame-gauge invariance”. A local gauge group, in turn, involves its own connection forms or gauge potentials. The exterior derivatives dn^a are not frame-gauge invariant. They have to be replaced by covariant derivatives $dn^a + \omega^a_b \wedge n^b$, where the connection matrix ω^a_b of 1-forms, under the frame change by Lorentz transformation $\Lambda : n' = \Lambda n$, transforms as $\omega' = \Lambda \omega \Lambda^{-1} + \Lambda d\Lambda^{-1}$. (We have omitted the indices and the wedge symbol when there is no confusion.) The covariant derivatives of the co-frame n^a are called the *torsion* 2-forms of the



Einstein-Cartan theory, given as:

$$T^a = dn^a + \omega^a_b \wedge n^b. \quad (1)$$

The frame-gauge fields, or *curvature 2-forms*, are determined by:

$$\Omega^a_b \equiv d\omega^a_b + \omega^a_c \wedge \omega^c_b. \quad (2)$$

The torsion and curvature (given in Eqs. (1) and (2) respectively) forms yield the *Bianchi identities* on exterior differentiation:

$$dT + \omega T = \Omega n, \quad (3)$$

$$d\Omega + \omega \Omega = \Omega \omega. \quad (4)$$

The nature of our gauge group (Lorentz group) is reflected in the antisymmetry of the connection matrix when one index is lowered by Minkowski metric η ,

$$\omega_{ab} \equiv \eta_{ac} \omega^c_b = -\omega_{ba}, \quad \text{and} \quad \Omega_{ab} \equiv \eta_{ac} \Omega^c_b = -\Omega_{ba}. \quad (5)$$

We have used constant matrices η 's for raising and lowering of indices.

3. EPS for fields

It has been suggested by the author that the fiber bundle structure of the extended phase-space in mechanics should be taken over for field systems by replacing the 1-dimensional base manifold by the 4-dimensional spacetime. The Poincare-Cartan form (or action) is now a 4-form [2]. Let ϕ be a scalar field. Then its canonical momentum is a differential 1-form p in this formalism. The Poincare-Cartan form has the structure:

$$\Xi_\phi = (*p) \wedge d\phi - H, \quad (6)$$

where the Hodge star operator [3] is used to convert a 1-form p into the 3-form $*p$ in order to obtain a 4-form of the type ' pdq '. The 'covariant Hamiltonian' $H = H(\phi, p)$, is a 4-form constructed from coordinate (0-form ϕ) and its canonical momentum (1-form p). The scalar field covariant Hamiltonian is:

$$H = \frac{1}{2}(*p) \wedge p + \frac{1}{2}m^2\phi^2(*1). \quad (7)$$

Here, we have used the notation $*1 = n^0 \wedge n^1 \wedge n^2 \wedge n^3$ for the Hodge dual of the constant function 1.

Allowed field configurations, here, are those four-dimensional sub-manifolds of the extended phase-space, which are sections on which the 4-form $i_X \circ d\Xi_\phi = 0$ for independent variational fields X . We calculate:

$$-d\Xi_\phi = [dp_a(*n^a) - m^2\phi(*1)] [p - d\phi] - \Theta^b T_b, \quad (8)$$

using formulas

$$p = p_a n^a, \quad dp = dp_a n^a + p_a dn^a, \quad d(*p) = dp_a \wedge (*n^a) + p_a d(*n^a),$$

$$d(*n^a) = *(n^a \wedge n^b) \wedge dn_b, \quad (*n^a) \wedge n^b = -\eta^{ab}(*1), \quad \text{and} \quad d(*1) = (*n^a) \wedge dn_a,$$

replacing dn_b by the torsion 2-form $T_a = dn_a + \omega_{ab}n^b$, because the added term containing 5 factors of 1-forms n in a four dimensional space is zero. The *stress-energy 3-form* Θ is given by:

$$\begin{aligned}\Theta^b &= p_a(*n^a n^b)d\phi + \frac{1}{2}p_a p^a(*n^b) - \frac{1}{2}m^2\phi^2(*n^b), \\ &= \left[p^b p^c - \eta^{bc} \left(\frac{1}{2}p_a p^a + \frac{1}{2}m^2\phi^2 \right) \right] (*n_c),\end{aligned}\tag{9}$$

$$\equiv T^{bc}(*n_c),\tag{10}$$

where T^{ab} is the familiar *stress-energy tensor*. The Hamiltonian equations for matter fields from the variation of fields ϕ and p while keeping frame fields and connection fixed are read off from the two factors of the first term:

$$p = d\phi, \quad dp_a(*n^a) - m^2\phi(*1) = 0.\tag{11}$$

If we substitute $p = d\phi$ in the second equation, we obtain the Klein-Gordon equation for ϕ in the curved background determined by the fixed frame field n^a .

It is worth emphasizing that the exterior derivative acts on the full phase-space, and $d\phi$ is independent of n^a (or dx^μ). The momentum is a differential 1-form $p = p_a n^a$, but p_a are not functions of x . They are coordinates in the extended phase-space just as in classical mechanics, and p_i are coordinates in the cotangent bundle $T^*(Q)$. Only when we look for a section, which makes action stationary, will independent phase-space coordinates p_a become functions of x .

The term $-\Theta^a T_a$ containing torsion of the gravitational field has been seen below to combine with the gravitational part of action when the frame field and connection are varied.

4. Gravitational field in EPS

The gravitational field is jointly determined by the co-frame fields n^a and the frame-gauge connection fields ω_{ab} . The “velocities” for these are the torsion $T = dn + \omega n$ and curvature $\Omega = d\omega + \omega\omega$ respectively. Since action is gauge invariant, the true variation fields X of co-frames n and gauge potentials ω contract with gauge-covariant T and Ω and not dn or $d\omega$.

We have taken the Poincare-Cartan 4-form for gravity as the Einstein-Hilbert action:

$$\Xi_{EH} = -\frac{1}{2\kappa} * (n^a n^b) \Omega_{ba} = \frac{1}{4\kappa} n_a E^a, \quad \kappa = 8\pi G_N,\tag{12}$$

where G_N is Newton’s gravitational constant, and we have defined the *Einstein 3-form* as:

$$E^a = *(n^a \wedge n^b \wedge n^c) \Omega_{bc} \equiv -2G^{ab}(*n_b).\tag{13}$$

The components G^{ab} are defined by this equation. They reduce to the usual symmetric Einstein tensor for the Riemannian geometry.

The variation of Einstein-Hilbert action is simply:

$$d\Xi_{EH} = \frac{1}{2\kappa} E T.\tag{14}$$

The proof goes as follows: Using formulas such as: $d*(n^a n^b) = *(n^a n^b n^c)dn_c$, and the Bianchi identity,

$$\begin{aligned}d[* (n^a n^b) \Omega_{ba}] &= *(n^a n^b n^c) dn_c \Omega_{ba} + *(n^a n^b) d\Omega_{ba} \\ &= *(n^a n^b n^c) \Omega_{ba} (T_c - \omega_{cd} n^d) + *(n^a n^b) (\Omega\omega - \omega\Omega)_{ba}.\end{aligned}$$

Furthermore,

$$*(n^a n^b n^c) n^d = -\eta^{ad} * (n^b n^c) + \eta^{bd} * (n^a n^c) - \eta^{cd} * (n^a n^b),$$

we can rearrange and simplify the terms not containing T . They all add up to

$$*(n^a n^b)(\Omega\omega + \omega\Omega)_{ba} = *(n^a n^b)(\Omega_{bd}\omega^d{}_a + \Omega_{ad}\omega^d{}_b) = 0,$$

being a sum over a product of a symmetric and an antisymmetric expression.

Combining with the derivative of the matter action given in Eq. (8) for the scalar field, we get:

$$-d(\Xi_\phi + \Xi_{EH}) = [dp(*n) - m^2\phi(*1)] [p - d\phi] - [E/2\kappa + \Theta]T. \quad (15)$$

The allowed field configurations given by the terms above can be read off easily. For variational fields in the direction of ϕ and p , we get the Klein-Gordon equations in curved background (determined by n and ω). In the second term, varying n and keeping ω constant, gives the Einstein equation:

$$E = -2\kappa\Theta \quad \text{or} \quad G^{ab} = (8\pi G_N)T^{ab}, \quad (16)$$

using Eqs. (13) and (10). And lastly, if ω is varied and n is kept fixed, we get the equation $T = 0$.

References

- [1] Arnold V I 1978 *Mathematical Principles of Classical Mechanics* (Springer-Verlag, New York, Section 9.C)
- [2] Pankaj Sharan [arxiv:1201.4092]
- [3] Pankaj Sharan 2009 *Spacetime, Geometry and Gravitation* (Hindustan Book Agency, New Delhi, Birkhauser, Basel, Section 6.6 for notation)