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Future non-linear stability of the Einstein-Vlasov system with Bianchi II symmetry

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Abstract. Assuming that the spacetime is close to the Collins-Stewart solution, which will play the role of the ω-limit, and that the maximal velocity of the particles is small, we have been able to show that for Bianchi II symmetry spacetimes with collisionless matter, the asymptotic behaviour at late times is close to the special case of dust. The key was a bootstrap argument.

1. The Einstein-Vlasov system

A cosmological model represents a universe at a certain averaging scale. It is described via a Lorentzian metric $g_{\alpha\beta}$ (we will use the signature $-+++$) on a manifold M, and a family of fundamental observers. The metric is assumed to be time-orientable, which means that at each point of M, the two halves of the light cone can be labelled past and future in a way which varies continuously from point to point. This enables to distinguish between future-pointing and past-pointing timelike vectors. This is physically a reasonable assumption from both a macroscopic point of view, e.g., the increase of entropy, and also from a microscopic point of view, e.g., the kaon decay. The interaction between the geometry and the matter is described by the Einstein's field equations (we use geometrized units, i.e., the gravitational constant G, and the speed of light in vacuum c are set equal to one) as:

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta} \ ,$$

where $G_{\alpha\beta}$ is the Einstein tensor, and $T_{\alpha\beta}$ is the energy-momentum tesnsor. The Einstein tensor satisfies:

$$\nabla^{\alpha} G_{\alpha\beta} = 0.$$

Thus, the energy-momentum tensor has to satisfy the same equation, which expresses the conservation of energy. For the matter model, we have taken the point of view of kinetic theory [1]. (The sign conventions of [2] are used. Also, the Einstein summation convention that repeated indices are to be summed over is used. Latin indices run from one to three and Greek ones from zero to three.)

Consider a particle with non-zero rest mass, which moves under the influence of the gravitational field. The mean field that we have mentioned in the introduction has been described now by the metric and the components of the metric connection. The worldline x^{α} of a particle is a timelike curve in spacetime. The unit future-pointing tangent vector to this curve is the

4-velocity v^α, and $p^\alpha = mv^\alpha$ is the 4-momentum of the particle. Let T_x be the tangent space at a point x^α in the spacetime M, then we have defined the *phase-space* P_m *for particles of mass* m as:

$$P_m = \{(x^\alpha,\ p^\alpha) : x^\alpha \in M,\ \ p^\alpha \in T_x,\ \ p_\alpha p^\alpha = -m^2,\ \ p^0 > 0\}.$$

We have considered from now on that all the particles have *equal* mass m. (For, how this relates to the general case of different masses, see [3].) We have chosen units such that $m = 1$, which means that a distinction between velocities and moments is not necessary. We have then that the possible values for the 4-momenta are all future-pointing unit timelike vectors. These values form the hypersurface:

$$P_1 = \{(x^\alpha,\ p^\alpha) : x^\alpha \in M,\ \ p^\alpha \in T_x,\ \ p_\alpha p^\alpha = -1,\ \ p^0 > 0\},$$

which we have called the *mass shell.* The collection of particles (galaxies or clusters of galaxies) will be described (statistically) by a non-negative real valued distribution function $f(x^\alpha,\ p^\alpha)$ on P_1. This function represents the density of particles at a given spacetime point with given 4-momentum. A free particle travels along a geodesic. Consider now a future-directed timelike geodesic parametrized by proper time s. The tangent vector is then at any time future-pointing unit timelike. Thus, the geodesic has a natural lift to a curve on P_1 by taking its position and tangent vector. The equations of motion, thus, define a flow on P_1, which is generated by a vector field L, which is called the *geodesic spray or Liouville operator.* The geodesic equations are:

$$\frac{dx^\alpha}{ds} = p^\alpha\ , \quad and \quad \frac{dp^\alpha}{ds} = -\Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma\ ,$$

where the components of the metric connection, i.e., $\Gamma_{\alpha\beta\gamma} = g(e_\alpha, \nabla_\gamma e_\beta) = g_{\alpha\delta}\Gamma^\delta_{\beta\gamma}$ can be expressed in the vector basis e_α as (1.10.3 of [4]):

$$\Gamma_{\alpha\beta\gamma} = \frac{1}{2}\Big(e_\beta(g_{\alpha\gamma}) + e_\gamma(g_{\beta\alpha}) + e_\alpha(g_{\gamma\beta}) + \eta^\delta_{\gamma\beta} g_{\alpha\delta} + \eta^\delta_{\alpha\gamma} g_{\beta\delta} - \eta^\delta_{\beta\alpha} g_{\gamma\delta}\Big)\ .$$

The commutator of the vectors e_α can be expressed with the following formula:

$$[e_\alpha,\ e_\beta] = \eta^\gamma_{\alpha\beta} e_\gamma\ ,$$

where $\eta^\gamma_{\alpha\beta}$ are called the *commutation functions.* The restriction of the Liouville operator to the mass shell is defined as:

$$L = \frac{dx^\alpha}{ds}\frac{\partial}{\partial x^\alpha} + \frac{dp^a}{ds}\frac{\partial}{\partial p^a}\ .$$

Using the geodesic equations, it has the followings form:

$$L = p^\alpha \frac{\partial}{\partial x^\alpha} - \Gamma^a_{\beta\gamma} p^\beta p^\gamma \frac{\partial}{\partial p^a}\ .$$

This operator is sometimes also called *geodesic spray.* If we denote, now, the phase-space density of collisions by $C(f)$, then the Boltzmann equation [5] in curved spacetime in our notation looks as follows:

$$L(f) = C(f)\ ,$$

which describes the evolution of the distribution function. Between collisions, the particles follow the geodesics. We have considered the collisionless case, which is described via the Vlasov equation, given as:

$$L(f) = 0\ .$$

The unknowns of our system are a 4-manifold M, a Lorentz metric $g_{\alpha\beta}$ on this manifold, and the distribution function f on the mass shell P_1 defined by the metric. We have the Vlasov equation defined by the metric for the distribution function and the Einstein's equations. It remains to define the energy-momentum tensor $T_{\alpha\beta}$ in terms of the distribution and the metric. Before that, we need a Lorentz invariant volume element on the mass shell. A point of the tangent space has the volume element $|g^{(4)}|^{\frac{1}{2}} dp^0 dp^1 dp^2 dp^3$, (where $g^{(4)}$ is the determinant of the spacetime metric), which is Lorentz invariant. Now, considering p^0 as a dependent variable, the induced (Riemannian) volume of the mass shell considered as a hypersurface in the tangent space at that point is:

$$\varpi = |p_0|^{-1} |g^{(4)}|^{\frac{1}{2}} dp^1 dp^2 dp^3 \ .$$

Now, we have defined the energy-momentum tensor as:

$$T_{\alpha\beta} = \int f(x^\alpha, \ p^\alpha) p_\alpha \ p_\beta \ \varpi \ .$$

One can show that $T_{\alpha\beta}$ is divergence-free and thus, it is compatible with the Einstein's equations. For collisionless matter, all the energy conditions hold. In particular, the dominant energy condition is equivalent to the statement that in any orthonormal basis, the energy density dominates the other components of $T_{\alpha\beta}$, i.e., $T_{\alpha\beta} \leq T_{00}$ for each α, β (p. 91 of [6]). Using the mass shell relation, one can see that this holds for collisionless matter. The non-negative sum pressure condition in our case is equivalent to $g_{ab} T^{ab} \geq 0$.

The Vlasov equation in a fixed spacetime can be solved by the method of characteristic (see Chapter 3.2 of [7]):

$$\frac{dX^a}{ds} = P^a \ , \quad and \quad \frac{dP^a}{ds} = -\Gamma^a_{\beta\gamma} P^\beta P^\gamma \ .$$

Let $X^a(s, \ x^\alpha, \ p^a)$, $P^a(s, \ x^\alpha, \ p^a)$ be the unique soultion of that equation with initial conditions $X^a(t, \ x^\alpha, \ p^a) = x^a$, and $P^a(t, \ x^\alpha, \ p^a) = p^a$. Then the solution of the Vlasov equation can be written as:

$$f(x^\alpha, \ p^a) = f_0(X^a(0, \ x^\alpha, \ p^a), \ P^a(0, \ x^\alpha, \ p^a)),$$

where f_0 is the restriction of f to the hypersurface $t = 0$. It follows that, if f_0 is bounded, the same is true for f. We have assumed that f has compact support in momentum space for each fixed t (note that it is not possible in the Boltzmann case). This property holds if the initial data f_0 has compact support, and if each hypersurface $t = t_0$ is Cauchy hypersurface [8].

2. Main result

Using the $3 + 1$ formulation, our initial data are: $(g_{ij}(t_0), \ k_{ij}(t_0), \ f(t_0))$, i.e., a Riemannian metric, a second fundamental form, and the distribution function of the Vlasov equation respectively, on a 3-dimensional manifold $S(t_0)$. This is the initial data set at $t = t_0$ for the Einstein-Vlasov system. We have decomposed the second fundamental form, by introducing σ_{ab} as the trace-free part: $k_{ab} = \sigma_{ab} - H g_{ab}$, where $H = -\frac{1}{3}k$ is the Hubble parameter. We have defined: $\Sigma^b_a = \frac{\sigma^b_a}{H}$, $\Sigma_+ = -\frac{1}{2}(\Sigma^2_2 + \Sigma^3_3)$ and $\Sigma_- = -\frac{1}{2\sqrt{3}}(\Sigma^2_2 - \Sigma^3_3)$, $N^j_i = \frac{R^j_i}{H^2}$, and $(N_1)^2 = -2\frac{R}{H^2}$, where R_{ij} is the Ricci tensor. We have a number (different from zero) of particles at possible different momenta, and we have defined P as the supremum of the absolute value of these momenta at a given time t as:

$$P(t) = sup \left\{ |p| = \left(g^{ab} p_a p_b \right)^{\frac{1}{2}} | f(t, p) \neq 0 \right\} \ .$$

We have obtained the following:

Theorem: *Consider any C^∞ solution of the Einstein-Vlasov system with Bianchi II symmetry, and with C^∞ initial data. Assume that* $|\Sigma_+(t_0)-\frac{1}{8}|$, $|\Sigma_-(t_0)|$, $|\Sigma_2^1(t_0)|$, $|\Sigma_3^1(t_0)|$, $|\Sigma_3^2(t_0)|$, $|\Sigma_2^3(t_0)|$, $|\Sigma_1^2(t_0)|$, $|\Sigma_1^3(t_0)|$, $|N_1(t_0) - \frac{3}{4}|$, $|N_2^1(t_0)|$, $|N_3^1(t_0)|$, *and* $P(t_0)$ *are sufficiently small. Then at late times, after possibly a basis change, the following estimates hold*:

$$\begin{aligned}
H(t) &= \frac{2}{3}t^{-1}(1+O(t^{-\frac{1}{2}})) \ , \\
\Sigma_+ - \frac{1}{8} &= O(t^{-\frac{1}{2}}) \ , \\
\Sigma_- &= O(t^{-1}) \ , \\
\Sigma_2^1 &= O(t^{-\frac{1}{2}}) \ , \\
\Sigma_3^1 &= O(t^{-\frac{1}{2}}) \ , \\
\Sigma_3^2 &= O(t^{-1}) \ , \\
\Sigma_2^3 &= O(t^{-1}) \ , \\
\Sigma_1^2 &= O(t^{-1}) \ , \\
\Sigma_1^3 &= O(t^{-1}) \ , \\
N_1 - \frac{3}{4} &= O(t^{-\frac{1}{2}}) \ , \\
N_2^1 &= O(t^{-\frac{1}{2}}) \ , \\
N_3^1 &= O(t^{-\frac{1}{2}}) \ , \quad \textit{and} \\
P(t) &= O(t^{-\frac{1}{2}}) \ .
\end{aligned}$$

We will present the proof in a different paper.

References

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